Characteristic Classes in Complex Algebraic Geometry
and
The Grothendieck Riemann Roch Theorem

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Introduction

In literature there are various definitions of characteristic classes of a vector bundle. Some are analytical in nature and other are purely algebraic. We introduce in various ways *Chern classes* of a vector bundle, and state the equivalence between these a priori different definitions. Then we define *Pontryagin classes* and the *Atiyah class* highlighting their relationships with Chern classes.
Furthermore we state some fundamental theorem, like *Hopf index theorem*, the *Atiyah-Singer index theorem*, and the *Grothendieck-Riemann-Roch theorem*. In particular we discuss this latter theorem in the case of curves, surfaces and three-folds.

CHAPTER 1

Vector Bundles

1.1. The Thom Isomorphism

Let $\pi : E \to M$ be an orientable vector bundle on the orientable manifold M. Let $\Omega^*_{cv}(E)$ be the complex of forms on E with *compact support in the vertical direction*, i.e. an n-form ω on E is in $\Omega^n_{cv}(E)$ if and only if for any compact set $K \subseteq M$ the set $\pi^{-1}(K) \cap Supp(\omega)$ is compact. The cohomology $H^*_{cv}(E)$ of this complex is called *compact vertical cohomology*.
Let $\{(U_i, \psi_i)\}$ be an oriented trivialization of E. Let $x_1, ..., x_m$, $y_1, ..., y_m$ be coordinate functions on U_i and U_j respectively, and let $t_1, ..., t_n$, $u_1, ..., u_n$ be fiber coordinates on $E_{|U_i}$ and $E_{|U_j}$. Since $\{(U_i, \psi_i)\}$ is oriented the fiber coordinates are related by a transformation in $GL^+(n, \mathbb{R})$. Let $\omega \in \Omega^*_{cv}(E)$ be a form, ω is a real linear combination of two types of forms:

(1) forms which do not contain as factor the n-form $dt_1 \wedge ... \wedge dt_n$,
(2) forms which contain as factor the n-form $dt_1 \wedge ... \wedge dt_n$.

We define a morphism $\pi_* : \Omega^*_{cv}(E) \to \Omega^{*-n}(M)$, to be zero on the first type of forms, and to be the integral on the component $dt_1 \wedge ... \wedge dt_n$ on the second type. More precisely let $\omega_i = (\pi^*\Gamma_i) f(x_1, ..., x_m, t_1, ..., t_n) dt^1 \wedge ... \wedge dt_n$, where $\Gamma_i \in \Omega^{*-n}(M)$, then we define

$$\pi_*\omega_i = \Gamma_i \int_{\mathbb{R}^n} f(x, t) dt_1 \wedge ... \wedge dt_n.$$

Since E is an oriented vector bundle $\pi_*\omega_i = \pi_*\omega_j$, and the $\{\pi_*\omega_i\}$ piece together to give a global form $\pi_*\omega$ on M. Furthermore the integration along the fibers π_* commutes with exterior differentials.

REMARK 1.1. A $n+k$-form on E can be written ad

$$\omega = \sum f(x_1, ..., x_m, t_1, ..., t_n) dx_I \wedge dt_J,$$

with $|I| + |J| = n + k$. If ω does not contain the term $dt_1 \wedge ... \wedge dt_n$, then $\pi_*\omega = 0$, if ω contains this term then it can be integrated along the fibers to give a well defined k-form $\pi_*\omega$ on M.
From the singular homology viewpoint the map π_* is the contraction of a $k+n$-simplex Δ in E to a k-simplex in M that maps the points in $\Delta \cap E_x$ to the point $x \in M$.

THEOREM 1.2. *(Thom Isomorphism) The integration along the fibers*

$$\pi_* : H^*_{cv}(E) \to H^{*-n}(M),$$

is an isomorphism. Its inverse $\mathcal{T} : H^*(M) \to H^{*+n}_{cv}(E)$ is called the Thom isomorphism.

The image of $1 \in H^0(M)$ determines a cohomology class $T = \mathcal{T}(1) \in H^n_{cv}(E)$ called the *Thom Class* of the oriented vector bundle E. In terms of the Thom class the Thom isomorphism can be written as

$$\mathcal{T}(\omega) = \pi^*(\omega) \wedge T.$$

REMARK 1.3. Locally we can write $\omega \in H^k(M)$ as

$$\omega(x_1, ..., x_m) = \sum f_{i_1,...,i_k}(x_1, ..., x_m) dx_{i_1} \wedge ... \wedge dx_{i_k}.$$

The pullback π^* gives a k-form on E, and the Thom isomorphism can be written locally as

$$\mathcal{T}(\omega) = \pi^*(\omega) \wedge T = \pi^*(\omega) \wedge dt_1 \wedge ... \wedge dt_n.$$

1.2. The Euler Class and The Thom Class

Let E be a rank 2 real orientable vector bundle on a orientable manifold M, and let $\{U_i\}$ be an open cover of M that trivializes E. Since E has a Riemannian structure, over each U_i we can choose a orthonormal frame, this defines on $E_{|U_i}$ polar coordinates r_i, θ_i. If $x_1, ..., x_n$ are local coordinates on U_i, then $\pi^* x_1, ..., \pi^* x_n, r_i, \theta_i$ are coordinates on $E_{|U_i}$. On $U_i \cap U_j$ the radii r_i, r_j are equal, but the angular coordinates θ_i, θ_j differ by a rotation. We define $\varphi_{i,j}$ (up to a multiple of 2π) as the angle of rotation in the counterclockwise direction form the i-coordinates to the j-coordinates:

$$\theta_j - \theta_i = \pi^* \varphi_{i,j},\ \varphi_{i,j} : U_i \cap U_j \to \mathbb{R}.$$

Note that on a triple intersection $\varphi_{i,j} + \varphi_{j,k} - \varphi_{i,k} = \theta_j - \theta_i + \theta_k - \theta_j - \theta_k + \theta_i = 0$, so $\varphi_{i,j} + \varphi_{j,k} - \varphi_{i,k} \in 2\pi\mathbb{Z}$. Clearly the 1-forms $\{d\varphi_{i,j}\}$ satisfy the cocycle condition. Consider now the 1-form ξ_i on U_i given by

$$\xi_i = \frac{1}{2\pi} \sum_k \rho_k d\varphi_{k,i},$$

where $\{\rho_k\}$ is a partition of unity subordinate to $\{U_k\}$. Then

$$\xi_j - \xi_i = \frac{1}{2\pi} \sum_k \rho_k d\varphi_{k,j} - d\varphi_{k,i} = \frac{1}{2\pi} d\varphi_{i,j} \sum_k \rho_k = \frac{1}{2\pi} d\varphi_{i,j}.$$

We see that $d\xi_i = d\xi_j$ on $U_i \cap U_j$. Hence $\{d\xi_i\}$ piece together to give a global, closed 2-form e on M. It is not necessarily exact since the ξ_i do not usually piece together to give a global 1-form. The cohomology class of e in $H^2(M)$ is called the *Euler Class* of the oriented vector bundle E.
If $\{\ni_i\}$ are a different choice of 1-forms such that $\frac{1}{2\pi} d\varphi_{i,j} = \overline{xi}_j - \overline{\xi}_i = \xi_j - \xi_i$, then $\overline{\xi}_j - \xi_j = \overline{\xi}_i - \xi_i = \xi$ is a global form. So $d\overline{\xi}_i$ and $d\xi_i$ differ by an exact global form. Then our construction of the Euler class is independent of the choice of ξ.
By the formulas $\frac{1}{2\pi} d\varphi_{i,j} = \xi_j - \xi_i$ and $\theta_j - \theta_i = \pi^* \varphi_{i,j}$, on $E^0 = E \setminus \{\textit{zero section}\}$ we have $d\theta_j - d\theta_i = \pi^* d\varphi_{i,j} = \pi^* 2\pi(\xi_j - \xi_i)$, so

$$\frac{d\theta_j}{2\pi} - \pi^* \xi_j = \frac{d\theta_i}{2\pi} - \pi^* \xi_i.$$

These forms then piece together to give a *global angular* 1-*form* ψ on E^0, whose restriction to each fiber is the angular form $\frac{1}{2\pi} d\theta$. Note that the global angular form is not closed:

$$d\psi = d(\frac{d\theta_i}{2\pi} - \pi^*\xi_i) = -\pi^* d\xi_i = -\pi^* d\xi_j.$$

So we have a relation between the global angular form and the Euler class

$$d\psi = -\pi^* e.$$

The Euler class $e(E)$ can be written in terms of the transition functions. Let $g_{i,j} : U_i \cap U_j \to SO(2)$ be the transition functions of E. We can identify $SO(2)$ as the unit circle in the complex plane via

$$\begin{pmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta) \end{pmatrix} = e^{i\theta}.$$

So $g_{i,j}$ can be thought as the complex valued function $g_{i,j} = e^{i\theta}$, and $i\theta = log(g_{i,j})$, so the angle from the j-coordinates to the i-coordinates is $\theta = \frac{1}{i}log(g_{i,j})$. We get

$$\theta_j - \theta_i = \pi^*\varphi_{i,j} = -\pi^*\frac{1}{i}log(g_{i,j}).$$

Since π_* is surjective, π^* is injective, so

$$\varphi_{i,j} = -\frac{1}{i}log(g_{i,j}).$$

Now let $\{\rho_k\}$ be a partition of unity relative to $\{U_k\}$. Then $\xi_i = \frac{1}{2\pi}\sum_k \rho_k d\varphi_{k,i} = -\frac{1}{2\pi i}\sum_k \rho_k d(log(g_{k,i}))$, and

$$e(E) = -\frac{1}{2\pi i}\sum_k d(\rho_k d(log(g_{k,i}))).$$

EXAMPLE 1.4. Consider the holomorphic line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}^2)$. Its transition function are $g_{0,1} = \frac{z_1}{z_0}$ and $g_{1,0} = \frac{z_0}{z_1}$, over the standard covering of $\mathbb{P}^1$. If $(1, z)$ is the coordinate in U_0 and $(w, 1)$ with $w = \frac{1}{z}$ is the coordinate on U_1, then the functions

$$\rho_0 = \frac{1}{1+|z|^2}, \ \rho_1 = \frac{|w|^2}{1+|w|^2},$$

are a partition of unity relative to our covering. We have $d(log(g_{0,1})) = d(log(z)) = \frac{1}{z}dz$, and $d(\rho_0\frac{1}{z}dz) = d(\frac{1}{(z+z^2\bar{z})}\frac{1}{z}dz) = -\frac{z^2}{(z+z^2\bar{z})^2}d\bar{z}\wedge dz = \frac{1}{(1+z\bar{z})^2}dz\wedge d\bar{z}$. So

$$e(\mathcal{O}(-1)) = \frac{1}{2\pi i}\frac{1}{(1+z\bar{z})^2}dz\wedge d\bar{z}.$$

PROPOSITION 1.5. *Let $f : N \to M$ be a smooth map, and let E be a rank 2 vector bundle on M, then*

$$e(f^{-1}E) = f^*e(E),$$

i.e. the Euler class is functorial.

