Kuznetsov's conjecture and rationality of cubic fourfolds

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Outline

1. Motivations
2. Generalities on the Hassett's theory of cubic fourfolds
3. Kuznetsov's conjecture on the rationality of cubic fourfolds
4. Rationality via congruences of $(3e-1)$-secant curves of degree e
5. Rationality via trisecant flops
6. Generalities on the Debarre-Iliev-Manivel's theory of Gushel-Mukai fourfolds
7. A new divisor of rational Gushel-Mukai fourfolds and the rationality of the cubics in $\mathcal{C}_{42}$

Section 1

Motivations

Rationality of smooth degree-d hypersurfaces in $\mathbb{P}^{n+1}_{\mathbb{C}}$

d	curves	surfaces	threefolds	fourfolds	5-folds	6-folds	7-folds
2	all are rational	all are rational	all are rational	all are rational	all are rational	all are rational	all are rational
3	all are irrational	all are rational	all are irrational	open problem (we have many rational examples and a precise conjecture)	open problem	open problem (we know just some very special rational examples)	open problem
4	all are irrational	all are irrational	all are irrational	the very general is irrational but rational ones are unknown	the very general is irrational but rational ones are unknown	open problem	open problem
5	all are irrational	all are irrational	all are irrational	all are irrational	the very general is irrational but rational ones are unknown	the very general is irrational but rational ones are unknown	the very general is irrational but rational ones are unknown
6	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	the very general is irrational but rational ones are unknown	the very general is irrational but rational ones are unknown
7	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	the very general is irrational but rational ones are unknown
8	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational	all are irrational

Classical and easy.

Easy cases: $K_X = -i\,H_X$ with $i = n+2-\deg(X) \leq 0$, hence $p_g(X) = h^0(X, K_X) \neq 0$.

Famous result of Clemens and Griffiths.

Hard result (contrib. by Noether, Fano, Segre, Iskovskikh, Manin, Corti, Pukhlikov, Cheltsov, de Fernex, Ein, Mustata, Zhuan).

Result of Totaro obtained by combining methods and results of Kollár, Voisin, Colliot, Thélène, Pirutka.

Recent result of Schreieder.

Very recent result of Nicaise and Ottem.

Rationality for all prime Fano fourfolds of coindex ≤ 3

Fourfold	Irrationality of very general	Description of the rational ones	Birational representation on $\mathbb{P}^4$
Quadric hypersurface in $\mathbb{P}^5$	no	all	projection from a point on it
Cubic hypersurface in $\mathbb{P}^5$	not known	just many examples but with a precise conjecture	(constructions due to Fano and Russo-S.)
Quartic hypersurface in $\mathbb{P}^5$	yes	no known examples	
Complete intersection of two quadrics in $\mathbb{P}^6$	no	all	projection from a line on it
Complete intersection of a quadric and a cubic in $\mathbb{P}^6$	yes	no known examples	
Complete intersection of three quadrics in $\mathbb{P}^7$	yes	just some examples	*e.g.*, when the fourfold contains a plane then one takes the projection from it
Linear section in $\mathbb{P}^7$ of $\mathbb{G}(1,4) \subset \mathbb{P}^9$	no	all	projection from the unique ρ-plane
GM fourfold (fourfold of degree 10, genus 6, and coindex 3 in $\mathbb{P}^8$)	not known	just many examples but with a precise conjecture	(constructions due to Roth, Debarre--Iliev-Manivel, and Hoff-Russo-S.)
Linear section in $\mathbb{P}^9$ of the spinorial $\mathbb{S}^{10} \subset \mathbb{P}^{15}$	no	all	projection from a tangent space
Linear section in $\mathbb{P}^{10}$ of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$	no	all	linear system of quadrics through a quintic del Pezzo surface contained in it
Linear section in $\mathbb{P}^{11}$ of the Lagrangian Grass. $LG(3,6) \subset \mathbb{P}^{13}$	no	all	linear system of hyperplane sections through a line and with one double point
Fourfold of degree 18, genus 10, and coindex 3 in $\mathbb{P}^{12}$	no	all	linear system of hyperplane sections through a conic and with one double point

Classical and easy. Classical results of Todd, Fano, and mainly Roth.

Result of Totaro obtained by combining methods and results of Kollár, Voisin, Colliot, Thélène, Pirutka.

Result of Hassett, Pirutka, and Tschinkel. Very recent result of Nicaise and Ottem.

Section 2

Generalities on the Hassett's theory of cubic fourfolds

Parameter spaces of cubic fourfolds

A *cubic fourfold* is just a smooth cubic hypersurface in $\mathbb{P}^5$.

Cubic fourfolds are parametrized by an open set U of the projective space $\mathbb{P}(\mathbb{C}[x_0, \ldots, x_5]_3) \simeq \mathbb{P}^{55}$. Its complementary set $\Delta = \mathbb{P}^{55} \setminus U$ is the *discriminant hypersurface*, an irreducible hypersurface of degree 192.

The moduli space of cubic fourfolds is the quotient

$$\mathcal{C} = [U/\mathrm{PGL}_6],$$

which is an irreducible quasi-projective variety of dimension

$$\dim(\mathcal{C}) = \dim(U) - \dim(\mathrm{PGL}_6) = 55 - 35 = 20.$$

Unirationality of cubic fourfolds

Let $X \subset \mathbb{P}^5$ be a general cubic fourfold, and let $L \subset X$ be a line (the family of lines contained in X is smooth of dimension 4).

