THE 2-SECANT VARIETIES OF THE VERONESE EMBEDDING

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ABSTRACT. We prove that a general polynomial $F \in k[x_0,...,x_n]_d$ admits a decomposition as sum of $h = 2$ powers of linear forms if and only if its second partial derivatives lie on a line. In this way we work out set-theoretical equations for the variety of secant lines $\mathbb{S}ec_2(V_d^n)$ of the Veronese variety V_d^n. In [Ka] *V. Kanev*, adopting a different approach, proved that the same equations cut out ideal-theoretically $\mathbb{S}ec_2(V_d^n)$.

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INTRODUCTION

A variation on the Waring problem (coming from a question in number theory stated by *E. Waring* in 1770, see [Wa], which states that every integer is a sum of at most 9 positive cubes) asked which is the minimum positive integer h such that the generic polynomial of degree d on $\mathbb{P}^n$ admits a decomposition as a sum of h d-powers of linear forms.
In 1995 *J. Alexander* and *A. Hirshowitz* solved completely this problem over an algebraically closed base field k of characteristic zero, see [AH]. They proved that the minimum integer h is the expected one $h = \lfloor \frac{1}{n+1}\binom{n+d}{d} \rfloor$, except in the following cases:

n	d	h
n	2	$2 \leq h \leq n$
2	4	5
3	4	9
4	3	7
4	4	14

Polynomials often appear in issues of applied mathematics, As instance in signal theory [CM], algebraic complexity theory [BCS], coding and information theory [Ro]. In particular issues related to decompositions of homogeneous polynomials in sums of powers are of particular interest in signal theory and clearly in pure mathematics. Indeed degree d homogeneous polynomials can be seen as points if the projective space $\mathbb{P}^N = \text{Proj}(k[x_0,...,x_n]_d)$, while d-powers of linear forms are

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parametrized by the Veronese variety $V_d^n \subset \mathbb{P}^N$. Therefore the geometric counterpart of this type of problems is the study of secant varieties to Veronese varieties. There is a line of research that studies varieties parametrizing decomposition of the form

$$F = L_1^d + ... + L_h^d$$

of a general homogeneous polynomial $F \in k[x_0, ..., x_n]_d$. These varieties, called varieties of sums of powers, VSP for short are the main object of a series of papers [DK], [IK], [RS], [MM] for a birational approach, and [Do] for a survey on the theme.
However, for applied sciences, is more interesting to determine:

- *whether a polynomial admits a decomposition into a number of linear forms,*
- *and eventually to calculate explicitly the decomposition.*

We focus the attention on the case $\mathbb{S}ec_h(V_d^n) \subsetneqq \mathbb{P}^N$ and adopt the philosophy dictated by the following trivial but crucial statement:
"If $F = \sum_{i=1}^h \lambda_i L_i^d$ then its partial derivatives of order l lie in the linear space $\langle L_1^{d-l}, ..., L_h^{d-l}\rangle$ for any $l = 1, ..., d-1$."
In the case $n = 2$ we prove that in order to establish if a homogeneous polynomials $F \in k[x_0, x_1]_d$ admits a decomposition as sum of h powers it is enough to verify that $\dim(H_\partial) \leq h-1$, where H_∂ is the linear space spanned by the partial derivatives of order $d-h$ of F. Furthermore, if $\dim(H_\partial) = h-1$ we get a method to work out the linear forms related to F, 2.13. Finally trying to extend the method in higher dimension we compute the dimension of the linear space of polynomials whose $(d-1)$-derivatives lie in a general linear subspace $H \subset (\mathbb{P}^N)^*$, this space is also called the $(d-1)$-prolongation of H. Consequently we find the formula for the dimension of $\mathbb{S}ec_h(V_2^n)$, and the secant defect of V_2^n. Furthermore we obtain a criterion to determine whether a polynomial admits a decomposition in the cases $d = 2$ and $d = 3, h = 2$.
Finally, in theorem 3.1, we will prove that a general polynomial $F \in k[x_0, ..., x_n]_d$ admits a decomposition as sum of $h = 2$ powers of linear forms if and only if its second partial derivatives lie on a line. In [Ka] *V. Kanev*, adopting a different approach, proved that the same conditions cut out ideal-theoretically $\mathbb{S}ec_2(V_d^n)$.

1. Preliminaries on secant varieties

Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non degenerate variety,

$$\Gamma_h(X) \subset X \times ... \times X \times \mathbb{G}(h-1, N),$$

the reduced closure of the graph of

$$\alpha : X \times ... \times X \dashrightarrow \mathbb{G}(h-1, N),$$

taking h general points to their linear span $\langle x_1, ..., x_h\rangle$. Observe that $\Gamma_h(X)$ is irreducible and reduced of dimension hn. Let $\pi_2 : \Gamma_h(X) \to \mathbb{G}(h-1, N)$ be the natural projection. Denote by

$$\mathbb{S}_h(X) := \pi_2(\Gamma_h(X)) \subset \mathbb{G}(h-1, N).$$

Again $\mathbb{S}_h(X)$ is irreducible and reduced of dimension hn. Finally let

$$\mathcal{I}_h = \{(x, \Lambda) \mid x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N),$$

with natural projections π_h and ψ_h onto the factors. Furthermore observe that $\psi_h : \mathcal{I}_h \to \mathbb{G}(h-1, N)$ is a $\mathbb{P}^{h-1}$-bundle on $\mathbb{G}(h-1, N)$.