PROOF. If $g_{i,j}$ are the transition function of E then $f^*g_{i,j}$ are the transition function of $f^{-1}E$. The Euler class of $f^{-1}E$ is given by

$$e(f^{-1}E) = -\frac{1}{2\pi i}\sum_k d(\rho_k d\, log\, (f^*g_{i,j})) = f^*e(E).$$

♠

Consider now the cohomology class

$$T = d(\rho(r) \cdot \psi) = d\rho(r) \cdot \psi + \rho(r) d\psi = d\rho(r) \cdot \psi - \rho(r)\pi^* e.$$

Note that the form T is a global form on E since $d\rho \equiv 0$ near the zero section, and T has the following properties:

- T has compact support in the vertical direction,
- T is closed: $dT = -d\rho(r) \cdot d\psi - d\rho(r)\pi^* e = 0$,
- the restriction of T to each fiber has integral 1, indeed on each fiber we have
$$\int_0^\infty \int_0^{2\pi} d\rho(r) \cdot \frac{d\theta}{2\pi} = \rho(\infty) - \rho(0) = 1,$$
- the cohomology class of T is independent of the choice of $\rho(r)$.

Therefore T defines the Thom class. If $s_0 : M \to E$ is the zero section of E, then $s_0^* T = s_0^*(d\rho(r) \cdot \psi - \rho(r)\pi^* e) = ds_0^*\rho(r) \cdot s_0^*\psi - \rho(0) s_0^* \pi^* e = d\rho(0) \cdot s_0^*\psi - \rho(0)(\pi \circ s_0)^*(e) = e$. This proves the following fact.

PROPOSITION 1.6. *The pullback of the Thom class to M via the zero section is the Euler class.*

Let $\{U_i\}$ be a trivializing cover for E, let $\{\rho_i\}$ be a partition of unity relative to this cover, and let $g_{i,j}$ be the transition function of E. Since

$$\psi = \frac{d\theta_i}{2\pi} - \pi^* \xi_i = \frac{d\theta_i}{2\pi} + \frac{1}{2\pi i}\pi^* \sum_k \rho_k d\, log\, g_{k,i},$$

we have

$$T = d(\rho(r) \cdot \psi) = d(\rho(r)\frac{d\theta_i}{2\pi}) + \frac{1}{2\pi i} d(\rho(r)\pi^* \sum_k \rho_k d\, log\, g_{k,i}).$$

This is an explicit formula for the Thom class.

1.2.1. The Thom Isomorphism for non orientable Vector Bundles. Let $\pi : E \to M$ be any rank n vector bundle on M, and let $\mathcal{U}$ be a good cover for M. We define a presheaf $\mathcal{H}^n_{cv}$ on M by

$$U \subseteq M \mapsto H^n_{cv}(\pi^{-1}(U)),$$

then there is an isomorphism (*the Thom isomorphism*)

$$H^*_{cv}(E) \cong H^{*-n}(\mathcal{U}, \mathcal{H}^n_{cv}).$$

1.2.2. Thom Class and Poincaré Duality. Let S be a closed oriented submanifold of dimension k in an oriented manifold M, with $dim(M) = n$. The Poincaré dual of S is the cohomology class of the closed $(n-k)$-form ω_S, characterized by the property

$$\int_S \alpha = \int_M \alpha \wedge \omega_S,$$

for any closed k-form α with compact support on M.
The submanifold S has a tubular neighborhood in M that is diffeomorphic to the normal bundle $N_{S/M}$ of S in M. Since S and M are orientable the tangent bundles T_S and T_M are orientable, and by the exact sequence

$$0 \mapsto T_S \to T_{M|S} \to N_{S/M} \mapsto 0,$$

we conclude that $N_{S/M}$ is also orientable. Let $j : N_{S/M} \to M$ be the inclusion if the tubular neighborhood in M. Since $N_{S/M}$ is a rank $(n-k)$-vector bundle on S we can apply the Thom isomorphism theorem. So we have a composition of maps

$$H^*(S) \overset{\wedge T}{\to} H^{*+n-k}_{cv}(N_{S/M}) \overset{j_*}{\to} H^{*+n-k}(M),$$

where T is the Thom class of the tubular neighborhood $N = N_{S/M}$, and j_* is the extension by zero.

PROPOSITION 1.7. *The Poincaré dual of a closed oriented submanifold S in a oriented manifold M is the Thom class of the normal bundle of S, more precisely*

$$\omega_S = j_*T, \text{ in } H^{n-k}(M).$$

Furthermore the Thom class of an oriented vector bundle $E \to M$ over an oriented manifold M and the Poincaré dual of the zero section of E can be represented by the same form.

PROOF. We have to show that j_*T satisfies the defining property of the Poincaré dual ω_S. Let α be a closed k-form with compact support on M, and let $i : S \to N$ be the inclusion, regarded as the zero section of the bundle $\pi : N \to S$. Since π is a deformation retraction, π^* and i^* are inverse morphism in cohomology, so on level of forms α and $\pi^*i^*\alpha$ differ by an exact form, i.e. $\alpha = \pi^*i^*\alpha + d\gamma$.We compute $\int_M \alpha \wedge i^*T = \int_N \alpha \wedge T$ because $Supp(j_*T) \subseteq N$. Furthermore $\int_N \alpha \wedge T = \int_N (\pi^*i^*\alpha + d\gamma) \wedge T = \int_N (\pi^*i^*\alpha) \wedge T$, since $\int_N (d\gamma \wedge T) = \int_N d(\gamma \wedge T) = 0$ by Stokes theorem. Now by projection formula we get

$$\int_N (\pi^*i^*\alpha) \wedge T = \int_S i^*\alpha \wedge \pi_*T = \int_S i^*\alpha,$$

because $\pi_*T = 1$.
Now suppose that E is an oriented vector bundle over an oriented manifold M. Embed M diffeomorphically in E as the zero section, there is an exact sequence

$$0 \mapsto T_M \to T_{E|M} \to E \mapsto 0,$$

so the normal bundle of M in E is E itself. By the first part of the proof we conclude that the Poincaré dual of M in E is the Thom class of E. ♠

1.2.3. The Euler Class and the general section. The preceding argument on rank 2 orientable vector bundle can be generalized to rank n orientable vector bundle, we define similarly the Thom and the Euler class, and again the Euler class turns out to be the pullback of the Thom class via the zero section.
Let $\pi : E \to M$ be a vector bundle, and let $S_0 \subseteq E$ be the image of the zero section. A section s of E is transversal if its image $S = s(M)$ intersects S_0 transversally.

PROPOSITION 1.8. *Let Z be the zero locus of a general section (in particular it is transversal). Then Z is a submanifold of M and its normal bundle in M is $N_{Z/M} \cong E_{|Z}$.*

PROOF. Let $S = s(M)$ be the image of the section s. The fact that Z is a submanifold of M is a consequence of the transversality. Note that since E is locally trivial its tangent bundle on S_0 can be written as

$$T_{E|S_0} = E_{|S_0} \oplus T_{S_0}.$$

By the transversality of $S \cap S_0$, we have $T_S + T_{S_0} = T_E = E \oplus T_{S_0}$ on $S \cap S_0$. So the projection $T_S \to E$ over $S \cap S_0$ is surjective with kernel $T_S \cap T_{S_0}$. But by transversality we have $T_S \cap T_{S_0} = T_{S \cap S_0}$. So we have an exact sequence

$$0 \mapsto T_{S \cap S_0} \to T_{S|S \cap S_0} \to E_{|S \cap S_0} \mapsto 0.$$

Then $N_{S \cap S_0/M} \cong E_{|S \cap S_0}$. ♠

PROPOSITION 1.9. *Let $\pi : E \to M$ be an oriented vector bundle over an oriented manifold M. The the Euler class $e(E)$ is Poincaré dual to the zero locus of a general section.*

PROOF. We identify M with its image S_0 in E via the zero section. Let S be the image of a general section $s : M \to E$, then $Z = S \cap S_0$ is a closed oriented submanifold of M, and its normal bundle is $N_{Z/M} \cong E_{|Z}$. Choose the Thom class T of E to have support enough close to the zero section such that T restricted to the tubular neighborhood $N_{Z/S}$ in S has compact support in the vertical direction. The pullback s^*T is the Thom class of the tubular neighborhood $N_{Z/M}$ in M. We know that $s^*T = e(E)$. Since the Thom class of $N_{Z/M}$ is Poincaré dual to Z in M, then the Euler class is Poincaré dual to Z in M. ♠

1.3. Chern Classes

A complex vector bundle of rank r is a bundle with fiber $\mathbb{C}^r$ and structure group $GL(r, \mathbb{C})$. The structure group of a real vector vector bundle can be reduces to the orthogonal group $O(r)$, and similarly the structure group of a rank r complex vector bundle can be reduced to the unitary group $U(r)$. Clearly if E is a complex vector bundle of rank $2r$ we can consider the underlying real vector bundle $E_{\mathbb{R}}$ of rank $2r$. Since $U(1)$ and $SO(2, \mathbb{R})$ are isomorphic as algebraic groups there is a bijective correspondence between complex line bundles and oriented rank r real bundles.