- Consider the family $W \subset \mathbb{G}(1,5)$ of lines of $\mathbb{P}^5$ which are tangent to X at some point of L. Then W is a rational fourfold (it is isomorphic to a cone of vertex a point over a smooth 3-dim. rational normal scroll of degree 5).
- We get a $2:1$ rational map $W \dashrightarrow X$ by sending a general line $[T] \in W$ to the point $T \cap X \setminus L$.
- This construction can be done explicitly once we have a line $L \subset X$ defined over the base field. So, we can get an explicit rational map

$$\mathbb{P}^1 \times \mathbb{P}^3 \overset{1:1}{\dashrightarrow} \{(p,t) : p \in L, t \in \mathbb{P}(T_pX)\} \overset{2:1}{\dashrightarrow} X \subset \mathbb{P}^5$$

  which is defined by biforms of bidegree $(7,3)$ and has multidegree $(4,16,15,12,6)$.

Special cubic fourfolds

For a cubic fourfold $X \subset \mathbb{P}^5$, we define

$$H^{2,2}(X,\mathbb{Z}) := H^4(X,\mathbb{Z}) \cap H^2(\Omega_X^2).$$

Theorem (Voisin, 1986)

If $[X] \in \mathcal{C}$ is very general, then $H^{2,2}(X,\mathbb{Z}) \simeq \mathbb{Z}\langle h^2\rangle$, where h denotes the class of a hyperplane section of X.

If $H^{2,2}(X,\mathbb{Z}) \supsetneq \mathbb{Z}\langle h^2\rangle$ then there exists an "algebraic surface" (cycle of dimension 2 with integral coefficients) $S \subset X$ such that $H^{2,2}(X,\mathbb{Z}) \supseteq K$, with $K = \langle h^2, S\rangle$ and $\mathrm{rk}(K) = 2$. In this case, X is said to be *special*.

Hassett (1999) defined the Noether-Lefschetz loci $\mathcal{C}_d$ as

$$\mathcal{C}_d = \{[X] \in \mathcal{C} : \exists K \subseteq H^{2,2}(X, \mathbb{Z}),\ h^2 \in K,\ \mathrm{rk}(K) = 2, |K| = d\}.$$

The discriminant $|K|$ of the sublattice K is defined as the determinant of the intersection form on K. If $K = \langle h^2, S \rangle$, with S an algebraic surface, then

$$|K| = \det \begin{pmatrix} h^4 & h^2 \cdot S \\ S \cdot h^2 & (S)^2_X \end{pmatrix} = \det \begin{pmatrix} 3 & \deg(S) \\ \deg(S) & (S)^2_X \end{pmatrix} = 3(S)^2_X - (\deg(S))^2.$$

When S has smooth normalization and only a finite number δ of nodes as singularities, the self-intersection $(S)^2_X$ can be explicitly calculated by the following formula:

$$(S)^2_X = 6h^2 + 3h \cdot K_S + K_S^2 - \chi_S + 2\delta.$$

First properties of Hassett's Noether-Lefschetz loci

Theorem (Hassett, 1999, 2000)

1. *$\mathcal{C}_d$ is either empty or an irreducible divisor in $\mathcal{C}$.*
2. *$\mathcal{C}_d \neq \emptyset$ if and only if $d > 6$ and $d \equiv 0, 2 \mod 6$.*
3. *If $[X] \in \mathcal{C}_d$ is very general, then $H^{2,2}(X, \mathbb{Z}) = \langle h^2, S \rangle$, for some algebraic surface S.*

In particular, the first values of d for which $\mathcal{C}_d$ is not empty are:

$$8,\ 12,\ 14,\ 18,\ 20,\ 24,\ 26,\ 30,\ 32,\ 36,\ 38,\ 42,\ 44,\ 48,\ 50,\ 54,\ 56,\ 60,\ 62, \ldots$$

Geometric descriptions of some Hassett's divisors

$\mathcal{C}_8$: Cubic fourfolds containing a plane

$$\mathcal{C}_8 = \{[X] \in \mathcal{C} : X \supset P, P \text{ plane } \},$$

$$K_8 = \begin{array}{c|cc} & h^2 & P \\ \hline h^2 & 3 & 1 \\ P & 1 & 3 \end{array}$$

$\mathcal{C}_{12}$: Cubic fourfolds containing a cubic scroll

$$\mathcal{C}_{12} = \overline{\{[X] \in \mathcal{C} : X \supset \Sigma_3, \Sigma_3 \text{ cubic scroll}\}},$$

$$K_{12} = \begin{array}{c|cc} & h^2 & \Sigma_3 \\ \hline h^2 & 3 & 3 \\ \Sigma_3 & 3 & 7 \end{array}$$

$\mathcal{C}_{14}$: Cubic fourfolds containing a quartic scroll

$$\begin{aligned}\mathcal{C}_{14} &= \overline{\{[X] \in \mathcal{C} : X \supset \Sigma_4, \Sigma_4 \text{ quartic scroll}\}} \\ &= \overline{\{[X] \in \mathcal{C} : X \supset T, T \text{ quintic de Pezzo surface}\}},\end{aligned}$$

$$K_{14} = \begin{array}{c|cc} & h^2 & \Sigma_4 \\ \hline h^2 & 3 & 4 \\ \Sigma_4 & 4 & 10 \end{array} \simeq \begin{array}{c|cc} & h^2 & T \\ \hline h^2 & 3 & 5 \\ T & 5 & 13 \end{array}, T + \Sigma_4 = 3h^2.$$

$\mathcal{C}_{20}$: Cubic fourfolds containing a Veronese surface

$$\mathcal{C}_{20} = \overline{\{[X] \in \mathcal{C} : X \supset V, V \text{ Veronese surface }\}},$$

$$K_{20} = \begin{array}{c|cc} & h^2 & V \\ \hline h^2 & 3 & 4 \\ V & 4 & 12 \end{array}$$

Section 3

Kuznetsov's conjecture on the rationality of cubic fourfolds

Rational cubic fourfolds

Let us define

$$\mathrm{Rat}(\mathcal{C}) = \{[X] \in \mathcal{C} : X \text{ is rational}\}.$$

As a consequence of a result of de Fernex and Fusi (2013) and improved by Kontsevich and Tschinkel (2019) we have that

Proposition

$\mathrm{Rat}(\mathcal{C})$ *and* $\mathrm{Rat}(\mathcal{C}) \cap \mathcal{C}_d$ *are countable unions of closed subsets.*

The general natural expectation is that

$$\mathrm{Rat}(\mathcal{C}) \subseteq \bigcup_d \mathcal{C}_d$$

In particular, *a very general cubic fourfold* $X \subset \mathbb{P}^5$ *should be not rational.*

No cubic fourfold is known to be irrational.