Definition 1.1. Let $X \subset \mathbb{P}^N$ be an irreducible and reduced, non degenerate variety. The *abstract h-Secant variety* is the irreducible and reduced variety

$$\text{Sec}_h(X) := (\psi_h)^{-1}(\mathbb{S}_h(X)) \subset \mathcal{I}_h.$$

While the *h-Secant variety* is

$$\mathbb{S}ec_h(X) := \pi_h(Sec_h(X)) \subset \mathbb{P}^N.$$

It is immediate that $\mathrm{Sec}_h(X)$ is a $(hn+h-1)$-dimensional variety with a $\mathbb{P}^{h-1}$-bundle structure on $\mathbb{S}_h(X)$. One says that X is *h-defective* if

$$\dim \mathbb{S}ec_h(X) < \min\{\dim \mathrm{Sec}_h(X), N\}.$$

2. Veronese embedding and homogeneous polynomials

Let $\nu : \mathbb{P}^n \to \mathbb{P}^{N_d}$ be the d-Veronese embedding, and let $V_d^n = \nu(\mathbb{P}^n)$ be its image. Let $[F] \in \mathbb{P}^N = \mathrm{Proj}(k[x_0,...,x_n]_d)$ be a degree d homogeneous polynomial. Fixed a positive integer h such that $\mathbb{S}ec_h(V_d^n) \neq \mathbb{P}^N$ we want to determine whether $[F] \in \mathbb{S}ec_h(V_d^n)$. We begin with the following simple observation:

Remark 2.1. If $F = \sum_{i=1}^h \lambda_i L_i^d$ then its partial derivatives of order l lie in the linear space $\langle L_1^{d-l},...,L_h^{d-l}\rangle$ for any $l = 1,...,d-1$.

The partial derivatives of order l are $\binom{n+l}{l}$ homogeneous polynomials of degree $d-l$, so the previous observation is meaningful when $h < \binom{n+l}{l}$ and $h < \binom{d-l+n}{n}$. The latter condition ensures that $\langle L_1^{d-l},...,L_h^{d-l}\rangle$ is a proper subspace of the projective space $\mathbb{P}^{N_{d-l}}$ parametrizing homogeneous polynomials of degree $d-l$. Consider the partial derivatives $F^l_{l_0,...,l_n} := \frac{\partial^l F}{\partial x_0^{l_0},...,\partial x_n^{l_n}}$ and the incidence variety

$$\mathcal{I}_{l,h} = \{(F,H) \mid \in F^l_{l_0,...,l_n} \in H,\ \forall\, l_0+...+l_n = l\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N_{d-l})$$

$$\mathbb{P}^N \xleftarrow{\ \pi_1\ } \mathcal{I}_{l,h} \xrightarrow{\ \pi_2\ } \mathbb{G}(h-1, N_{d-l})$$

Let $\mathbb{S}_h V_{d-l}^n \subseteq \mathbb{G}(h-1, N_{d-l})$ be the abstract h-secant variety of V_{d-l}^n. Note that when $h < \binom{n+l}{l}$ the map π_1 is generically injective. Let $X_{l,h} = \pi_1(\mathcal{I}_{l,h}) \subseteq \mathbb{P}^N$ be its image. By remark 2.1 we have $\mathbb{S}ec_h(V_d^n) \subseteq X_{l,h}$. We want to find cases when the equality holds in order to get a simple criterion to establish whether $[F] \in \mathbb{S}ec_h(V_d^n)$.

Remark 2.2. The equality holds trivially when $d = 2$. Let $F \in k[x_0,...,x_n]_2$ be a polynomial and let $\mathcal{M}_F$ be the matrix of the quadratic symmetric form associated to F. Then $F \in \mathbb{S}ec_h(V_2^n)$ if and only if $\mathrm{rank}(\mathcal{M}_F) \leq h$. On the other hand the rows of $\mathcal{M}_F$ are exactly the partial derivatives of F.