DEFINITION 1.10. The *first Chern class* of a complex line bundle L over a manifold M is the Euler class of the underlying real vector bundle $L_{\mathbb{R}}$,

$$c_1(L) = e(L_{\mathbb{R}}) \in H^2(M).$$

REMARK 1.11. Recall that for the Euler class of a real rank 2 vector bundle we have $e(E) = -\frac{1}{2\pi i} \sum_k d(\rho_k d(log(g_{k,i})))$. If L and $L^{'}$ are two complex line bundle given by the transition functions $\{g_{i,j}\}$ and $\{g^{'}_{i,j}\}$, then $L \otimes L^{'}$ has $\{g_{i,j} g^{'}_{i,j}\}$ has transition functions. Since $d\, log(g_{i,j} g^{'}_{i,j}) = d\, log(g_{i,h}) + d\, log(g^{'}_{i,j})$ we get

$$c_1(L \otimes L^{'}) = c_1(L) + c_1(L^{'}).$$

If L^* is the dual of L we know $Hom(L, L) = L \otimes L^*$. Since $Hom(L, L)$ has a nowhere vanishing section given by the identity map, $L \otimes L^*$ is the trivial bundle. So $c_1(L \otimes L^*) = c_1(L) + c_1(L^*) = 0$, and

$$c_1(L^*) = -c_1(L).$$

Let V be a $\mathbb{C}$-vector space of dimension n. On the projective space $\mathbb{P}(V)$ we have the product bundle $\mathbb{P}(V) \times V$ and the tautological bundle S given by

$$S = \{(l, v) \in \mathbb{P}(V) \times V \mid v \in l\}.$$

Furthermore we have the universal quotient bundle Q defined by the exact sequence
$$0 \mapsto S \to \mathbb{P}(V) \times V \to Q \mapsto 0.$$
This sequence is called the tautological sequence and S^* is the hyperplane bundle. Consider the natural projection $\pi : S \to V$, if $v \neq 0$ the fiber $\pi^{-1}(v)$ consists of a single point (l, v), where l is the line generated by v. But for $v = 0$ we have $\pi^{-1}(0) = \mathbb{P}(V)$. So S can be obtained from V by separating all the lines through the origin in V, in other words $\pi : S \to V$ is the blow-up of V at the origin.
Now we choose an hermitian structure on V, and let E be the unit sphere bundle of the universal bundle S,
$$E = \{(l, v) \mid v \in l, \parallel v \parallel = 1\}.$$
Note that $\pi^{-1}(0)$ is the zero section of the universal bundle S. Since S and V are birational, and $S \setminus \sigma^{-1}(0)$ is diffeomorphic to $V \setminus \{0\}$, we have that E is diffeomorphic to the sphere S^{2n-1} in V. The map $\overline{\pi} : E \to \mathbb{P}(V)$ gives a fibration
$$S^1 \to S^{2n-1} \cong E \to \mathbb{P}(V).$$
Let $x = c_1(S^*) = -c_1(S)$ be the first Chern class of the hyperplane bundle, using this fibration and and Leray spectral sequence we get that the cohomology ring $H^*(\mathbb{P}(V))$ is generated by x and
$$H^*(\mathbb{P}(V)) = \mathbb{R}[x]/(x^n),$$
where $n = dim(V)$. Recall that the Poincaré series of a manifold M is
$$P_t(M) = \sum_{j=0}^{\infty} dim(H^j(M))t^j.$$
Since the projective space $\mathbb{P}(V) = \mathbb{P}^{n-1}$ has cohomology only in even degree, and more precisely $dim(H^j(\mathbb{P}(V))) = 0$ for j odd, and $dim(H^j(\mathbb{P}(V))) = 1$ for j even, with $j = 0, ..., 2n-2$, we get
$$P_t(\mathbb{P}(V)) = 1 + t^2 + t^4 + ... + t^{2n-2} = \frac{1 - t^{2n}}{1 - t^2}.$$
Let $E \to M$ be a complex vector bundle and let $\pi : \mathbb{P}(E) \to M$ be the projectivized bundle. On $\pi : \mathbb{P}(E) \to M$ there is the tautological bundle $\pi^{-1}E$, the universal bundle S and the universal quotient bundle Q defined by the sequence
$$0 \mapsto S \to \pi^{-1}E \to Q \mapsto 0.$$
Let $x = c_1(S^*)$ be the Chern class of the hyperplane bundle, then x is a cohomology class in $H^2(\mathbb{P}(E))$. The restriction S to a fiber $\mathbb{P}(E_p)$ is the universal bundle S_p on the projective space $\mathbb{P}(E_p)$, then $c_1(S_p)$ is the restriction of $-x$ to $\mathbb{P}(E_p)$. So the cohomology classes $1, x, ..., x^{n-1}$ are cohomology classes on $\mathbb{P}(E)$ whose restrictions to each fiber $\mathbb{P}(E_p)$ freely generate the cohomology of the fiber $\mathbb{P}(E_p)$. Recall now the Leray-Hirsch theorem.

THEOREM 1.12. *(Leray-Hirsch) Let E be a fiber bundle over a manifold M with fiber F, suppose that M admits a finite good cover. If there are global cohomology classes $x_1, ..., x_n$ on E such that their restriction to each fiber freely generated the cohomology of the fiber, then $H^*(E)$ is a free module over $H^*(M)$ with basis $\{x_1, ..., x_n\}$, i.e.*
$$H^*(E) = H^*(M) \otimes \mathbb{R}\{x_1, ..., x_n\} = H^*(M) \otimes H^*(F).$$

By Leray-Hirsch theorem x^n can be written uniquely as a linear combination of $1, x, ..., x^{n-1}$ with coefficients in $H^*(M)$,

$$x^n + c_1(E)x^{n-1} + ... + c_n(E) = 0.$$

The coefficients $c_j(E) \in H^{2j}(M)$ are the *Chern classes* of E, their sum

$$c(E) = 1 + c_1(E) + ... + c_n(E) \in H^*(M)$$

is the *total Chern class*. Then the cohomology ring of $\mathbb{P}(E)$ is given by

$$H^*(\mathbb{P}(E)) = H^*(M)[x]/(x^n + c_1(E)x^{n-1} + ... + c_n(E)).$$

Furthermore by Leray-Hirsch we have $H^*(\mathbb{P}(E)) = H^*(M) \otimes H^*(\mathbb{P}^{n-1})$, and the Poincaré series of $\mathbb{P}(E)$ is

$$P_t(\mathbb{P}(E)) = P_t(M)\frac{1-t^{2n}}{1-t^2}$$

Note that for a line bundle L, $\mathbb{P}(L) = M$ and $\pi^{-1}L = L$. The universal bundle S is L itself. So $x = e(S^*_{\mathbb{R}}) = -e(S_{\mathbb{R}}) = -e(L_{\mathbb{R}})$, and $x + e(L_{\mathbb{R}}) = 0$ proves that $c_1(L) = e(L_{\mathbb{R}})$. We see that our two definitions of first Chern class of a line bundle are equivalent.

1.3.1. Properties of the Chern Classes.

- Let $f : Y \to X$ be a map end let $E \to X$ be complex vector bundle, then $c_j(f^{-1}E) = f^*c_j(E)$ for any j.
- (Whitney Formula) For the direct sum one has $c(E \oplus F) = c(E)c(F)$.
- If E has a non-vanishing section then its top Chern class is zero.
  To see this consider $s : X \to E$ a non-vanishing section, and take the section $f : X \to \mathbb{P}(E)$ induced by s. Then $f^{-1}S_E$ is a line bundle on X and its fiber at p is the line in E_p spanned by $s(p)$. So $f^{-1}S_E$ is a line bundle with a non-vanishing section, then it is trivial. For the Chern class we get $f^*c_1(S_E) = c_1(f^{-1}S_E) = 0$. This implies $f^*x = f^*c_1(S^*_E) = 0$, and applying f^* to $x^n + c_1x^{n-1} + ... + c_n = 0$ we get $f^*c_n = 0$, and finally $c_n = 0$.
- (Splitting Principle) Let $E \to M$ be a complex vector bundle of rank r over a manifold M. Then there exists a manifold $F(E)$ called a *split manifold of* E, with a map $\sigma : F(E) \to M$ such that:
  (1) the pullback of E to $F(E)$ splits into a direct sum of line bundles $\sigma^{-1}E = L_1 \oplus ... \oplus L_n$;
  (2) σ^* embeds $H^*(M)$ in $H^*(F(E))$.
- Let $E = L_1 \oplus ... \oplus L_r$ be a splitting. By Whitney product $c(E) = c(L_1)...c(L_r) = (1 + c_1(L_1))...(1 + c_1(L_r))$. For the dual bundle we have $E^* = L_1^* \oplus ...L_r^*$, and $c(E^*) = (1 - c_1(L_1))...(1 - c_1(L_r))$. Comparing the two expressions we get the formula

$$c_j(E^*) = (-1)^j c_j(E).$$

Now we apply the Whitney formula and the splitting principle to state the relation between the top Chern class of E and its Euler class.

PROPOSITION 1.13. *Let $E \to M$ be a complex vector bundle of rank r. The top Chern class $c_r(E)$ of E is the Euler class $e(L_{\mathbb{R}})$ of the underlying real vector bundle $L_{\mathbb{R}}$.*

PROOF. Let $\sigma : F(E) \to M$ be a splitting manifold for E, and let $\sigma^{-1}E = L_1 \oplus ... \oplus L_r$ be the splitting. Then $\sigma^* c_n(E) = c_n(\sigma^{-1}E) = c_1(L_1)...c_1(L_r) = e((L_1)_{\mathbb{R}})...e((L_r)_{\mathbb{R}}) = e((L_1)_{\mathbb{R}}) \oplus ... \oplus (L_r)_{\mathbb{R}})) = e((\sigma^{-1}E)\mathbb{R}) = \sigma^* e(E_{\mathbb{R}})$. By the injectivity of σ^* we get $c_n(E) = e(E_{\mathbb{R}})$. ♠

1.4. Pontrjagin Classes

The Chern classes are invariants of complex bundles but the can be used to define invariants for real vector bundles. Let $E \to M$ be areal vector bundle of rank r, and let $E_{\mathbb{C}} = E \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified vector bundle. We define the *Pontrjagin classes* of E as the Chern classes of $E_{\mathbb{C}}$:

$$p_j(E) = c_j(E \otimes_{\mathbb{R}} \mathbb{C}),$$

the total Pontrjagin class is

$$p(E) = 1 + p_1(E) + ... + p_r(E).$$

By the properties of the Chern classes we have the Whitney formula for Pontrjagin classes $p(E \oplus F) = p(E)p(F)$. The Pontrjagin classes of a manifold M are the Pontrjagin classes of its tangent bundle T_M.
Note that since E is real the transition function of $E_{\mathbb{C}}$ are the same as those of E, they are real valued and so $E \times_{\mathbb{R}} \mathbb{C}$ is isomorphic to its conjugate $\overline{E \otimes_{\mathbb{R}} \mathbb{C}}$. So $c_j(E \times_{\mathbb{R}} \mathbb{C}) = c_j(\overline{E \otimes_{\mathbb{R}} \mathbb{C}}) = (-1)^j c_j(E \otimes_{\mathbb{R}} \mathbb{C})$. Then $2c_j(E \otimes_{\mathbb{R}} \mathbb{C}) = 0$ for j odd. We see that the odd Pontrjagin classes are zero in the De Rham cohomology, and have torsion of order two in the integral cohomology.