All the known rational cubic fourfolds belong to some divisor $\mathcal{C}_d$.

Associated K3 surfaces and admissible integers

Motivated by the proof of Clemens and Griffiths (1972) for the irrationality of cubic 3-folds, Hassett (1999) considered an *associated K3 surface* to a cubic fourfold $[X] \in \mathcal{C}_d$ with rank two sublattice $K \subseteq H^{2,2}(X, \mathbb{Z})$ of discriminant d.

A polarized K3 surface (S, f) of degree d is associated to X if there exists an isomorphism $H^2(S, \mathbb{Z})(-1) \supset f^\perp \xrightarrow{\simeq} K^\perp$ respecting Hodge structures.

Theorem (Hassett, 1999)

$[X] \in \mathcal{C}_d$ *admits an associated K3 surface if and only if* d *is admissible.*

Definition

An even integer $d > 6$ is admissible if it is not divisible by 4, 9 or any odd prime congruent to 2 module 3.

Admissible values < 140

8 12 **14** 18 20 24 **26** 30 32 36 **38** **42** 44 48 50 54 56 60 **62** 66 68 72 **74** **78** 80 84 **86** 90 92 96 **98** 102 104 108 110 **114** 116 120 **122** 126 128 132 **134** 138

Kuznetsov's conjecture

Kuznetsov (2010) introduced another *associated K3 surface* to a cubic fourfold $[X] \in \mathcal{C}$ in terms of derived categories of coherent sheaves on X.

Kuznetsov conjectures that X is rational if and only if it admits this new associated K3 surface.

Theorem (Addington and Thomas, 2014)

If a cubic fourfold X has an associated K3 surface in the sense of Kuznetsov, then it has an associated K3 surface in the sense of Hassett, i.e. $[X] \in \mathcal{C}_d$ with d admissible. Conversely, a generic $[X] \in \mathcal{C}_d$ with d admissible has an associated K3 surface in the sense of Kuznetsov.

Moreover, it is recently shown by Bayer-Lahoz-Macrì-Nuer-Perry-Stellari (2019) that in the above result the word "*generic*" can be replaced by "*every*".

Equivalent form of Kuznetsov's conjecture

A cubic fourfold $[X] \in \mathcal{C}$ is rational if and only if $[X] \in \mathcal{C}_d$ with d admissible.

Census of rational cubic fourfolds

Theorem (Morin, 1940; Fano, 1943)

A generic $[X] \in \mathcal{C}_{14}$ is rational. (The word "generic" can be replaced by "every", by [Bolognesi, Russo, S., 2019] or [Kontsevich and Tschinkel, 2019].)

Theorem (Hassett, 1999, for $\mathcal{C}_8$; A., H., T., V.-A., 2016, for $\mathcal{C}_{18}$)

There exists a countably infinite union of codimension-two subloci $\mathcal{D}_8 = \bigcup_m \mathcal{D}_m \subset \mathcal{C}_8$ (resp. $\mathcal{D}_{18} \subset \mathcal{C}_{18}$) parametrizing rational cubic fourfolds.

1. $\mathcal{D}_8$ and $\mathcal{D}_{18}$ parametrize cubic fourfolds X with $rk(H^{2,2}(X,\mathbb{Z})) \geq 3$.
2. $\mathcal{D}_8$ corresponds to the cubic fourfolds X containing a plane P such that the induced quadric fibration $X \dashrightarrow \mathbb{P}^2$ obtained by projecting from P admits a rational section ($\mathcal{D}_{18}$ has a similar description).

Kuznetsov's conjecture for $d = 14, 26, 38, 42$

Until July 2017, these were all the cubic fourfolds known to be rational. So, Kuznetsov's conjecture was known to be true only for the first admissible value $d = 14$ by the classical works of Morin (1940) and Fano (1943).

Theorem (Russo and S., 2019a (preprint in 2017))

A generic cubic fourfold in $\mathcal{C}_{26}$ and $\mathcal{C}_{38}$ is rational. (As before, "generic" can be replaced by "every", by [Kontsevich and Tschinkel, 2019].)

This result follows from the discovery of so-called "*congruences of 5-secant conics*", which will be illustrated in the following slides.

Moreover, very recently we also showed the following:

Theorem (Russo and S., 2019b)

A generic (hence every) cubic fourfold in $\mathcal{C}_{42}$ is rational.

Thus we have:

$$\mathcal{C}_{14} \cup \mathcal{C}_{26} \cup \mathcal{C}_{38} \cup \mathcal{C}_{42} \subset \mathrm{Rat}(\mathcal{C}).$$

Section 4

Rationality via congruences of $(3e-1)$-secant curves of degree e

Congruences of $(3e-1)$-secant curves of degree e

Let $\mathcal{H}$ be an irreducible proper family of rational curves of degree $e \geq 1$ in $\mathbb{P}^5$ whose general element $[C] \in \mathcal{H}$ is irreducible.

We have a universal family $\mathcal{D}$ and two natural projections:

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "\pi"'] \arrow[dr, "\psi"] & \\ \mathcal{H} & & \mathbb{P}^5 \end{tikzcd} \tag{1}$$

such that $\psi(\pi^{-1}([C])) = C \subset \mathbb{P}^5$.

Definition

Let $S \subset \mathbb{P}^5$ be an irreducible surface. We say that (1) is a *congruence of $(3e-1)$-secant curves of degree e* to S if the following hold:

1. ψ is birational;
2. for $[C] \in \mathcal{H}$ general, the intersection $C \cap S$ consists of $3e-1$ points.