The Waring rank. Let $\overline{h}$ be the smallest integer such that $\mathbb{S}ec_h(V_d^n) = \mathbb{P}^N$. By a dimensions computation we expect:

$$\overline{h} = \left\lceil \tfrac{1}{n+1}\tbinom{n+d}{d} \right\rceil.$$

This is almost always true, *J. Alexander* and *A. Hirschowitz* in [AH] proved that the following are the only exceptional cases:

d	n	$\overline{h}$
2	arbitrary	$n+1$
3	4	8
4	2	6
4	3	10
4	4	15

2.1. **Catalecticant Varieties.** Let us look closer at the variety $X_{l,h}$. This variety parametrizes polynomials $F \in k[x_0,...,x_n]_d$ whose partial derivatives of order l span a $(h-1)$-plane. Let $\mathcal{M}_{l,h}$ be the $\binom{n+l}{l} \times \binom{n+d-l}{d-l}$ matrix whose lines are the l-th derivatives of $F = \sum_{i_0+...+i_n=d} \alpha_{i_0,...,i_n} x_0^{i_0}...x_n^{i_n}$. Then $X_{l,h}$ is the determinantal variety defined in $\mathbb{P}^N$ by $\mathrm{rank}(\mathcal{M}_{l,h}) \leq h$, where the $\alpha_{i_0,...,i_n}$ are the homogeneous coordinates on $\mathbb{P}^N$. Let $\mathbb{P}^M$ be the projective space parametrizing $\binom{n+l}{l} \times \binom{n+d-l}{d-l}$ matrices, and let $M_h \subset \mathbb{P}^M$ be the variety of matrices of rank less or equal than h. Then M_h is a irreducible variety of dimension $M - \left(\binom{n+l}{l} - h\right) \cdot \left(\binom{n+d-l}{d-l} - h\right)$. Clearly the variety $X_{l,h}$ is a special linear section of M_h.

Lemma 2.3. *The varieties $X_{l,h}$ and $X_{d-l,h}$ are isomorphic.*

Proof. The matrix $\mathcal{M}_{d-l,h}$ whose lines are the $(d-l)$-th partial derivatives of F is the $\binom{n+d-l}{d-l} \times \binom{n+l}{l}$ matrix given by

$$\mathcal{M}_{d-l,h} = \mathcal{M}_{l,h}^t,$$

where $\mathcal{M}_{l,h}^t$ is the transposed matrix of $\mathcal{M}_{d-l,h}$. Then the assertion follows. □

Example 2.4. *Consider a polynomial of degree three in three variables*

$$F = a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xz^2 + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3.$$

The variety $X_{1,2}$ is defined by

$$\mathrm{rank}\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \mathrm{rank}\begin{pmatrix} 3a_0 & 2a_1 & 2a_2 & a_3 & a_4 & a_5 \\ a_1 & 2a_3 & a_4 & 3a_6 & 2a_7 & a_8 \\ a_2 & a_4 & 2a_5 & a_7 & 2a_8 & 3a_9 \end{pmatrix} \leq 2.$$

Consider the projective space $\mathbb{P}^{17}$ of 3×6 matrix with homogeneous coordinates

$$X_{0,0},...,X_{0,5},X_{1,0},...,X_{1,5},X_{2,0},...,X_{2,5}.$$

The determinantal variety M_2 defined by

$$\mathrm{rank}\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} & X_{0,3} & X_{0,4} & X_{0,5} \\ X_{1,0} & X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} & X_{1,5} \\ X_{2,0} & X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} \end{pmatrix} \leq 2$$

is irreducible of dimension $17 - 4 = 13$. The linear space

$$H = \begin{cases} 2X_{1,0} - X_{0,1} = 0, \\ 2X_{2,0} - X_{0,2} = 0, \\ 2X_{0,3} - X_{1,1} = 0, \\ X_{0,4} - X_{1,2} = 0, \\ 2X_{0,5} - X_{2,2} = 0, \\ 2X_{2,3} - X_{1,4} = 0, \\ 2X_{2,4} - X_{1,5} = 0, \\ X_{0,4} - X_{2,1} = 0. \end{cases}$$

cuts out on M_2 the variety $X_{1,2}$, which is irreducible of dimension $5 = \dim(\mathbb{S}ec_2(V_3^2))$.

Remark 2.5. Considering a polynomial $F \in k[x,y,z]_4$ and proceeding as in example 2.4 one get $\dim(X_{1,2}) = 6$, so

$$\mathbb{S}ec_2(V_4^2) \subsetneqq X_{1,2}.$$

Proposition 2.6. *Let $d = 2k$ be an even integer such that $\binom{n+k}{k} \geq N_{d-k}$, where $N_{d-k} = \binom{d-k+n}{n} - 1$. The variety $X_{k,N_{d-k}}$ is an irreducible hypersurface of degree $\binom{n+k}{k}$ in $\mathbb{P}^N$.*

Proof. The map $\pi_2 : \mathcal{I}_{k,N_{d-k}} \to \mathbb{G}(N_{d-k}-1, N_{d-k}) \cong \mathbb{P}^{N_{d-k}}$ is dominant, so $\mathcal{I}_{k,N_{d-k}}$ and $X_{k,N_{d-k}}$ are irreducible. The assertion follows observing that $X_{k,N_{d-k}}$ is defined by the vanishing of the determinant of a $\binom{n+k}{k} \times \binom{n+k}{k}$ matrix. □

Let us look at some consequences of the previous proposition.