1.5. The Hopf Index Theorem

Let M be a compact oriented manifold and let $e(T_M)$ be its Euler class. We define the Euler number as $\int_M e(T_M)$. One can prove that the Euler number is equal to the Euler characteristic:

$$\int_M e(T_M) = \chi(M) = \sum (-1)^j dim\, H^j(M).$$

Let $E \to M$ be a $(n-1)$-sphere bundle over a compact orientable manifold M of dimension n. Let s be a section of E over a punctured neighborhood D_r of a point $x \in M$, and choose D_r sufficiently small such that it is diffeomorphic to a punctured dick in $\mathbb{R}^n$ and trivializes E. Choose an orientation on S^{n-1} such that $E_{|D_r} \cong D_r \times S^{n-1}$ is a preserving orientation isomorphism. We have a map

$$\delta\overline{D_r} \cong S^{n-1} \xrightarrow{s} E_{|\overline{D_r}} \cong \overline{D_r} \times S^{n-1} \xrightarrow{\pi} S^{n-1},$$

where π is the projection. It make sense to consider the degree of the composition $\pi \circ s$, and we define the local degree in x of the section s as

$$locdeg_x(s) = deg(\pi \circ s).$$

One can prove that if $E \to M$ is an oriented $(n-1)$-sphere bundle over a compact oriented manifold M of dimension n, and if E has a section s over $M \setminus \{x_1, ..., x_k\}$, then the Euler number is the sum of the local degrees of s at $x_1, ..., x_k$,

$$\int_M e(T_E) = \sum locdeg_{x_j}(s).$$

Let $V : M \to TM$ be a vector field on a manifold M of dimension n with isolated zeros $x_1, ..., x_k$. Then $rank(T_M) = n$ and we can consider the $(n-1)$-sphere bundle $S(T_M)$ of T_M. Now the function

$$s_V : M \to S(T_M),\ x \mapsto \frac{V(x)}{\| V(x) \|},$$

is a section of the unit tangent bundle of M relative to some Riemannian metric on M, and it is defined on $M \setminus \{x_1, ..., x_k\}$. We define the index of the vector field V in x_j as $ind_{x_j} V = locdeg_{x_j} s_V$. Then the following theorem follows

THEOREM 1.14. *(Hopf Index Theorem) The sum of the indexes of a vector field V on a compact oriented manifold M is the Euler characteristic of M,*

$$\sum_{j=1}^{k} ind_{x_j} V = \int_M e_{T_M} = \chi(M).$$

REMARK 1.15. The sum of the indexes of a vector field V seems to depend strictly from the differentiable structure on the manifold M, but the Hopf index theorem reveals that it is equal to the Euler characteristic, so it depends only on the topology of M. Recall that the Euler characteristic depends only on the topology because by *De Rham's theorem* there is an isomorphism between De Rham cohomology and singular homology.

CHAPTER 2

The Complex Analytic Viewpoint

Let E be a complex vector bundle over a real manifold M.

DEFINITION 2.1. An *hermitian structure* h on $E \to M$ is an hermitian scalar product h_x on each fiber $E(x)$ for any $x \in M$ which depends smoothly on x. The pair (E, h) is called an hermitian vector bundle.

EXAMPLE 2.2. Let $L = \mathcal{O}(-1)$ be the tautological line bundle on $\mathbb{P}^1$. Over a point $z = [z_0 : z_1] \in \mathbb{P}^1$ the fiber L_z is the line generated by the vector $(z_0, z_1) \in \mathbb{C}^2$, i.e.

$$L_z = \{(\lambda z_0, \lambda z_1) \mid \lambda \in \mathbb{C}\}.$$

On the open set $U_0 = \{z_0 \neq 0\}$ we have a local trivialization

$$\varphi_0 : L_{|U_0} \to U_0 \times \mathbb{C},\ (\lambda z_0, \lambda z_1) \mapsto ((1 : w), \lambda w),$$

where $w = \frac{z_1}{z_0}$. We can define an hermitian product on the fiber as

$$h(\lambda) = \frac{|\lambda w|^2}{1 + |w|^2}.$$

2.1. Connections

We denote by $\bigwedge^{p,q} M \otimes E$ the twisted form bundle, and by $A^{p,q}(E)$ its sheaf of sections.

DEFINITION 2.3. A *connection* on a vector bundle E is a $\mathcal{C}$-linear morphism of shaves $\nabla : A^0(E) \to A^1(E)$, which satisfies the *Leibniz rule*

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s),$$

for any local function f on M and any local section s of E. A section s of E is called flat with respect to ∇ if $\nabla(s) = 0$.

Now let ∇ be a connection on E, and let $a \in A^1(End(E))$. Then a acts on $A^0(E)$ by multiplication on the form part (i.e on $\bigwedge^0 M$), and by evaluation on $End(E) \times E \to E$ on the bundle part. We compute $(\nabla + a)(f \cdot s) = \nabla(f \cdot s) + a(f \cdot s) = df \otimes s + f \cdot \nabla(s) + da(s) = df \otimes s + f \cdot (\nabla + a)(s)$. Thus $\nabla + a$ is again a connection. We conclude that:
The set of all connections on a vector bundle E is an affine space over the complex vector space $A^1(End(E))$.
Suppose E to be the trivial bundle $E = M \times \mathbb{C}^r$, then a section of $\bigwedge^k X \otimes E$ is of the form $s = (s_1, ..., s_r)$ with $s_i \in A^k_M$. We can define a connection on E by the usual Cartan differential $d : A^0(E) \to A^1(E)$. Any other connection is of the form $\nabla = d + A$, with $A \in A^1(End(E))$.

DEFINITION 2.4. Let (E, h) be an hermitian vector bundle. A connection ∇ on E is an hermitian *connection* with respect to h if for any local sections s_1, s_2, one has

$$d(h(s_1, s_2)) = h(\nabla(s_1), s_2) + h(s_1, \nabla(s_2)).$$

In what follows we consider a holomorphic vector bundle E on a complex manifold X. In this case we have a $\overline{\delta}$-operator $\overline{\delta} : A^0(E) \to A^{0,1}(E)$. Using the decomposition $A^1(E) = A^{1,0}(E) \oplus A^{0,1}(E)$ we can decompose the connection $\nabla = \nabla^{1,0} \oplus \nabla^{0,1}$ with

$$\nabla^{1,0} : A^0(E) \to A^{1,0}(E), \quad \nabla^{0,1} : A^0(E) \to A^{0,1}(E).$$

Note that $\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s) = (\delta + \overline{\delta}) f \otimes s + f \cdot \nabla(s) = \delta f \otimes s + \overline{\delta} f \otimes s + f \cdot \nabla^{1,0}(s) + f \cdot \nabla^{0,1}(s) = (\delta f \otimes s + f \cdot \nabla^{1,0} s) + (\overline{\delta} f \otimes s + f \cdot \nabla^{0,1}(s)) = \nabla^{1,0}(f \cdot s) + \nabla^{0,1}(f \cdot s)$. So $\nabla^{0,1}$ behaves similarly to $\overline{\delta}$.

DEFINITION 2.5. A connection ∇ on a holomorphic vector bundle E is *compatible with the holomorphic structure* if $\nabla^{0,1} = \overline{\delta}$.

As for arbitrary connections for connections compatible with the holomorphic structure we have that
The space of connections ∇ on a holomorphic vector bundle E compatible with the holomorphic structure forms an affine space over the complex vector space $A^{1,0}(End(E))$.

THEOREM 2.6. *Let (E, h) be a holomorphic vector bundle with an hermitian structure. Then there exists a unique hermitian connection ∇ that is compatible with the holomorphic structure. This connection is called the Chern connection.*

PROOF. The problem is local, so assume E to be the trivial holomorphic vector bundle $E = X \times \mathbb{C}^r$. The connection is of the form $\nabla = d + A$, where $A = (a_{i,j})$ is a matrix valued one-form on X. The hermitian structure on E is given by a map H that associates to any $x \in X$ a positive definite hermitian matrix $H(x) = (h_{i,j}(x))$. Let e_i be the constant unit vector considered as a section of E. Then $\nabla(e_i) = de_i + \sum a_{k,i} e_k$, and since ∇ has to be compatible with the hermitian structure we have $dh(e_i, e_j) = h(\sum a_{k,i} e_k, e_j) + h(e_i, \sum a_{l,j}) e_l$, or in matrix form

$$dH = A^t \cdot H + H \cdot \overline{A}.$$

The connection ∇ is compatible with $\overline{\delta}$, so the matrix A has to be of type $(1, 0)$. By $\delta H + \overline{\delta} H = A^t H + H \cdot \overline{A}$ we get $\overline{\delta} H = H \cdot \overline{A}$, and after conjugation

$$A = \overline{H}^{-1} \delta(\overline{H}).$$

Thus A and a fortiori ∇ are uniquely determined by H. ♠

REMARK 2.7. If $E = L$ is a holomorphic line bundle, then an hermitian structure h on L is given by a positive real valued function. In this case $H = (h)$ is a 1×1 matrix, and $\delta \, log(h) = \frac{1}{h} \delta \, h = \overline{H}^{-1} \delta(\overline{H})$. So the Chern connection on L is locally given by

$$\nabla = d + \delta \, log(h).$$

EXAMPLE 2.8. Consider the example 2.2. The hermitian metric is given by

$$h = \frac{|w|^2}{1 + |w|^2} = \frac{w\overline{w}}{1 + w\overline{w}}.$$

We compute $\delta\, log(h) = \frac{1+w\overline{w}}{w\overline{w}} \frac{\overline{w}}{(1+|w|^2)^2} dw$. The Chern connection is

$$\nabla = d + \frac{1}{w(1+|w|^2)} dw.$$

2.2. The Atiyah Class

We discuss the notion of holomorphic connection, which should not be confused with the notion compatible with the holomorphic structure. This notion is much more restrictive, but it generalize to pure algebraic setting.

DEFINITION 2.9. Let E be a holomorphic vector bundle on a complex manifold X. A *holomorphic connection* on E is a $\mathbb{C}$-linear of sheaves $D : E \to \Omega_X \otimes E$ such that

$$D(f \cdot s) = df \otimes s + f \cdot D(s),$$

for any local holomorphic function f on X and any local holomorphic section s of E.