Let $S \subset \mathbb{P}^5$ be a surface admitting a congruence of $(3e-1)$-secant curves of degree e parametrized by $\mathcal{H}$, and let $X \in |H^0(\mathcal{I}_S(3))|$ be an irreducible cubic hypersurface containing S.

If $[C] \in \mathcal{H}$ is general, from Bézout's theorem there exists a unique point $p \in X$ such that $\{p\} = (C \cap X) \setminus (C \cap S)$. Thus we have a rational map:

$$\alpha : \mathcal{H} \dashrightarrow X, \quad \alpha([C]) = (C \cap X) \setminus (C \cap S).$$

If $p \in X$ is a general point, from Zariski's main theorem there exists a unique $[C_p] \in \mathcal{H}$ such that $p \in C_p$. Thus we have another rational map:

$$\beta : X \dashrightarrow \mathcal{H}, \quad \beta(p) = [C_p].$$

We say that X is *trasversal to the congruence* $\mathcal{H}$ if for the general point $p \in X$ the curve $\beta(p)$ is not contained in X, *i.e.*, if the composition $\alpha \circ \beta$ is a well-defined rational map.

If X is transversal to the congruence, then α and β are inverse to each other.

Genericity implies transversality

Proposition

Let $S \subset \mathbb{P}^5$ be a surface admitting a congruence of $(3e-1)$-secant curves of degree e parametrized by $\mathcal{H}$. Assume that the linear system $|H^0(\mathcal{I}_S(3))|$ of cubics through S defines a birational map $\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^N$. Then a general $X \in |H^0(\mathcal{I}_S(3))|$ is trasversal to $\mathcal{H}$, in particular X is birational to $\mathcal{H}$.

The proof follows easily from the fact that Φ induces a 1–1 correspondence:

$$\bigcup_{e\geq 1} \left\{ \begin{array}{c} (3e-1)\text{-secant curves of degree } e \text{ to } S \\ \text{passing through a general point } p \in \mathbb{P}^5 \end{array} \right\} \to \left\{ \begin{array}{c} \text{lines contained in } Z \\ \text{passing through } \Phi(p) \end{array} \right\}$$

How to apply congruences

Let $S \subset \mathbb{P}^5$ be a surface admitting a congruence of $(3e-1)$-secant curves of degree e parametrized by $\mathcal{H}$, as in the previous Proposition.

Suppose we are able to show at least one of the following:

1. $\mathcal{H}$ is rational (resp. irrational);
2. there exists a particular singular cubic hypersurface $X \in |H^0(\mathcal{I}_S(3))|$ trasversal to $\mathcal{H}$ which is rational (resp., irrational), for instance $\mathrm{Sing}(X)$ is a double point (resp., a triple point).

Then we can conclude that a general cubic hypersurface $X \in |H^0(\mathcal{I}_S(3))|$ is rational (resp., irrational).

Fano's classical applications of congruences ($e = 1$)

The surfaces $S \subset \mathbb{P}^5$ admitting a congruence of 2-secant lines are usually called OADP (*one apparent double point*).

The OADP surfaces have been completely classified in the works by Severi (1901), Russo (2000), and Ciliberto and Russo (2011). Those contained in a cubic fourfold are only the following:

- quintic del Pezzo surfaces,
- smooth quartic rational normal scrolls.

Cubic fourfolds through these surfaces describe the divisor $\mathcal{C}_{14}$ as it was firstly remarked by Fano (1943).

A congruence of 5-secant conics for the rationality of cubic fourfolds in $\mathcal{C}_{26}$

The first example of congruence of $(3e-1)$-secant curves of degree $e>1$

Let $S\subset\mathbb{P}^5$ be a rational septimic scroll with three nodes, which is the projection of the rational normal scroll $S'=\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(3)\oplus\mathcal{O}_{\mathbb{P}^1}(4))\subset\mathbb{P}^8$ from a plane intersecting the secant variety $\mathrm{Sec}(S')$ at three general points.

Farkas and Verra (2018) showed that the cubic fourfolds containing such a surface S describe the divisor $\mathcal{C}_{26}$.

1. A surface $S\subset\mathbb{P}^5$ as above is cut out by 13 cubics which give a birational map $\Phi:\mathbb{P}^5\dashrightarrow Z\subset\mathbb{P}^{12}$ onto a fivefold $Z\subset\mathbb{P}^{12}$ such that through a general point $\Phi(p)$, $p\in\mathbb{P}^5$ general, there pass 8 lines contained in Z.
2. Since $S\subset\mathbb{P}^5$ has only 7 secant lines through p, there is an "*excessing line*" $L\subset Z$ passing through $\Phi(p)$ which does not come from secant lines to S. This line is the image of a 5-secant conic to S.

Section 5

Rationality via trisecant flops

To showing the rationality of cubic fourfolds via (only) congruences of $(3e-1)$-secant curves of degree e has two main disadvantages:

- it does not give information about the explicit birational map to $\mathbb{P}^4$;
- it does not clarify the relation with the associated K3 surfaces.

To solve these disadvantages, in [Russo and S., 2019b] we introduced the trisecant flop for the study of the rationality of cubic fourfolds. We explain this new method in the following slides.

General (simplified) assumptions

The trisecant locus

Let us fix a smooth irreducible non-degenerate surface $S \subset \mathbb{P}^5$ cut out scheme-theoretically by cubics which define a birational map

$$\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^N.$$

For simplicity, we can assume that the Koszul syzygies of the cubics defining S are generated by the linear ones (*Vermeire's $\mathcal{K}_3$ condition*).

This leads to the simplification that the exceptional locus of Φ coincides set-theoretically with the trisecant locus of $S \subset \mathbb{P}^5$, $\mathrm{Trisec}(S) \subset \mathbb{P}^5$, which is defined as the closure of the union of all the trisecant lines to S.