Example 2.7. *Consider a polynomial*

$$F = a_0x^4 + a_1x^3y + a_2x^3z + a_3x^2y^2 + a_4x^2yz + a_5x^2z^2 + a_6xy^3 + a_7xy^2z + a_8xyz^2 \\ +a_9xz^3 + a_{10}y^4 + a_{11}y^3z + a_{12}y^2z^2 + a_{13}yz^3 + a_{14}z^4.$$

The map $\pi_2 : \mathcal{I}_{2,4} \to \mathbb{G}(3,5)$ is dominant, so $X_{2,4}$ is irreducible. Let $Z_0, Z_1, Z_2, Z_3, Z_4, Z_5$ be homogeneous coordinates on $\mathbb{P}^5$ corresponding to $x^2, xy, xz, y^2, yz, z^2$ respectively. To compute the dimension of the general fiber of π_2 we can take the $3-plane$ $H = \{Z_0 = Z_3 = 0\}$ which intersect V_2^2 in a subscheme of dimension zero. Computing the second partial derivatives of F it turns out that

$$\pi_2^{-1}(H) = \{a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_{10} = a_{11} = a_{12} = 0\}.$$

So $\dim(\pi_2^{-1}(H)) = 14 - 11 = 3$ *and* $\dim(X_{2,4}) = 3 + 8 = 11$. *Since* $\dim(\mathbb{S}ec_4V_4^2) = 11$ *we get*

$$\mathbb{S}ec_4V_4^2 = X_{2,4}.$$

Consider now $\pi_2 : \mathcal{I}_{2,5} \to \mathbb{P}^5$. This map is dominant, so $X_{2,5}$ is irreducible. We have $\dim(\pi_2^{-1}(H)) = 14 - 6 = 8$, *where $H = \{Z_0 = 0\}$. So* $\dim(X_{2,5}) = 13$ *and*

$$\mathbb{S}ec_5V_4^2 = X_{2,5}$$

is an hypersurface of degree 6 in $\mathbb{P}^{14}$.
Consider now the case $d = 4, n = 3, h = 9$ and the second partial derivatives. The map $\pi_2 : \mathcal{I}_{2,9} \to \mathbb{P}^9$ is dominant and $X_{2,9}$ is irreducible. The general fiber of π_2 has dimension 24. Then $\dim(X_{2,9}) = 24 + 9 = 33$ *and*

$$\mathbb{S}ec_9V_4^3 = X_{2,9}$$

is an hypersurface of degree 10 in $\mathbb{P}^{34}$.
Finally in the case $d = 4, n = 4, h = 14$ as before one can verify that $X_{2,14}$ is irreducible of dimension 68, so

$$\mathbb{S}ec_{14}V_4^4 = X_{2,14}$$

is an hypersurface of degree 15 in $\mathbb{P}^{69}$.

Example 2.8. *Consider now a polynomial $F \in k[x,y,z]_6$ and the partial derivative of order 3. For $h = 8, 9$ the map π_2 is dominant, so $X_{3,8}$ and $X_{3,9}$ are irreducible. First let us take $h = 8$. Proceeding as before we get* $\dim(\pi_2^{-1}(H)) = 27 - 19 = 8$ *and* $\dim(X_{3,8}) = 24$. *So $\mathbb{S}ec_8V_6^2 \subset X_{3,8}$ is a divisor.*
In the case $h = 9$ we have $\dim(\pi_2^{-1}(H)) = 27 - 10 = 17$ *and* $\dim(X_{3,9}) = 17 + 9 = 26$. *So*

$$\mathbb{S}ec_9V_6^2 = X_{3,9}$$

is an hypersurface of degree 10 in $\mathbb{P}^{27}$.

2.2. **Secant varieties of rational normal curves.** We begin with the simplest case $n = 1$. We denote by $C_d \subset \mathbb{P}^d$ the degree d rational normal curve, in this case $\mathbb{S}ec_h(C_d) \neq \mathbb{P}^d$ if and only if $h \leq \frac{d}{2}$.

Lemma 2.9. *Let $F = \sum_{i+j=d} \alpha_{i,j}x_0^ix_1^j \in k[x_0,x_1]_d$ be a homogeneous polynomial, and let $c = c(\alpha_{i,j})$ be the coefficient of x_0^h in the partial derivative $\frac{\partial^{d-h}F}{\partial x_0^m \partial x_1^s}$, with $h \geq 1$. Then $c = C \cdot \alpha_{d-s,s}$, where C is a constant.*

Proof. Since the only monomial of F producing c is $x_0^{d-s}x_1^s$ the assertion follows. □

Theorem 2.10. *For any $h \leq \frac{d}{2}$ we have $\mathbb{S}ec_h(C_d) = X_{d-h,h}$. Consequently if the partial derivatives of order $d-h$ of a homogeneous polynomial $F \in k[x_0, x_1]_d$ lie in a hyperplane of $\mathbb{P}^h$ then $[F]$ lies in $\mathbb{S}ec_h(C_d)$.*