Note that if f is a holomorphic function then $\delta(f)$ is holomorphic, in fact $\overline{\delta}\delta(f) = -\delta\overline{\delta}(f) = 0$. Locally any holomorphic connection can be written as $D = \delta + A$, where A is a holomorphic section of $\Omega_X \otimes End(E)$. Thus D looks like the $(1,0)$-part of an ordinary connection. Indeed $\nabla = D + \overline{\delta}$ defines an ordinary connection. However the $(1,0)$-part of an arbitrary connection need not to be a holomorphic connection in general. It might map a holomorphic section of E to a non-holomorphic section of $A^{1,0}(E)$. We want to introduce a cohomology class whose vanishing decides whether a holomorphic connection on a holomorphic vector bundle can be found.
Let $\{U_i\}$ be an open covering of X such that there are holomorphic trivializations $\psi_i : E_{|U_i} \to U_i \times \mathbb{C}^r$, and transition function $\psi_{i,j} = \psi_i\psi_j^{-1} : \mathbb{C}^r \to \mathbb{C}^r$. Consider the differentials $d\psi_{i,j} : \mathbb{C}^r \to \mathbb{C}^r$, and the compositions $\psi_j^{-1}(\psi_{i,j}^{-1} d\psi_{i,j})\psi_j$. Since $\{\psi_{i,j}\}$ is a cocycle also $\{\psi_j^{-1}(\psi_{i,j}^{-1} d\psi_{i,j})\psi_j\}$ is a cocycle. The class given by the Čeach cocycle $\{\psi_j^{-1}(\psi_{i,j}^{-1} d\psi_{i,j})\psi_j\}$ is denoted by

$$A(E) = \{U_{i,j}, \psi_j^{-1}(\psi_{i,j}^{-1} d\psi_{i,j})\psi_j\} \in H^1(X, \Omega_X \otimes End(E)),$$

and is called the *Atiyah Class* of the holomorphic vector bundle E.

PROPOSITION 2.10. *A holomorphic vector bundle E admits a holomorphic connection if and only if its Atiyah class $A(E) \in H^1(X, \Omega_X \otimes End(E))$ is trivial.*

PROOF. The $\psi_{i,j}$ are holomorphic, so $d\psi_{i,j} = \delta\psi_{i,j}$. The local holomorphic connection on $U_i \times \mathbb{C}^r$ are of the form $\delta + A_i$, and can be glued together if and only if $\psi_i^{-1}(\delta + A_i)\psi_i = \psi_j^{-1}(\delta + A_j)\psi_j$ on $U_{i,j}$. Then we have

$$\psi_i^{-1}\delta\psi_i - \psi_j^{-1}\delta\psi_j = \psi_j^{-1}A_j\psi_j - \psi_i^{-1}A_i\psi_i.$$

The left hand side of this equation can be written as $\psi_j^{-1}(\psi_{i,j}^{-1}\delta\psi_i\psi_j^{-1})\psi_j - \psi_j^{-1}\delta\psi_j = \psi_j^{-1}(\psi_{i,j}^{-1}\delta\psi_{i,j} - \delta)\psi_j = \psi_j^{-1}(\psi_{i,j}^{-1}\delta\psi_{i,j})\psi_j$. The right hand side is the coboundary of $\{S_i \in \Gamma(U_i, \Omega \otimes End(E))\}$, where $S_i = \psi_i^{-1}A_i\psi_i$. We conclude that a global holomorphic connection on E exists if and only if the cocycle $\{U_{i,j}, \psi_j^{-1}(\psi_{i,j}^{-1}\delta\psi_{i,j})\psi_j\}$ is a coboundary, in other words if and only if the Atiyah class $A(E)$ of E is trivial. ♠

2.3. Curvature

Let ∇ be a connection on a vector bundle E. In general ∇ does not satisfy $\nabla^2 = 0$. The obstruction for a connection to be a differential is measured by its curvature.
Let E be a vector bundle on a manifold M, and let $\nabla : A^0(E) \to A^1(E)$ be a connection on E. We consider the natural extension $\nabla : A^k(E) \to A^{k+1}(E)$ defined as follows, if α is a local k-form on M and s is a local section of E then

$$\nabla(\alpha \otimes s) = d\alpha \otimes s + (-1)^k \alpha \wedge \nabla(s).$$

In particular the composition

$$F_\nabla = \nabla \circ \nabla : A^0(E) \to A^1(E) \to A^2(E),$$

is called the curvature of the connection ∇ on the vector bundle E. Note that for any local section s of E and for any local function f on M we have $F_\nabla(f \cdot s) = \nabla(df \otimes s + f \cdot \nabla(s)) = d^2 f \otimes s - df \wedge \nabla(s) + df \wedge \nabla(s) + f\nabla(\nabla(s)) = f(\nabla \circ \nabla)(s) = f \cdot F_\nabla(s)$, so the curvature homomorphism is A^0-linear.
Now let $E = M \times \mathbb{C}^r$ be the trivial bundle, if $\nabla = d$ is the trivial connection, then $F_\nabla = d^2 = 0$. We can write any other connection in the form $\nabla = d + A$ where A is a matrix of one-form on M with coefficients in $End(E)$. If s is a section of E we get $F_\nabla(s) = (d+A)(d+A)(s) = (d+A)(ds + A \cdot s) = d^2 s + d(A \cdot s) + A \cdot ds + A(A(s)) = d(A \cdot s) + A \cdot ds + A(A(s)) = d(A)(s) + (-1)^1(A \cdot ds) + A \cdot ds + A(A(s)) = d(A)(s) + (A \wedge A)(s)$, so the curvature homomorphism can be written as

$$F_\nabla = d(A) + A \wedge A, \text{ where } \nabla = d + A.$$

Note that for a line bundle $A \wedge A = 0$, and the curvature $F_\nabla = d(A)$ is an ordinary two-form.
If $a \in A^1(M, End(E))$ the two-form $a \wedge a \in A^2(M, End(E))$ is given by wedge product in the form part and composition on $End(E)$. If ∇ is a connection then $F_{\nabla+a}(s) = (\nabla + a)(\nabla s + a \cdot s) = \nabla^2 s + \nabla(a)s + (-1)^1 a\nabla(s) + (a \wedge a)(s)$, so

$$F_{\nabla+a} = F_\nabla + \nabla(a) + a \wedge a.$$

Let E be a holomorphic vector bundle with an hermitian structure h. One can prove that the curvature of the Chern connection ∇ is of type $(1,1)$, real, and skewhermitian.
The hermitian structure on E is locally given by a matrix H, and we know that the Chern connection is of the form $\nabla = d + \overline{H}^{-1}\partial\overline{H}$. So the curvature is $F_\nabla = d(\overline{H}^{-1}\partial\overline{H}) + (\overline{H}^{-1}\partial\overline{H}) \wedge (\overline{H}^{-1}\partial\overline{H}) = \overline{\partial}\overline{H}^{-1}\partial\overline{H} + \partial\overline{H}^{-1}\partial\overline{H} + (\overline{H}^{-1}\partial\overline{H}) \wedge (\overline{H}^{-1}\partial\overline{H})$, but the last two therms are of type $(2,0)$ and then vanish. We get

$$F_\nabla = \overline{\partial}(\overline{H}^{-1}\partial(\overline{H})).$$

In particular if E is a line bundle the hermitian matrix is just a positive real function h, in this case we have

$$F_\nabla = \overline{\partial}\partial \, log(h).$$

REMARK 2.11. Let (X, g) be an hermitian manifold. Then the hermitian structure on X induces an hermitian structure on the tangent bundle T_X. The curvature of the Chern connection on T_X is called the *curvature of the hermitian manifold* (X, g).

If X is a Kähler manifold then the Chern connection and the Levi-Civita connection on the tangent bundle coincide, so the curvature of the Kähler manifold X is the curvature of the underlying Riemannian manifold.

EXAMPLE 2.12. Consider again the metric $h = \frac{w^2}{1+|w|^2}$ over the line bundle $\mathcal{O}(1) \to \mathbb{P}^1$. Then $\partial \, log(h) = \frac{1}{w(1+w\overline{w})} dw$, and $F_\nabla = \overline{\partial}\partial \, log(h) = -\frac{1}{(1+w\overline{w})^2} dw \wedge d\overline{w}$. Now recall that the Fubini Study metric over a standard open set on $\mathbb{P}^1$ is given by $h = 1 + |w|^2$ and the corresponding Kähler form is $\omega_{FS} = \frac{i}{2\pi}\partial\overline{\partial} log(1 + |w|^2) = \partial(\frac{w}{1+w\overline{w}} d\overline{w}) = -\frac{1}{(1+w\overline{w})^2} dw \wedge d\overline{w}$. We conclude that

$$\omega_{FS} = \frac{i}{2\pi} F_\nabla.$$

REMARK 2.13. Clearly the previous assertion holds on $\mathbb{P}^n$. If L is a holomorphic line bundle on a complex manifold X generated by global sections $s_0, ..., s_n$, and if h is the hermitian structure induced by these sections, then h is the pullback of the hermitian structure on $\mathcal{O}(1)$ under the morphism $\psi_L : X \to \mathbb{P}^n$, $x \mapsto [s_0(x) : ... : s_n(x)]$. If F_∇ is the curvature of the Chern connection on L one has $\frac{i}{2\pi} F_\nabla = \psi_L^* \omega_{FS}$, where ω_{FS} is the Fubini Study form on $\mathbb{P}^n$.

By the Bianchi identity $\nabla(F_\nabla) = 0$, for the Chern connection of a holomorphic hermitian bundle we have $(\nabla(F_\nabla))^{1,2} = \overline{\partial}(F_\nabla) = 0$. So F_∇ as an element of $A^{1,1}(X, End(E))$ is $\overline{\partial}$-closed. So the curvature yields a Dolbeault cohomology class $[F_\nabla] \in H^1(X, \Omega_X \otimes End(E))$. The following proposition shows that this cohomology class does not depend on the chosen hermitian structure.