Expected trisecant behaviour

Let $\operatorname{Al}^3 S$ be the Hilbert scheme of length 3 aligned subschemes of $S \subset \mathbb{P}^5$.
If $\operatorname{Al}^3 S \neq \emptyset$ then every irreducible component of $\operatorname{Al}^3 S$ has dimension at least 2 with expected dimension 2. The smoothness of $\operatorname{Al}^3 S$ is related to the tangential behaviour of S at the points of intersection of a general trisecant line. More precisely we have:

Proposition (Gruson and Peskine, 2013)

Let $L \subset \mathbb{P}^5$ be a proper trisecant line to S, with $[L \cap S]$ belonging to a 2-dimensional irreducible component A of $\operatorname{Al}^3 S$. Then $\operatorname{Al}^3 S$ is smooth at $[L \cap S]$ if and only if the tangent planes to S at the points in $L \cap S$ are in general linear position.
In this case, A is generically smooth, and the irreducible component of $\operatorname{Trisec}(S)$ corresponding to A has dimension 3.

We say that $S \subset \mathbb{P}^5$ has the expected trisecant behaviour if $\operatorname{Al}^3 S$ is of pure dimension two and every irreducible component is generically smooth.

Induced flop small contractions

Let $X \subset \mathbb{P}^5$ be a general cubic fourfold containing S. The restriction to X of the map $\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^N$ is a birational map

$$\varphi : X \dashrightarrow Y \subset \mathbb{P}^{N-1},$$

where Y is the corresponding hyperplane section of Z.

Let $\lambda : X' = \mathrm{Bl}_S X \to X$ be the blow-up of X along S. Then the induced morphism

$$\tilde{\varphi} : X' \to Y$$

is defined by the linear system

$$|3\,\lambda^*(H) - E| = |\lambda^*(-K_X) - E| = |-K_{X'}|.$$

If S has the expected trisecant behaviour, then $\tilde{\varphi}$ is a flop small contraction. Indeed its exceptional locus consists of the strict trasforms of the trisecant lines to S contained in X (which is at most 2-dimensional), and if $L \subset X'$ is such a curve, we have

$$K_{X'} \cdot L = (E - 3\lambda^*(H)) \cdot L = 3 - 3 = 0.$$

Existence of the trisecant flop

Theorem

Under the above assumptions, if the exceptional locus of $\tilde{\varphi}$ is a smooth irreducible surface T, then there exists a smooth projective fourfold W', a smooth surface $R \subset W'$, and a small contraction $\tilde{\psi} : W' \dashrightarrow Y$ defined by $|-K_{W'}|$, such that we have a diagram:

$$\begin{tikzcd}
 & \mathrm{Bl}_T X' = \mathrm{Bl}_R W' \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X' \arrow[rr, dashed, "\tau"'] \arrow[d, "\lambda"'] \arrow[dr, "\tilde{\varphi}"'] & & W' \arrow[dl, "\tilde{\psi}"] \\
X & Y &
\end{tikzcd}$$

where $\sigma^*(K_{X'}) = \omega^*(K_{W'})$.

The key idea in the proof is that, under the assumptions, the pull back via σ of the strict transform of a trisecant line to S is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and hence it admits another projection onto a smooth rational curve.

Extremal contraction of the congruence

Theorem

In the previous hypothesis, assume also that $S \subset \mathbb{P}^5$ admits a congruence of $(3e-1)$-secant rational curves of degree $e \geq 2$. Then there exists a $\mathbb{Q}$-factorial Fano variety W with $\mathrm{Pic}(W) \simeq \mathbb{Z}$ and index $i(W)$, and a birational morphism $\nu : W' \dashrightarrow W$ which is generically the blow-up of an irreducible surface $U \subset W$, such that the previous diagram can be completed as follows:

$$\begin{tikzcd}
 & \mathrm{Bl}_T X' = \mathrm{Bl}_R W' \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X' \arrow[rr, dashed, "\tau"] \arrow[d, "\lambda"'] \arrow[dr, "\tilde{\varphi}"] & & W' \arrow[d, "\nu"] \arrow[dl, "\tilde{\psi}"'] \\
X \arrow[r, dashed, "\varphi"] \arrow[rr, dashed, bend right, "\mu"'] & Y & W \arrow[l, dashed, "\psi"'] \arrow[ll, dashed, bend right, "\mu^{-1}"']
\end{tikzcd}$$

$$\psi : |H^0(\mathcal{I}_U(i(W)))|$$

$$\mu : |H^0(\mathcal{I}_S^e(3e-1))|$$

$$\mu^{-1} : |H^0(\mathcal{I}_U^e(i(W)e-1))|.$$

A trisecant flop for the rationality of the cubics in $\mathcal{C}_{38}$

Trisecant locus of a Coble surface

Now we specialize the previous setting to the smooth rational surface $S \subset \mathbb{P}^5$ of degree 10 and sectional genus 6 obtained as the image of $\mathbb{P}^2$ via the linear system of curves of degree 10 with 10 general triple points.

By a parameter count, Nuer (2015) showed that the cubic fourfolds containing a surface S as above describe the divisor $\mathcal{C}_{38}$.

In [Russo and S., 2019a] we proved that S admits a conguence of 5-secant conics. Moreover, the map defined by $|H^0(\mathcal{I}_S^2(5))|$ is dominant onto $\mathbb{P}^4$.

The surface S is cut out by 10 cubics satisfying the $\mathcal{K}_3$ condition, and the map

$$\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^9,$$

defined by $|H^0(\mathcal{I}_S(3))|$, is birational onto its image Z.

The exceptional locus of Φ, *i.e.* $\mathrm{Trisec}(S)$, consists of

- ten planes (contracted to ten points), and
- an irreducible threefold B (contracted to a Veronese surface), which is a degeneration of the Bordiga scroll of degree 6 and sectional genus 3.