Proof. The partial derivatives of order $d-h$ of F are $d-h+1$ homogeneous polynomials of degree h. If $F = \sum_{i=1}^{h} \lambda_i L_i^d$ the partial derivatives lie in $\langle L_1^h, ..., L_h^h \rangle$ which is a hyperplane h-secant to C_h, but $\deg(C_h) = h$ and the latter condition is irrelevant. Let H be a general hyperplane in $\mathbb{P}^h$, forcing the partial derivatives of a degree d polynomial $G = \sum_{i+j=d} \alpha_{i,j} x_0^i x_1^j \in k[x_0, x_1]_d$ to lie in H gives $d-h+1$ linear equations in the coefficients of G. Without loss of generality we can suppose H to be the defined by the vanishing of the first homogeneous coordinate on $\mathbb{P}^h$, then by 2.9 the fiber of π_2 is the linear subspace of $\mathbb{P}^N$ defined by

$$\pi_2^{-1}(H) = \{\alpha_{d-s,s} = 0,\ \forall\ s = 0, ..., d-h\}.$$

The equations of $\pi_2^{-1}(H)$ are independent so

$$\dim(\pi_2^{-1}(H)) = d - (d-h+1) = h-1,$$

and the dimension of $X_{d-h,h}$ is

$$\dim(X_{d-h,h}) = \dim(\mathcal{I}_{d-h,h}) = h-1+h = 2h-1.$$

Finally $\dim(\mathbb{S}ec_h(C_d)) = h + h - 1 = 2h-1$ yields $\mathbb{S}ec_h(C_d) = X_{d-h,h}$. □

Remark 2.11. The partial derivatives of order $d-h$ of a homogeneous polynomial $F \in k[x_0, x_1]_d$ depend on $d+1$ parameters. We consider the matrix $\mathcal{M}_{d,h}$ whose lines are the partial derivatives. From 2.10 we get equations for $\mathbb{S}ec_h(C_d)$ imposing $\mathrm{rank}(\mathcal{M}_{d,h}) \leq h$, that is the classical determinantal description of $\mathbb{S}ec_h(C_d)$.

Proposition 2.12. *If $[F] \in \mathbb{S}ec_h(C_d)$ is general then its decomposition in powers of linear forms is unique.*

Proof. Let $H_\partial \subset \mathbb{P}^h$ be the hyperplane spanned by the partial derivatives of order $d-h$ of F. Since $\deg(C_h) = h$ and F is general we have $H_\partial \cdot C_h = \{L_1^h, ..., L_h^h\}$. Then $\{L_1, ..., L_h\}$ is the unique h-polyhedron of F. □

Theorem 2.10 and proposition 2.12 immediately suggest an algorithm.

Construction 2.13. Given $F \in k[x_0, x_1]_d$ to establish if F admits a decomposition in $h \leq \frac{d}{2}$ linear forms, and eventually to find it we proceed as explained in the following diagram.

$\{Compute\ dim(H_\partial)\}$ $\xrightarrow{dim(H_\partial)=h-1}$ $\{F\ admits\ a\ h-polyhedron\}$ $\longrightarrow$ $\{Compute\ H_\partial \cdot C_h\}$

$\{Compute\ dim(H_\partial)\}$ $\xrightarrow{dim(H_\partial)=h}$ $\{F\ does\ not\ admit\ a\ h-polyhedron\}$

Then $H_\partial \cdot C_h = \{L_1^h, ..., L_h^h\}$ and $F = \sum_{i=1}^{h} \lambda_i L_i^d$.

Example 2.14. *Consider the case $d = 4, h = 2$ and write $F = \sum_{i_0+i_1=4} \alpha_{i,j} x_0^i x_1^j$. Forcing $\frac{\partial^2 F}{\partial x_0 \partial x_1} \in \langle \frac{\partial^2 F}{\partial x_0^2}, \frac{\partial^2 F}{\partial x_1^2} \rangle$ we get*

$$\mathbb{S}ec_2(C_4) = \{54\alpha_{3,1}^2\alpha_{0,4} - 18\alpha_{3,1}\alpha_{2,2}\alpha_{1,3} - 144\alpha_{4,0}\alpha_{2,2}\alpha_{0,4} + 4\alpha_{2,2}^3 + 54\alpha_{4,0}\alpha_{1,3}^2 = 0\}.$$

Now consider the polynomial

$$F = 9(x_0^4 + x_0^3 x_1 + x_0^2 x_1 + x_0 x_1^3) + 4x_1^4.$$

The second partial derivatives of F lie on the line

$$H_\partial = \{X_0 - 3X_1 + 3X_2 = 0\} \subset \mathrm{Proj}(k[x_0,x_1]_2).$$

Now we have to compute the intersection $H_\partial \cdot C_2$, where $C_2 = \{X_1^2 - 4X_0X_2 = 0\}$ is the conic parametrizing squares of linear forms, we have

$$H_\partial \cdot C_2 = \{[15+6\sqrt{6} : 6+2\sqrt{6} : 1], [15-6\sqrt{6} : 6-2\sqrt{6} : 1]\}.$$

Finally we compute the linear forms giving the decomposition

$$L_1 = 5.44948x_0 + x_1 \text{ and } L_2 = 0.55051x_0 + x_1.$$

Now we consider the variety $X_{d-1,h}$. First we compute the dimension of the general fiber of $\pi_2 : \mathcal{I}_{d-1,h} \to \mathbb{G}(h-1,n)$.