PROPOSITION 2.14. *For the curvature F_∇ of the Chern connection on an hermitian holomorphic vector bundle (E, h) one has*

$$[F_\nabla] = A(E) \in H^1(X, \Omega_X \otimes End(E)).$$

PROOF. Consider the following diagram

$$\begin{tikzcd}
& \Omega_X \otimes End(E) \arrow[r] \arrow[d] & C^0(\{U_i\}, \Omega_X \otimes End(E)) \arrow[r] \arrow[d] & C^1(\{U_i\}, \Omega_X \otimes End(E)) \arrow[d, "G"] \\
& A^{1,0}(End(E)) \arrow[r] \arrow[d] & C^0(\{U_i\}, A^{1,0}(End(E))) \arrow[r, "\delta_1"] \arrow[d, "\overline{\partial}"] & C^1(\{U_i\}, A^{1,0}(End(E))) \\
& C^0(\{U_i\}, A^{1,1}(End(E))) \arrow[r, "\delta_0"] & C^0(\{U_i\}, A^{1,1}(End(E))) &
\end{tikzcd}$$

Let $\{U_i\}$ be a trivializing covering for E with isomorphisms $\psi_i : E_{|U_i} \to U_i \times \mathbb{C}^r$. With respect to the trivialization on U_i the hermitian structure h is given by an hermitian matrix H_i. The curvature of the Chern connection on E is given by $F_{\nabla|U_i} = \psi_i^{-1}(\overline{\partial}(\overline{H}_i^{-1}\partial\overline{H}_i))\psi_i$. So

$$\delta_0(F_\nabla) = \{U_i, \psi_i^{-1}(\overline{\partial}(\overline{H}_i^{-1}\partial\overline{H}_i))\psi_i\} = \overline{\partial}\{U_i, \psi_i^{-1}(\overline{H}_i^{-1}\partial\overline{H}_i)\psi_i\},$$

because the ψ_i are holomorphic. Now one has to show that

$$G(\{U_{i,j}, \psi_j^{-1}(\psi_{i,j}^{-1} d\psi_{i,j})\psi_j\}) = \delta_1(\{U_i, \psi_i^{-1}(\overline{H}_i^{-1}\partial\overline{H}_i)\psi_i\}),$$

because the left hand side is the Atiyah class of E. This follows from the definition of δ_1, the equality $\psi^t_{i,j} H_i \overline{\psi}_{i,j} = H_j$, and $\partial \overline{\psi_{i,j}} = 0$. ♠

2.4. Another definition of Chern classes

Let $E \to X$ be a rank r vector bundle with a connection ∇. Consider the degree j homogeneous polynomials P_j defined as

$$det(Id + Y) = 1 + P_1(Y) + ... + P_r(Y).$$

One can prove that these polynomials are invariant, and we define the Chern forms of E with respect to ∇ as

$$c_j(E, \nabla) = P_j(\frac{i}{2\pi} F_\nabla) \in A^{2k}(X).$$

Their cohomology classes are independent from the connection, we define the Chern classes of E as

$$c_j(E) = [c_j(E, \nabla)].$$

Note that $c_0(E) = 1$ and $c_j(E) = 0$ for $j > rank(E)$.

REMARK 2.15. The following two remarks are fundamental in the theory of Chern classes.

- Let E be an hermitian bundle and let ∇ be an hermitian connection on E. With respect to the hermitian trivialization of E the curvature satisfies $F^*_\nabla = F^t_\nabla = -F_\nabla$. So $\overline{\frac{i}{2\pi} F_\nabla} = \frac{i}{2\pi} F^t_\nabla$. We get $c(E, \nabla) = det(Id + \frac{i}{2\pi} F_\nabla) = det(I + \frac{i}{2\pi} F^t_\nabla) = \overline{det(Id + \frac{i}{2\pi} F_\nabla)} = \overline{c(E, \nabla)}$, we see that $c(E, \nabla)$ is a real form and
$$c(E) \in H^*(M, \mathbb{R}).$$
- If E is a holomorphic vector bundle over a complex manifold X we can choose an hermitian connection ∇ that is compatible with the holomorphic structure on E. Since in this case the curvature F_∇ is a $(1,1)$-form then also the Chern forms $c_j(E, \nabla)$ are of type (j, j).
Furthermore if X is compact Kähler we have the Hodge decomposition at level of cohomology $H^{2j}(X, \mathbb{C}) = \bigoplus_{p+q=2j} H^{p,q}(X)$. Then for the Chern classes we get
$$c_j(E) \in H^{2j}(X, \mathbb{R}) \cap H^{j,j}(X).$$

Recall that for the first Chern class there exists another definition. Consider the exponential sequence

$$0 \mapsto \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}^*_X \mapsto 0,$$

and the long exact sequence in cohomology

$$... \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}^*_X) \xrightarrow{c_1} H^2(X, \mathbb{Z}) \to ...$$

since $H^1(X, \mathcal{O}^*_X) \cong Pic(X)$ we get a map

$$c_1 : Pic(X) \to H^2(X, \mathbb{Z}),$$

and the image $c_1(L) \in H^2(X, \mathbb{Z})$ is the first Chern class of the line bundle L.

2.5. Comparison with the definitions of First Chern Class

For a holomorphic line bundle L on a complex manifold X we have three definitions of Chern class:

(1) Using the curvature $c_1(L) = [c_1(L, \nabla)] = [\frac{i}{2\pi} F_\nabla]$ where ∇ is a connection on L.
(2) Via the Atiyah class $A(L) \in H^1(X, \Omega_X) = H^1(X, \Omega_X \otimes End(E))$.
(3) Via the exponential sequence and the boundary map $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbb{Z})$.

By proposition 2.14 the first two definitions are compatible. Indeed we can choose on L an hermitian structure and the Chern connection ∇, so $A(L) = [F_\nabla]$. If X is compact Kähler we can embed $H^1(X, \Omega_X) = H^{1,1}(X) \subseteq H^2(X, \mathbb{C})$, so we get

$$c_1(L) = [\frac{i}{2\pi} F_\nabla] = \frac{i}{2\pi} A(L).$$

We consider the comparison between (1) and (3) in the more general context of a complex line bundle L on a differentiable manifold M. In this case L is describer by a cocycle $\{U_{i,j}, \psi_{i,j}\} \in H^1(M, \mathcal{C}_\mathbb{C}^*)$. The smooth exponential sequence

$$0 \mapsto \mathbb{Z} \to \mathcal{C}_\mathbb{C} \to \mathcal{C}_\mathbb{C}^* \mapsto 0$$

induce a boundary isomorphism $\delta : H^1(M, \mathcal{C}_\mathbb{C}^*) \to H^2(M, \mathbb{Z})$ because $\mathcal{C}_\mathbb{C}$ is a soft sheaf. Now $H^2(M, \mathbb{Z})$ maps to $H^2(M, \mathbb{R}) \subseteq H^2(M, \mathbb{C})$, then we can compare $c_1(L)$ and $\delta(L)$.

PROPOSITION 2.16. *Let L be a complex line bundle on a differentiable manifold M. The image of $\delta(L)$ under the map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{C})$ is equal to $-c_1(L)$.*

PROOF. We consider the resolutions of the constant sheaf $\mathbb{C}$ over M given by de Rham and Čeach complexes.

$$\begin{tikzcd}
\mathbb{C} \arrow[r] \arrow[d] & C^0(\{U_i\}, \mathbb{C}) \arrow[r] & C^1(\{U_i\}, \mathbb{C}) \arrow[r] & C^2(\{U_i\}, \mathbb{C}) \arrow[d, "i"] \\
A^0 \arrow[d] & & C^1(\{U_i\}, A^0) \arrow[r, "\delta_2"] \arrow[d, "d"] & C^2(\{U_i\}, A^0) \\
A^1 \arrow[d] & C^0(\{U_i\}, A^1) \arrow[r, "\delta_1"] \arrow[d, "d"] & C^1(\{U_i\}, A^1) & \\
A^2 \arrow[r, "\delta_0"] & C^0(\{U_i\}, A^2) & &
\end{tikzcd}$$

Let $M = \bigcup U_i$ be an open covering trivializing L, let $\{\psi_i\}$ be the trivializing isomorphisms and $\psi_{i,j} = \psi_i \psi_j^{-1}$ are sections of $\mathcal{C}_\mathbb{C}^*(U_{i,j})$. We can choose a branch of the logarithm for any $U_{i,j}$ and $\varphi_{i,j} \in \mathcal{C}_\mathbb{C}(U_{i,j})$ with $exp(2\pi i \varphi_{i,j}) = \psi_{i,j}$. The boundary map $\delta(L) = \delta\{\psi_{i,j}\}$ is given by $\{U_{i,j,k}, \varphi_{j,k} - \varphi_{i,k} + \varphi_{i,j}\}$ which takes values in the locally constant sheaf $\mathbb{Z}$. Let ∇ be a connection on L, locally with respect to ψ_i we have $\nabla = d + A_i$. The compatibility condition yields $\psi_{i,j}^{-1} d(\psi_{i,j}) +$

$\psi_{i,j}^{-1} A_i \psi_{i,j} = A_j$, so $A_j - A_i = \psi_{i,j}^{-1} d(\psi_{i,j}) = (2\pi i) d(\varphi_{i,j})$ because $rank(L) = 1$ implies $\psi_{i,j}^{-1} A_i \psi_{i,j} = A_i$. Since L is a line bundle for the curvature we have $F_\nabla = d(A_i) + A_i \wedge A_i = d(A_i)$. We compute

- $\delta_0(\frac{i}{2\pi} F_\nabla) = \{U_i, \frac{i}{2\pi} d(A_i)\} = d\{U_i, \frac{i}{2\pi} A_i\}$,
- $\delta_1\{U_i, \frac{i}{2\pi} A_i\} = \{U_{i,j}, \frac{i}{2\pi}(A_j - A_i)\} = -d\{U_{i,j}, \varphi_{i,j}\}$,
- $-\delta_2\{U_{i,j}, \varphi_{i,j}\} = -\{U_{i,j,k}, \varphi_{jk} - \varphi_{ik} + \varphi_{ij}\}$.

This conclude the proof. ♠

CHAPTER 3

The Grothendieck Riemann Roch Theorem

3.1. Segre and Chern Classes from the pure algebraic viewpoint

Let L be a line bundle on a scheme X. For any subvariety Y of X the restriction $L_{|Y}$ of L on Y is isomorphic to $\mathcal{O}_Y(D)$ for some Cartier divisor D on Y. The Weyl divisor $[D]$ determines an element in $A_{k-1}X$, which we denoted by $c_1(L) \cap [Y] = [D]$. Extending by linearity we obtain a morphism

$$c_1(L) : A_kX \to A_{k-1}X,\ \alpha \mapsto c_1(L) \cap \alpha.$$

Let $E \to X$ be a rank $r+1$ vector bundle on the scheme X, let $\mathbb{P}(E)$ be the projective bundle of lines in E, and let $\pi : \mathbb{P}(E) \to X$ be the projection. On $\mathbb{P}(E)$ there is a canonical line bundle $\mathcal{O}(1)$.
Let $\alpha \in A_kX$ be a k-cycle on X, then $\pi^*\alpha$ is a $k+r$-cycle on $\mathbb{P}(E)$ and $\pi_*(c_1(\mathcal{O}(1))^{r+i} \cap \pi^*\alpha)$ is a $(k-i)$-cycle on X. We define a morphism

$$A_kX \to A_{k-i}X,\ \alpha \mapsto s_i(E) \cap \alpha,$$

where $s_i(E) \cap \alpha = \pi_*(c_1(\mathcal{O}(1))^{r+i} \cap \pi^*\alpha)$. The morphisms $s_i(E)$ are called the *Segre Classes* of E. Note that if E is a line bundle on X then $\mathbb{P}(E) = X$ and $\mathcal{O}_E(-1) = E$, so $\mathcal{O}_E(1) = \check{E}$ and

$$s_1(E) \cap \alpha = c_1(\mathcal{O}_E(1)) \cap \alpha = -c_1(E) \cap \alpha.$$

Now consider the formal power series

$$s_t(E) = \sum_{i=0}^{\infty} s_i(E)t^i = 1 + s_1t + s_2t^2 + ... + s_nt^n + ...$$

and let

$$c_t(E) = \sum_{i=0}^{\infty} c_i(E)t^i = 1 + c_1t + c_2t^2 + ... + c_nt^n + ...$$

be its inverse power series. One can prove that $c_t(E)$ is a polynomial called the *Chern Polynomial* of E. Explicitly we have

- $c_0(E) = 1$,
- $c_1(E) = -s_1(E)$,
- $c_2(E) = s_1(E)^2 - s_2(E)$,
- $c_n(E) = -s_1(E)c_{n-1}(E) - s_2(E)c_{n-2}(E) - ... - s_n(E)$.