A trisecant flop for the rationality of the cubics in $\mathcal{C}_{38}$

Flop small contraction and congruence of 5-secant conics for a Coble surface

Let $X \subset \mathbb{P}^5$ be a general cubic fourfold containing a surface S as above, and let $\varphi : X \dashrightarrow Y = Z \cap \mathbb{P}^8 \subset \mathbb{P}^8$ be the restriction of $\Phi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^9$.

- X is a general cubic fourfold in $\mathcal{C}_{38}$; in particular, it cannot contain any of the ten planes in the exceptional locus of Φ.
- The exceptional locus of φ is a rational scroll surface of degree 8 with 6 nodes.
- The exceptional locus of the induced morphism

$$\tilde{\varphi} : X' = \mathrm{Bl}_S X \dashrightarrow Y \subset \mathbb{P}^8$$

  is a smooth rational surface which is contructed to a rational normal quartic curve.

In particular, $\tilde{\varphi}$ is a flop small contraction and all the hypotesis for the existence of a trisecant flop are satisfied.

Associated K3 surfaces for the cubics in $\mathcal{C}_{38}$

In conclusion, we have a commutative diagram:

$$\begin{tikzcd}
 & \operatorname{Bl}_T X' = \operatorname{Bl}_R W' \arrow[dl, "\sigma"'] \arrow[dr, "\omega"] & \\
X' \arrow[rr, dashed, "\tau"] \arrow[d, "\lambda"'] \arrow[dr, "\tilde{\varphi}"] & & W' \arrow[d, "\nu"] \arrow[dl, "\tilde{\psi}"'] \\
X \arrow[r, dashed, "\varphi"] \arrow[rr, dashed, bend right, "\mu"'] & Y & \mathbb{P}^4 \arrow[l, dashed, "\psi"'] \arrow[ll, dashed, bend right, "\mu^{-1}"']
\end{tikzcd}$$

$$\psi : |H^0(\mathcal{I}_U(5))|$$

$$\mu : |H^0(\mathcal{I}_S^2(5))|$$

$$\mu^{-1} : |H^0(\mathcal{I}_U^2(9))|.$$

It turns out that the surface $U \subset \mathbb{P}^4$ is a smooth surface of degree 12 and sectional genus 14 cut out scheme-theoretically by 9 quintics.

The surface U had been also constucted by Decker, Ein, and Schreyer (1993), who showed that it is the blow-up at eleven points of a minimal K3 surface of degree 38 and genus 20 in $\mathbb{P}^{20}$.

Section 6

Generalities on the Debarre-Iliev-Manivel's theory of Gushel-Mukai fourfolds

Gushel-Mukai fourfolds

Definition

A Gushel-Mukai fourfold $X \subset \mathbb{P}^8$, GM fourfold for short, is a (smooth) quadratic section of a 5-dimensional linear section of the cone over the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$. Equivalently, X is a degree-10 Fano fourfold with $\mathrm{Pic}(X) = \mathbb{Z}\langle \mathcal{O}_X(1) \rangle$ and $K_X \in |\mathcal{O}_X(-2)|$.

There are two types of GM fourfolds:

- quadratic sections of hyperplane sections of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ (*Mukai or ordinary fourfolds*, [Mukai, 1989]);
- double covers of $\mathbb{G}(1,4) \cap \mathbb{P}^7$ branched along its intersection with a quadric (*Gushel fourfold*, [Gushel, 1982]).

There is a 24-dimensional coarse moduli space $\mathcal{M}_4^{GM}$ of GM fourfolds, where the locus of Gushel fourfolds is of codimension 2. Moreover, we have a *period map* $\mathfrak{p} : \mathcal{M}_4^{GM} \to \mathcal{D}$ onto a 20-dimensional quasi-projective variety $\mathcal{D}$, which is dominant with 4-dimensional irred. fibers [Debarre, Iliev, and Manivel, 2015].

Unirationality of general GM fourfolds

Let $Y = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ be a smooth del Pezzo fivefold (note that all smooth del Pezzo fivefolds are projectively equivalent), and let $X = Y \cap Q \subset \mathbb{P}^8$ be a general quadratic section of Y, hence a general ordinary GM fourfold.

- Y contains exactly one linear 3-dimensional space P; the intersection $S = P \cap Q \subset X$ is a quadric surface, the so-called *σ-quadric surface*.
- If p is a general point of Y, then the Hilbert scheme $\mathrm{Hilb}_p^{t+1}(Y)$ of lines contained in Y and passing through p is a smooth cubic scroll surface; but if $p \in S$, $\mathrm{Hilb}_p^{t+1}(Y)$ is the union of a fixed plane (parametrizing lines of P) with a smooth quadric Ξ_p which intersects the plane along a line.
- The $(\mathbb{P}^1 \times \mathbb{P}^1)$-bundle $W = \bigcup_{p \in S} \Xi_p$ over S is a rational fourfold (if $p \in S$ is the generic point, the quadric Ξ_p is rational over the function field of S).
- The general line $[L] \in W$ is 2-secant to X, and we get a $2:1$ rational map $W \dashrightarrow X$ by sending $[L] \in W$ to $L \cap X \setminus S$.
- This construction can be done explicitly once we have a point on S. It turns out that the composition $\mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow W \dashrightarrow X$ is given by biforms of bidegree $(22, 4)$, while $\mathbb{P}^4 \dashrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \dashrightarrow X$ by forms of degree 26.

GM fourfolds vs cubic fourfolds

As in the case of cubic fourfolds, it is known that *all* GM fourfolds are unirational. Some rational examples are classical and easy to construct, but no examples have yet been proved to be irrational.

Similarly to Hassett's analysis of cubic fourfolds, in the works [Debarre, Iliev, and Manivel, 2015] and [Debarre and Kuznetsov, 2016, 2018a, 2018b] the authors studied GM fourfolds, by highlighting how the picture is very similar to what we have for cubic fourfolds.