Theorem 2.15. *The fiber of $\pi_2 : \mathcal{I}_{d-1,h} \to \mathbb{G}(h-1,n)$ on a general $(h-1)$-plane $H \in \mathbb{G}(h-1,n)$ is a linear subspace of $\mathbb{P}^N$ of dimension*

$$\dim(\pi_2^{-1}(H)) = \binom{d+h-1}{d} - 1.$$

Furthermore the dimension of X_{d-1} is given by

$$\dim(X_{d-1,h}) = h(n-h+1) + \binom{d+h-1}{d} - 1.$$

Proof. We can suppose $H = \{X_0 = ... = X_{n-h} = 0\}$, where $\{X_0,...,X_n\}$ are homogeneous coordinates on $\mathbb{P}^n$. We write a general polynomial $[F] \in \mathbb{P}^N$ in the form

$$F = \sum_{i_0+...+i_n=d} \alpha_{i_0,...,i_n} x_0^{i_0} ... x_n^{i_n}.$$

The fiber $\pi_2^{-1}(H)$ is the linear subspace of $\mathbb{P}^N$ defined by the vanishing of the coefficients of $x_0,...,x_{n-h}$ in the derivatives of F. Many of these equations are redundant, the difficulty is in counting the exact number of independent equations. We prove that this number is $\binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+h-1}{d}$ by induction on $n-h$. If $n-h=0$ then H is an hyperplane and the condition on the derivatives are all independent, so the number of conditions is exactly the number of derivatives $\binom{d-1+n}{d-1}$. Furthermore our formula for $n-h=0$ gives $\binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+n-1}{d} = \binom{d+n-1}{d-1}$, and the case $n-h=0$ is verified. Consider now the general case, let $\overline{H} = \{X_0 = ... = X_{n-h-1} = 0\}$, let C_{n-h-1} the number of independent conditions obtained forcing the partial derivatives to lie in $\overline{H}$. Adding the condition $\{X_{n-h} = 0\}$ gives new equations coming from the coefficients of the form $\alpha_{0,...,0,i_{n-h},i_{n-h+1},...,i_n}$, with $i_{n-h} \neq 0$. These corresponds to monomials of degree d in the variables $x_{n-h},...,x_n$ that contain the variable x_{n-h}. Now the monomials of degree d not containing x_{n-h} are the monomials of degree d in $x_{n-h+1},...,x_n$. So in the final step we are adding

$$\binom{d+h}{d} - \binom{d+h-1}{d}$$

conditions. Then the number if independent equations is $C_{n-h} = C_{n-h-1} + \binom{d+h}{d} - \binom{d+h-1}{d}$, by induction hypothesis

$$C_{n-h-1} = \binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+n-(n-h-1)-1}{d}.$$

So $C_{n-h} = \binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+n-(n-h-1)-1}{d} + \binom{d+h}{d} - \binom{d+h-1}{d} = \binom{d+n-1}{d-1} + \binom{d+n-1}{d} - \binom{d+h-1}{d}$. Finally we have $\dim(X_{d-1,h}) = \dim(\mathbb{G}(h-1,n)) + \dim(\pi_2^{-1}(H)) = h(n-h+1) + \binom{d+h-1}{d} - 1$. □

Proposition 2.16. *If $h \leq n$. The variety $X_{1,h}$ is irreducible.*

Proof. By Lemma 2.3 it is equivalent to prove that $X_{d-1,h}$ is irreducible. Consider the map $\pi_2 : \mathcal{I}_{d-1,h} \to \mathbb{G}(h-1,n)$. By Theorem 2.15 the general fiber of π_2 is a linear subspace of $\mathbb{P}^N$ of dimension $\dim(\pi_2^{-1}(H)) = \binom{d+h-1}{d} - 1$ and π_2 is surjective on $\mathbb{G}(h-1,n)$, so $X_{d-1,h}$ is irreducible. □

In the cases $d = 2$ and $d = 3, h = 2$ we have that $\dim(X_{1,h}) = \dim(\mathbb{S}ec_h(V_d^n))$, since $X_{1,h}$ is irreducible we get $\mathbb{S}ec_h(V_d^n) = X_{1,h}$. So if the first partial derivatives of a polynomial F span a linear space of dimension $h-1$ then F can be decomposed into a sum of h powers of linear forms.

Remark 2.17. Consider the case $d = 2$. By Alexander-Hirshowitz theorem, see [AH], $\mathbb{S}ec_h(V_2^n) \neq \mathbb{P}^N$ if and only if $h \leq n$. By theorem 2.15 and remark 2.2 we recover the effective dimension of $\mathbb{S}ec_h(V_2^n)$,

$$\dim(\mathbb{S}ec_h(V_2^n)) = \frac{2nh - h^2 + 3h - 2}{2},$$

and consequently the formula for the h-secant defect of V_2^n,

$$\delta_h(V_2^n) = \frac{h(h-1)}{2}.$$

Up to now we have a complete description for polynomials of arbitrary degree in two variables and for polynomials of degree two in any number of variables. So we concentrate on the case $n \geq 2$ and $d \geq 3$.