We interpret the s_i and the c_i as endomorphisms of the Crow ring A_*X, and we write $c_i(E) \cap \alpha$ for the element of $A_{k-i}X$ obtained by applying the endomorphism $c_i(E)$ to $\alpha \in A_kX$.
We define the *total Chern class* as the sum

$$c(E) = 1 + c_1(E) + ... + c_r(E),\ r = rank(E),$$

then $c(E) \cap \alpha = \sum_{i=0}^{r} c_i(E) \cap \alpha$. In the same way we define the *total Segre class* as

$$s(E) = 1 + s_1(E) + ... + s_r(E) + ...,$$

and $s(E) \cap \alpha = \sum_{i=0}^{\infty} s_i(E) \cap \alpha$.
The Chern classes of a rank r vector bundle satisfy the following properties.

(1) (Vanishing) $c_i(E) = 0$ for all $i > r$,
(2) (Whitney sum) If $0 \mapsto E^{'} \to E \to E^{''} \mapsto 0$ is a short exact sequence of vector bundles on X, then

$$c_t(E) = c_t(E^{'}) \cdot c_t(E^{''}).$$

Suppose that E has a filtration

$$E = E_r \supseteq E_{r-1} \supseteq ... \supseteq E_0 = \{0\},$$

such that the quotient $L_i = E_i/E_{i-1}$ are line bundle. Then the Chern polynomial of E i factored as

$$c_t(E) = \prod_{i=0}^{r}(1 + \alpha_i t),$$

where $\alpha_i = c_1(Li)$ are called the Chern roots of E. Since if the formula holds in this special case and the relation among the bundles are preserved by flat pull-back, the formula holds in general. From this fact we can deduce the following properties

- $c_i(\check{E}) = (-1)^i c_1(E)$,
- $c_1(\bigwedge^r E) = c_1(E)$.

The *Chern Character* and the *Todd Class* of E are defined as

$$ch(E) = \sum_{i=1}^{r} exp(\alpha_i),\ td(E) = \prod_{i=1}^{r} \frac{\alpha_i}{1 - exp(\alpha_i)}.$$

If $0 \mapsto E^{'} \to E \to E^{''} \mapsto 0$ is an exact sequence of vector bundles we have

$$ch(E) = ch(E^{'}) + ch(E^{''}),\ td(E) = td(E^{'}) \cdot td(E^{''}),$$

for tensor product we have

$$ch(E \otimes E^{'}) = ch(E) \cdot ch(E^{'}).$$

The Chern character and the Todd class of a smooth scheme X are defined as

$$ch(X) = ch(T_X),\ td(X) = td(T_X).$$

EXAMPLE 3.1. Consider the case $rank(E) = 2$. The Chern polynomial is

$$c_t(E) = (1 + \alpha_1 t)(1 + \alpha_2 t) = 1 + (\alpha_1 + \alpha_2)t + \alpha_1\alpha_2 t^2 = 1 + c_1 t + c_2 t^2.$$

Then the relations among the Chern classes and the Chern roots are $c_1 = \alpha_1 + \alpha_2$, $c_2 = \alpha_1\alpha_2$. The Chern character is of the form $ch(E) = exp(\alpha_1) + exp(\alpha_2) = 1 + \alpha_1 + \frac{\alpha_1^2}{2} + \frac{\alpha_1^3}{6} + ... + 1 + \alpha_2 + \frac{\alpha_2^2}{2} + \frac{\alpha_2^3}{6} + ... = 2 + \frac{1}{2}(c_1^2 - 2c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + 3c_3) + ...$ and for the Todd class $td(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + ...$

EXAMPLE 3.2. Consider the Euler exact sequence

$$0 \mapsto \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \to T_{\mathbb{P}^n} \mapsto 0.$$

For the Chern polynomial of the tangent bundle we have $c_t(T_{\mathbb{P}^n}) = c_t(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)}) = (c_t(\mathcal{O}_{\mathbb{P}^n}(1)))^{\oplus(n+1)} = (c_0 + c_1 t)^{\oplus(n+1)} = (1+Ht)^{\oplus(n+1)}$, where H is the hyperplane section of $\mathbb{P}^n$.

EXAMPLE 3.3. Let $i : X \to Y$ be a closed embedding of codimension d of smooth varieties. We have the exact sequence

$$0 \mapsto T_X \to T_{Y|X} \to N_{X/Y} \mapsto 0,$$

and $c_t(T_X) = c_t(T_{Y|X})/c_t(N_{X/Y})$. Consider the case $X = D_1 \cap ... \cap D_d$ i.e. X is intersection of d divisors on Y. Then the normal bundle has a decomposition $N_{X/Y} = (\mathcal{O}_X(D_1) \oplus ... \oplus \mathcal{O}_X(D_d))_X$. So

$$c_t(N_{X/Y}) = \prod_{i=1}^{d} c_t(\mathcal{O}_X(D_i)) = \prod_{i=1}^{d}(1 + c_1(\mathcal{O}_X(D_i)_{|X})t).$$

As instance take $Y = \mathbb{P}^N$ and D_i divisors of degree m_i, then $c_t(N_{X/Y}) = \prod_{i=1}^{d}(1 + m_iHt))$, and

$$c_t(T_X) = c_t(T_{Y|X})/c_t(N_{X/Y}) = (1+Ht)^{N+1}/\prod_{i=1}^{d}(1+m_iHt).$$

Let $X \subseteq \mathbb{P}^3$ be a smooth quartic curve complete intersection of two smooth quadric surfaces Q_1 and Q_2. Then $c_t(T_X) = (1+Ht)^4/(1+2Ht)^2$ and $4H + c_1(T_X) = 16H$. Taking degrees we get $deg(c_1(T_X)) = 0$, on the other hand $c_1(T_X) = -c_1(K_X)$ and since X is a smooth quartic curve on a quadric surface Q_1 in $Pic(Q_1)$ it is a divisor of type $(a,b) = (2,2)$, then its genus is $g = (a-1)(b-1) = 1$, and $deg(K_X) = 2g - 2 = 0$.

EXAMPLE 3.4. If C is an effective Cartier divisor an a surface X, then $C^2 = c_1(N)_{|C}$, where N is the normal bundle of C in X. If C and X are smooth, then $N = T_{X|C}/T_C$, so $C^2 = (c_1(T_{X|C}) - c_1(T_C))_{|C}$ and $C \cdot (C+K) = 2g-2$ (*adjunction formula*), where $K = -c_1(T_X)$ is the canonical class of X and g is the genus of C. If X is a smooth hypersurface of degree d in $\mathbb{P}^3$ we get $C^2 = 2g - 2 - deg(K)deg(C) = 2g - 2 + (4-d)deg(C)$, because $K_X = \mathcal{O}_X(d-4)$. In particular for a line L one has $L^2 = 2 - d$, and a surfaces admits lines with negative self-intersection if and only if $deg(X) \geq 3$.

EXAMPLE 3.5. Let $\nu : \mathbb{P}^n \to \mathbb{P}^N$, with $N = \binom{n+d}{d} - 1$ be the Veronese embedding induced by $\mathcal{O}_{\mathbb{P}^n}(d)$, and let H be the hyperplane class of $\mathbb{P}^n$. Then

$$c_t(N) = (1+dHt)^{N+1}/(1+h)^{n+1}.$$

In particular for the Veronese surface $V \subseteq \mathbb{P}^5$ we get $(1 + c_1t + c_2t^2 + c_3t^3)(1 + ht)^3 = 1 + 12ht + 60h^2t^2$, then $c_3 = 0$, $3h + c_1 = 12h$, $3h^2 + 3c_1h + c_2 = 60h^2$, and $c_3 = 0$, $c_1 = 9h$, $c_2 = 30h^2$. The Chern polynomial of the normal bundle $N_{V/\mathbb{P}^5}$ is

$$c_t(N) = 1 + 9ht + 30h^2t^2.$$

3.1.1. Riemann Hurwitz Theorem. Let $\phi : X \to Y$ be a morphism of smooth varieties of the same dimension $dim(X) = dim(Y) = n$. Let $R(\phi)$ be the subset of points of X where the induced map on the tangent spaces is not an isomorphism. Clearly the scheme structure on $R(\phi)$ is given locally by the vanishing of the Jacobian determinant, $R(\phi)$ is the zero scheme of the map $\bigwedge^n df : \bigwedge^n T_X \to \bigwedge^n \phi^* T_T$, or equivalently the zero scheme of a section of the line bundle $\bigwedge^n \phi^* T_Y \otimes \bigwedge^n \check{T_X}$. Then

$$[R(\phi)] = (c_1(\phi^* T_Y) - c_1(T_X)) \cap [X].$$

As instance take $n = 1$ and degrees of both sides. We get $deg(R(\phi)) = deg(\phi)(2 - 2g_Y) - (2 - 2g_X)$, and the Riemann-Hurwitz formula

$$2g_X - 2 = deg(\phi)(2g_Y - 2) + deg(R(\phi)).$$

3.2. Algebraic and Analytic definitions of first Chern Class

Let $D \subseteq X$ be a smooth cycle of codimension one in a compact complex manifold X of dimension n. We can interpret D as real codimension two subvariety of X, by Poincaré duality its fundamental class $[D]$ defines a two form in $H^2(X, \mathbb{R})$. We can associate to D the line bundle $\mathcal{O}_X(D)$. From the algebraic viewpoint the first Chern class of $\mathcal{O}_X(D)$ is the class $[D]$, but we can also consider the Chern class $c_1(\mathcal{O}_X(D))$ from the analytic viewpoint. A priori it is not clear that $c_1(\mathcal{O}_X(D)) = [D]$.