We will briefly illustrate this in the following slides.

Special GM fourfolds

It has been shown by Debarre, Iliev, and Manivel (2015) that for a very general GM fourfold X, the natural inclusion

$$A(\mathbb{G}(1,4)) = H^4(\mathbb{G}(1,4),\mathbb{Z}) \cap H^{2,2}(\mathbb{G}(1,4)) \subseteq A(X) = H^4(X,\mathbb{Z}) \cap H^{2,2}(X)$$

of middle Hodge classes is an equality.

A GM fourfold is said to be *special* if the above inequality is strict. This means that the fourfold contains a surface whose cohomology class "does not come" from the Grassmannian $\mathbb{G}(1,4)$.

The special GM fourfolds correspond via the period map $\mathfrak{p} : \mathcal{M}_4^{GM} \to \mathcal{D}$ to a countable union of hypersurfaces $\mathcal{D}_d \subset \mathcal{D}$, labelled by the integers $d > 8$ with

$$d \equiv 0, 2, \text{ or } 4 \pmod 8.$$

If $d \equiv 0$ (mod 4), then $\mathcal{D}_d$ is irreducible; while if $d \equiv 2$ (mod 8), $\mathcal{D}_d$ is the union of two irreducible hypersurfaces $\mathcal{D}'_d$ and $\mathcal{D}''_d$.

Computing the value of the *discriminant* d.

Following [Debarre, Iliev, and Manivel, 2015], let $X \subset \mathbb{P}^8$ be an ordinary GM fourfold containing a smooth surface S such that $[S] \in A(X) \setminus A(\mathbb{G}(1,4))$.

We may write $[S] = a\sigma_{3,1} + b\sigma_{2,2}$ in terms of Schubert cycles in $\mathbb{G}(1,4)$ for some integers a and b. We then have that $[X] \in \mathfrak{p}^{-1}(\mathcal{D}_d)$ and d is the determinant (or *discriminant*) of the intersection matrix in the basis $(\sigma_{1,1|X}, \sigma_{2|X} - \sigma_{1,1|X}, [S])$. That is

$$d = \det \begin{pmatrix} 2 & 0 & b \\ 0 & 2 & a-b \\ b & a-b & (S)^2_X \end{pmatrix} = 4(S)^2_X - 2(b^2 + (a-b)^2),$$

where

$$(S)^2_X = 3a + 4b + 2K_S \cdot \sigma_{1|S} + 2K_S^2 - 12\chi(\mathcal{O}_S).$$

Associated K3 surfaces

For some values of the discriminant d, the non-special cohomology of the GM fourfold $X \in \mathfrak{p}^{-1}(\mathcal{D}_d)$ looks like the primitive cohomology of a K3 surface. In this case, analogously to the case of cubic fourfolds, one says that X has an associated K3 surface. (See [Debarre, Iliev, and Manivel, 2015] for precise definition and results.)

The first values of d that satisfy the condition for the existence of an associated K3 surface are:

10 12 16 18 20 24 26 28 32 34 36 40 42 44 48 50 52 56 58 60 64 66 68 72 74 76 80 82 84 88 90 92 96 98 100 104 106 108 112 114 116 120 122 124 128

One could ask if a similar statement to the Kuznetsov's conjecture for cubic fourfolds could be true for GM fourfolds.

Census of rational GM fourfolds

The first of two basic examples

A τ*-quadric* surface in $\mathbb{G}(1,4)$ is a linear section of $\mathbb{G}(1,3) \subset \mathbb{G}(1,4)$; its class is $\sigma_1^2 \cdot \sigma_{1,1} = \sigma_{3,1} + \sigma_{2,2}$.

Theorem (Debarre, Iliev, and Manivel (2015))

The closure inside $\mathcal{M}_4^{GM}$ of the family of fourfolds containing a τ-quadric surface coincides with $\mathfrak{p}^{-1}(\mathcal{D}'_{10})$. A general (hence every) member of $\mathfrak{p}^{-1}(\mathcal{D}'_{10})$ is rational.

The rationality for a general fourfold $[X] \in \mathfrak{p}^{-1}(\mathcal{D}'_{10})$ also follows from the fact that a τ-quadric surface S, inside the unique del Pezzo fivefold $Y \subset \mathbb{P}^8$ containing X, admits *a congruence of* 1*-secant lines*, that is through the general point of Y there passes just a line contained in Y which intersects S.

So, a τ-quadric in a del Pezzo fivefold looks like a del Pezzo surface in $\mathbb{P}^5$, and GM fourfolds in $\mathfrak{p}^{-1}(\mathcal{D}'_{10})$ look like cubic fourfolds in $\mathcal{C}_{14}$.

Census of rational GM fourfolds

The second of two basic examples

A quintic del Pezzo surface can be obtained as a linear section of $\mathbb{G}(1,4)$; its class is $\sigma_1^4 = 3\sigma_{3,1} + 2\sigma_{2,2}$.

Theorem (Roth, 1949; Debarre, Iliev, and Manivel, 2015)

The closure inside $\mathcal{M}_4^{GM}$ of the family of fourfolds containing a quintic del Pezzo surface coincides with $\mathfrak{p}^{-1}(\mathcal{D}''_{10})$. A general (hence every) member of $\mathfrak{p}^{-1}(\mathcal{D}''_{10})$ is rational.

The proof of the rationality of a general fourfold $[X] \in \mathfrak{p}^{-1}(\mathcal{D}''_{10})$ is very classical. Indeed Roth (1949) remarked that the projection from the linear span of a quintic del Pezzo surface contained in X induces a dominant map

$$p : X \dashrightarrow \mathbb{P}^2$$

whose generic fibre is a quintic del Pezzo surface. By a result of Enriques (1897), a quintic del Pezzo surface defined over an infinite field K is K-rational. Thus the fibration p admits a rational section and X is rational.