Theorem 2.18. *Let $n \geq 2, d \geq 3, h \leq n$ be positive integers. Then $\mathbb{S}ec_h(V_d^n)$ is a subvariety of $X_{d-1,h}$ of codimension*

$$\operatorname{codim}_{\mathbb{S}ec_h(V_d^n)}(X_{d-1,h}) = \binom{d+h-1}{d} - h^2.$$

Proof. Since $n \geq 2$, $d \geq 3$, and $h \leq n$, by Alexander-Hirshowitz theorem the effective dimension of $\mathbb{S}ec_h(V_d^n)$ is the expected one

$$\dim(\mathbb{S}ec_h(V_d^n)) = \min\{hn + (h-1), N_d\}.$$

Furthermore $n \geq 2$, $d \geq 3$, $h \leq n$ implies $hn + (h-1) < N_d$. So

$$\dim(\mathbb{S}ec_h(V_d^n)) = hn + (h-1).$$

Finally $\operatorname{codim}_{\mathbb{S}ec_h(V_d^n)}(X_{d-1,h}) = h(n-h+1) + \binom{d+h-1}{d} - 1 - hn - (h-1) = \binom{d+h-1}{d} - h^2$. □

Corollary 2.19. *If $d = 3$ then $\mathbb{S}ec_2(V_3^n) = X_{2,2}$ for any $n \geq 2$. Consequently if the second partial derivatives of a homogeneous polynomial $F \in k[x_0, ..., x_n]_3$ lie in a line of $\mathbb{P}^n$ then $[F]$ lies in $\mathbb{S}ec_2(V_3^n)$.*

Proof. For $h = 2, d = 3$ we have $\binom{d+h-1}{d} - h^2 = 0$. We conclude by theorem 2.18. □

3. The first secant variety of V_d^n

We focus on the case $h = 2$ without any assumptions on d and n. We will use the equality

$$\sum_{k=0}^{n} \binom{d-1+k}{d-1} = \binom{d+n}{d},$$

which can be easily proved by induction on n.

Theorem 3.1. *If $h = 2$ for the first secant variety of V_d^n we have*

$$\mathbb{S}ec_2(V_d^n) = X_{2,d-2}$$

for any n and $d \geq 3$.

Proof. Consider the diagram

$$\begin{array}{ccc} & \mathcal{I}_{2,d-2} = \{(F,H) \mid \in F^{l}_{l_0,...,l_n} \in H,\ \forall\, l_0+...+l_n = d-2\} \subset \mathbb{P}^N \times \mathbb{G}(1,N_2) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ \mathbb{P}^N & & \mathbb{G}(1,N_2) \end{array}$$

clearly $\mathbb{S}_2V_2^n \subseteq Im(\pi_2)$. Let $F \in k[x_0,...,x_n]_d$ be a polynomial whose partial derivatives of order $d-2$ lie on a line $H \subset \mathbb{P}^{N_2}$. The derivatives of order $d-3$ of F are cubic polynomials whose first partial derivatives are collinear. By 2.19 $X_{2,1} = X_{2,2} = \mathbb{S}ec_2V_3^n$, so if we denote by G a partial derivative of order $d-3$ of F we get a decomposition $G = L_1^3 + L_2^3$. Then $G_{x_0},...,G_{x_n}$ (which are partial derivatives of order $d-2$ of F) lie on the line $\langle L_1^2, L_2^2\rangle$, and so the line containing the partial derivative of order $d-2$ of F is exactly the secant line to V_2^n given by $\langle L_1^2, L_2^2\rangle$. This means that

$$\mathbb{S}_2V_2^n = Im(\pi_2).$$

Since the fiber of π_2 are linear spaces we conclude that $\mathcal{I}_{2,d-2}$ and $X_{2,d-2}$ are irreducible.
We compute now the dimension of the fiber of π_2. We fix on $\mathbb{P}^{N_2}$ homogeneous coordinates $Z_0,...,Z_{N_2}$ corresponding to the monomials in lexicographic order $x_0^2, x_0x_1,...,x_n^2$, and consider the line $H = \{Z_0 = Z_1 = ... = Z_{N_2-2} = 0\}$.
First consider monomials containing x_0. Forcing the derivatives to lie in $\{Z_0 = 0\}$ we get $\binom{d-2+n}{n}$ conditions (the monomials containing x_0^2, whose number is equal to the number of degree $d-2$ monomials in $x_0,...,x_n$). Imposing $\{Z_1 = 0\}$ we get $\binom{d-2+n-1}{n-1}$ conditions (the monomials containing x_0x_1, whose number is equal to the number of degree $d-2$ monomials in $x_1,...,x_n$). Proceeding in this way forcing $\{Z_n = 0\}$ we get $\binom{d-2+n-n}{n-n} = 1$ condition (the monomials containing x_0x_n, whose number is equal to the number of degree $d-2$ monomials in x_n). Up to now we have