The previous two results exhaust all the examples of rational GM fourfolds known before three month ago. Moreover, there was no explicit descritions of codimension-one loci in $\mathcal{M}_4^{GM}$ parameterizing special GM fourfolds of discriminant $d > 12$.

In the recent preprint [Hoff and S., 2019], with also the help of F. Russo, we provided an explicit geometric description of the irreducible divisor $\mathfrak{p}^{-1}(\mathcal{D}_{20})$ and moreover we showed that its general member is rational.

So this new result is in agreement with the fact that the Kuznetsov's conjecture could be true even in the case of the GM fourfolds.

We give some few details in the next slides.

Section 7

A new divisor of rational Gushel-Mukai fourfolds and the rationality of the cubics in $\mathcal{C}_{42}$

A new divisor of rational GM fourfolds

- In [Hoff and S., 2019], using MACAULAY2 we costructed a smooth rational surface $S \subset \mathbb{P}^8$ of degree 9 and sectional genus 2 containied in a del Pezzo fivefold $Y = \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ with class $9\sigma_{3,1} + 3\sigma_{2,2}$.
- We showed that inside Y there is a 25-dimensional family of such surfaces, from which we deduced that the closure of the family of GM fourfolds containing such a surface coincides with $\mathfrak{p}^{-1}(\mathcal{D}_{20})$.
- The surface S inside Y admits a congruence of 3-secant conics, so that the rationality of the general GM fourfold X with $S \subset X \subset Y$ follows.
- Moreover, we have a commutative diagram

$$\begin{tikzcd}
 & M \subset \mathbb{P}^{12} & \\
X \arrow[ur, dashed, "m_1"] \arrow[rr, dashed, "|H^0(\mathcal{I}_S^2(3))|"] & & \mathbb{P}^4 \arrow[ul, dashed, "m_2"']
\end{tikzcd}$$

where m_1 and m_2 are birational maps defined, respectively, by the linear system of quadrics through S, and by the linear system of quintics through the projection in $\mathbb{P}^4$ of a minimal K3 surface of degree 20 in $\mathbb{P}^{11}$.

Birational representations in $\mathbb{P}^5$ of the surface of degree 9 and sectional genus 2

Recall that inside a del Pezzo fivefold $Y \subset \mathbb{G}(1,4) \cap \mathbb{P}^8 \subset \mathbb{P}^8$ there are two types of planes: a 3-dimensional family of planes with class $\sigma_{2,2}$, and a 4-dimensional family of planes with class $\sigma_{3,1}$.

- By projecting from a plane of the first type, we get a birational map $p_{\sigma_{2,2}} : Y \dashrightarrow \mathbb{P}^5$ whose inverse is defined by the quadrics through a cubic scroll in $\mathbb{P}^4 \subset \mathbb{P}^5$.
- By projecting from a plane of the second type, we get a dominant rational map $p_{\sigma_{3,1}} : Y \dashrightarrow Q \subset \mathbb{P}^5$ onto a smooth quadric hypersurface $Q \subset \mathbb{P}^5$ whose general fibre is a line.

Let $S \subset Y$ be a rational surface of degree 9 and sectional genus 2 as before.

- A general projection of the first type sends S into a surface of degree 9 and sectional genus 2 cut out 9 by cubics and having 5 non-normal nodes.
- A general projection of the second type sends S into a surface of degree 9 and sectional genus 2 cut out by 4 cubics and one quadric, and having 6 non-normal nodes.

A mysterious relationship between the new divisor of rational GM fourfolds, $\mathcal{C}_{42}$, and $\mathcal{C}_{48}$

From the number of non-normal nodes of the projected surfaces in $\mathbb{P}^5$ and from a standard parameter count (see [Russo and S., 2019b]), it follows that:

$$\mathcal{C}_{42} = \overline{\{[X] \in \mathcal{C} : X \text{ contains a general projection } p_{\sigma_{2,2}}(S) \text{ of a general } S \subset Y\}}$$

$$\mathcal{C}_{48} = \overline{\{[X] \in \mathcal{C} : X \text{ contains a general projection } p_{\sigma_{3,1}}(S) \text{ of a general } S \subset Y\}}$$

In particular, we have that $\mathcal{C}_{42}$ and $\mathcal{C}_{48}$ are uniruled. ($\mathcal{C}_{42}$ is actually unirational by a very recent result of Farkas and Verra (2019).)

There are no explicit descriptions of $\mathcal{C}_d$ with $d > 48$, and no alternative descriptions of $\mathcal{C}_{48}$.

Rationality for $\mathcal{C}_{42}$ using the surface $S_{42} = p_{\sigma_{2,2}}(S)$

Let $S_{42} = p_{\sigma_{2,2}}(S) \subset \mathbb{P}^5$ be a general projection of a general surface $S \subset Y$.
If $p \in \mathbb{P}^5$ is a general point, we have

- 9 2-secant lines to S_{42} passing through p;
- 7 5-secant conics to S_{42} passing through p;
- one single 8-secant twisted cubic to S_{42} passing through p.

In particular, the surface S_{42} admits a congruence of 8-secant twisted cubics, from which we deduce the rationality for the general cubic fourfold $[X] \in \mathcal{C}_{42}$.

Moreover, the linear system $|H^0(\mathcal{I}^3_{S_{42}}(8))|$ of hypersurfaces of degree 8 with points of multiplicity 3 along S_{42} gives a dominant rational map

$$\mu : \mathbb{P}^5 \dashrightarrow W = \mathbb{G}(1,4) \cap \mathbb{P}^7 \subset \mathbb{P}^7.$$

The restriction of μ to a general cubic $X \supset S_{42}$ is a birational map $X \dashrightarrow W$ whose inverse is defined by the restriction to W of the linear system of hypersurfaces of degree 8 with points of multiplicity 3 along a special projection of a minimal K3 surface of degree 42 and genus 22.

Section 8

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Some references

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