$$\sum_{k=0}^{n}\binom{d-2+k}{k} = \binom{d-1+n}{d-1}$$

conditions.
Consider now the monomials containing x_1. Forcing $\{Z_{n+1} = 0\}$ we get $\binom{d-2+n-1}{n-1}$ conditions (the monomials containing x_1^2, whose number is equal to the number of degree $d-2$ monomials in $x_1,...,x_n$). Imposing $\{Z_{n+2} = 0\}$ we get $\binom{d-2+n-2}{n-2}$ conditions (the monomials containing x_1x_2, whose number is equal to the number of degree $d-2$ monomials in $x_2,...,x_n$). Proceeding in this way we get

$$\sum_{k=0}^{n-1}\binom{d-2+k}{k} = \binom{d-1+n-1}{d-1}$$

conditions.
At the step x_{n-2} we have

$$\sum_{k=0}^{2}\binom{d-2+k}{k} = \binom{d-1+2}{d-1}$$

more conditions. At the step x_{n-1} we have only to force $\{Z_{N_2-2} = 0\}$, and we get $\binom{d-1}{1} = d-1$ conditions.
Summing up the fiber $\pi_2^{-1}(H)$ is a linear subspace of $\mathbb{P}^N$ defined by

$$\sum_{k=2}^{n}\binom{d-1+k}{d-1} + d - 1 = \sum_{k=0}^{n}\binom{d-1+k}{d-1} - 1 - d + d - 1 = \binom{d+n}{d} - 2$$

equations. So the fiber has dimension

$$\dim(\pi_2^{-1}(H)) = N - \binom{d+n}{d} + 2 = \binom{d+n}{d} - 1 - \binom{d+n}{d} + 2 = 1.$$

Finally we look at the map $\pi_2 : \mathcal{I}_{2,d-2} \rightarrow \mathbb{S}_2 V_2^n$, since π_2 is dominant we have

$$\dim(X_{2,d-2}) = \dim(\mathcal{I}_{2,d-2}) = 2n+1.$$

Since $\dim(\mathbb{S}ec_2 V_d^n) = 2n+1$ the assertion follows. □

The case $n=2, h=4$. In the same spirit of Theorem 3.1 we obtain the following result.

Theorem 3.2. *If* $n = 2, h = 4$ *for the variety of* 4*-secant* 3*-planes of* V_d^2 *we have*

$$\mathbb{S}ec_4(V_d^2) = X_{4,\lfloor \frac{d}{2} \rfloor}$$

for any $d \geq 2$.

Proof. The case $d = 4$ is example 2.7. Consider now the case $d = 5$. The map $\pi_2 : \mathcal{I}_{4,3} \rightarrow \mathbb{G}(3,5)$ is dominant, so $X_{4,3}$ and hence $X_{4,2}$ are irreducible. Let $F \in k[x,y,z]_5$ be a polynomial, looking at the proof of theorem 3.1 we get that forcing the partial derivatives of order 3 of F to lie in $\{Z_0 = Z_3 = 0\}$ gives

$$\binom{5-2+2}{2} + \binom{5-2+2}{2} - \sharp\{\text{monomials containing } x^2y^2\} = 20 - 3 = 17$$

conditions. Since $\dim(X_{4,2}) = \dim(X_{4,3}) = 20 - 17 + \dim(\mathbb{G}(3,5)) = 11$ we conclude

$$\mathbb{S}ec_4(V_5^2) = X_{4,2}.$$

Consider the case $d = 6$ and the partial derivative of order 3. If the 3-th derivatives of F lie in a 3-plane then the first partial derivative of F are degree 5 polynomials whose second partial derivatives lie in a 3-plane. By the same trick of Theorem 3.1 we prove that the 3-plane containing the 3-th partial derivative has to be 4 secant to V_3^2. So $X_{4,3}$ is irreducible, and as usual by counting dimension we get the equality

$$\mathbb{S}ec_4(V_6^2) = X_{4,3}.$$

Now we treat the general case by induction on d. Let $F \in k[x,y,z]_d$ be a polynomial whose $\lfloor \frac{d}{2} \rfloor$-th derivative lies in a 3-plane. Then the first partial derivative of F are polynomials of degree $d-1$ whose $\lfloor \frac{d-1}{2} \rfloor$-th derivatives lie in a 3-plane. So F_x, F_y, F_z can be decomposed as sums of four powers of linear forms. As before we conclude that the map $\pi_2 : \mathcal{I}_{4,\lfloor \frac{d}{2} \rfloor} \rightarrow \mathbb{G}(3, N_{d-\lfloor \frac{d}{2} \rfloor})$ is dominant, so $X_{4,\lfloor \frac{d}{2} \rfloor}$ is irreducible. By combinatorial calculations similar to previous we compute $\dim(X_{4,\lfloor \frac{d}{2} \rfloor}) = \dim(\mathbb{S}ec_4(V_d^2))$. □

Remark 3.3. In a completely analogous way one can show that $\mathbb{S}ec_5(V_d^2)$ is defined by size 6 minors of the matrix of partial derivatives of order $\lfloor \frac{d}{2} \rfloor$ for $d = 4$ and $d \geq 6$.

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