THEOREM 3.6. *Let X be a compact complex manifold and let $D \subseteq X$ be an irreducible cycle. The fundamental class $[D]$ of D is in fact contained in the image of $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$. Consider the line bundle $\mathcal{O}_X(D)$ associated to D. Then the algebraic and the analytic definitions of Chern class are the same, i.e.*

$$c_1(\mathcal{O}_X(D)) = [D].$$

PROOF. We consider the Chern connection ∇ on $L = \mathcal{O}_X(D)$ with respect to a hermitian structure h. To prove that $[\frac{i}{2\pi} F_\nabla] = c_1(L)$ is equal to $[D]$ one has to show that for any closed real form $\alpha \in H^{2n-2}(M, \mathbb{R})$

$$\int_D \alpha = \frac{i}{2\pi} \int_X F_\nabla \wedge \alpha,$$

where n is the complex dimension of X. Fix a trivializing covering $\psi_i : L_{|U_i} \to U_i \times \mathbb{C}$. On U_i the hermitian structure h is given by the function $h_i : U_i \to \mathbb{R}_+$, $h(s(x)) = h(s(x), s(x)) = h_i(x)|\psi_i(s(x))|^2$ for any local section s.
Let s be a holomorphic section on U_i vanishing on D, then $\overline{\partial}\partial\, log(h \circ s) = \overline{\partial}\partial\, log(h_i)$ on $U_i \setminus D$, where we used the fact that ψ_i holomorphic implies $\overline{\partial}\partial\, log(\psi_i \circ s) = \overline{\partial}\partial\, log(\overline{\psi_i} \circ s) = 0$.
Let $s \in H^0(X, L)$ be the global holomorphic section defining D i.e. $D = \{x \in X \,|\, s(x) = 0\}$, and denote with $D_\epsilon = \{x \in X \,|\, |h(s(x)) < \epsilon|\}$ the tubular neighborhood of D of radius ϵ. Then
$\frac{i}{2\pi} \int_X F_\nabla \wedge \alpha = \lim_{\epsilon \to 0} \frac{i}{2\pi} \int_{X \setminus D_\epsilon} F_\nabla \wedge \alpha = \lim_{\epsilon \to 0} \frac{i}{2\pi} \int_{X \setminus D_\epsilon} \overline{\partial}\partial\, log(h \circ s) \wedge \alpha = \lim_{\epsilon \to 0} \frac{i}{4\pi} \int_{X \setminus D_\epsilon} d(\partial - \overline{\partial})\, log(h \circ s) \wedge \alpha = \lim_{\epsilon \to 0} \frac{i}{4\pi} \int_{\delta D_\epsilon} (\partial - \overline{\partial})\, log(h \circ s) \wedge \alpha$, in the latter equality we use Stokes theorem and $d\alpha = 0$.
On U_i we can write $(\partial - \overline{\partial})log(h \circ s) = (\partial - \overline{\partial})log(\psi_i \circ s) + (\partial - \overline{\partial})log(\overline{\psi_i} \circ s) + (\partial - \overline{\partial})log(h_i) = \partial\, log(\psi_i \circ s) - \overline{\partial\, log(\psi_i \circ s)} + (\partial - \overline{\partial})log(h_i) = (2i)Im(\partial\, log(\psi_i \circ$

$s)) + (\partial - \overline{\partial})log(h_i)$. Note that h_i is bounded from below by some $\delta > 0$ so the second summand does not contribute to the integral for $\epsilon \mapsto 0$. Thus it suffices to show

$$lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\delta D_\epsilon \cap U_i} m(\partial\, log(\psi_i \circ s)) \wedge \alpha = -\int_{D\cap U_i}\alpha.$$

Since this is a local statement we may assume that D_ϵ is given by $z_1 = 0$ in a polydisc B. Furthermore $|h(z_1)| = h_i|z_i|$, where h is given by h_i in U_i. So $\delta D_\epsilon = \{z \mid |z_1| = \frac{\epsilon}{h_i}\}$. Moreover $\partial\, log(\psi_i \circ s) = \partial\, log(z_1) = \frac{dz_1}{z_1}$ and $\alpha = f(dz_2 \wedge ... \wedge dz_n) \wedge (d\overline{z_2} \wedge ... \wedge d\overline{z_n}) + dz_1 \wedge \beta + d\overline{z_1} \wedge \beta$. Note that $\partial\, log(\psi_i \circ s) \wedge (dz_1 \wedge \beta) = 0$ and $\partial\, log(\psi_i \circ s) \wedge (d\overline{z_1} \wedge \overline{\beta}) = (dz_1 \wedge d\overline{z_1}) \wedge (\frac{1}{z_1}\overline{\beta})$ does not contribute to the integral over δD_ϵ. We compute
$\int_{z_1=0}\alpha = \int f(0, z_2, ..., z_n)(dz_2 \wedge ... \wedge dz_n) \wedge (d\overline{z_2} \wedge ... d\overline{z_n})$ and
$\int_{\delta D_\epsilon} \partial\, log(\psi_i \circ s) \wedge \alpha = -\int_{|h(z_1)|=\epsilon} \frac{f}{z_1} dz_1 \wedge (dz_2 \wedge ... \wedge dz_n) \wedge (d\overline{z_2} \wedge ... d\overline{z_n})$.
The minus sign appears as we initially integrated over the exterior domain. By Cauchy integral formula
$lim_{\epsilon\to 0}\int_{\delta D_\epsilon} \partial\, log(\psi_i \circ s) \wedge \alpha = -lim_{\epsilon\to 0}\int_{|z_1=\epsilon/h_i|}(\int_{z_i>1} f \cdot (dz_2 \wedge ... \wedge dz_n \wedge d\overline{z_2} \wedge ... \wedge d\overline{z_n}))\frac{dz_1}{z_1} = (-2\pi i)\int_{z_1=0} f(0, z_2, ..., z_n)(dz_2 \wedge ... \wedge dz_n \wedge d\overline{z_2} \wedge ... \wedge d\overline{z_n}) = -2\pi i \int_{z_1=0}\alpha$,
so we get

$$lim_{\epsilon\to 0}\frac{1}{2\pi}\int_{\delta D_\epsilon} Im(\partial\, log(\psi_i \circ s) \wedge \alpha) = -Im(\int_{z_1=0} i \cdot \alpha) = -\int_{z_1=0}\alpha.$$

♠

REMARK 3.7. In the proof we assume D smooth but the same argument can be adjusted to the general case. Furthermore since the first Chern class and taking the fundamental class are both linear the assertion is true for arbitrary divisor i.e. $c_1(\mathcal{O}_X(\sum n_i D_i)) = \sum n_i [D_i]$.

3.3. Grothendieck Riemann Roch

3.3.1. Grothendieck Groups. Let $\mathcal{A}$ be an essentially small abelian category, and let F be the free abelian group generated by the set of isomorphism classes of objects in $\mathcal{A}$. For each exact sequence $0 \mapsto A' \to A \to A'' \mapsto 0$, consider the element $[A'] + [A''] - [A] \in F$, and let R be the subgroup of F generated by the elements of this form. The quotient group $K(\mathcal{A}) = F/R$ is called the Grothendieck group of $\mathcal{A}$.
In particular if X is a noetherian scheme, then $K(X) = K(\mathfrak{Coh}(X))$ is the Grothendieck group of X. Similarly we define $K_1(X)$ as the quotient F_1/R_1, where F_1 is the free abelian group generated by isomorphism classes of locally free shaves on X, and R_1 is generated by the expressions $[E'] + [E''] - [E]$ for any short exact sequence $0 \mapsto E' \to E \to E'' \mapsto 0$ of locally free shaves. Clearly there is a morphism $\epsilon : K_1(X) \to K_0(X)$, the unexpected fact is that ϵ is an isomorphism from a smooth scheme X. Then if X is smooth we can compute $K(X)$ using only locally free shaves.
Because of the additivity property of the Chern classes, the Chern polynomial defines a map

$$c_t : K(X) \to A(X)[t],$$

and the Chern character extends to a map

$$ch : K(X) \to A(X) \otimes \mathbb{Q}.$$

Furthermore $K(X)$ has a ring structure given by $(\mathcal{E}, \mathcal{F}) \to \mathcal{E} \otimes \mathcal{F}$, and ch is a ring homomorphism.

3.3.2. Grothendieck Riemann Roch. Let $f : X \to Y$ be a morphism of smooth schemes, then there is a ring homomorphism

$$f^! : K(Y) \to K(X),\ \mathcal{E} \mapsto f^*\mathcal{E},$$

for $\mathcal{E}$ locally free shaves. Then if $f : X \to Y$ is a proper morphism, we define an additive map

$$f_! : K(X) \to K(Y),\ \mathcal{F} \mapsto \sum(-1)^i R^i f_*(\mathcal{F}).$$

The map $f^!$ commutes with the Chern character, however the map $f_!$ does not commute with ch. A measure of which it fails to commute is the generalized Riemann-Roch theorem of Grothendieck.

THEOREM 3.8. *(Grothendieck-Riemann-Roch) Let $\pi : X \to Y$ be a proper morphism of schemes with smooth base Y, then*

$$ch(\pi_!(E)) \cdot td(Y) = \pi_*(ch(E) \cdot td(X)),$$

for any $E \in K(X)$, where $td(Y) = td(T_Y)$ and $td(X) = td(T_X)$.

REMARK 3.9. Using the fact that the Todd class is multiplicative on exact sequences, we have $td(T_X)/td(T_B) = td(T_\pi)$, where T_π is the relative tangent bundle of π. Then we can rewrite the formula as

$$ch(\pi_!(E)) = \pi_*(ch(E) \cdot td(T_\pi)).$$

Recall that if $f : X \to Y$ is a continuous map between topological spaces, we define the *higher direct image* functors $R^i f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ to be the right derived functors of the direct image functor f_*, this make sense, in fact f_* is left exact and $\mathfrak{Ab}(X)$ has enough injective. Furthermore for each $i \geq 0$ and each $\mathcal{E} \in \mathfrak{Ab}(X)$, we have that $R^i f_* \mathcal{E}$ is the sheaf associated to the presheaf

$$V \mapsto H^i(f^{-1}V, \mathcal{E}_{|f^{-1}(V)}),$$

on Y. In particular if $\pi : X \to Spec(k)$ is a morphism of X on a point, then $R^i\pi_*\mathcal{E}$ is the sheaf $Speck \mapsto H^i(\pi^{-1}(Spec(k)), \mathcal{E}_{|\pi^{-1}(Spec(k))}) = H^i(X, \mathcal{E})$, and

$$\pi_!(\mathcal{E}) = \sum(-1)^i R^i \pi_*(\mathcal{E}) = \sum H^i(X, \mathcal{E}) = \chi(\mathcal{E}),$$

is the Euler characteristic of $\mathcal{E}$. In this case we have also $td(\mathcal{E} = 1$, and from the Grothendieck-Riemann-Roch formula we get the *Hirzebruch-Riemann-Roch* formula.

THEOREM 3.10. *(Hirzebruch-Riemann-Roch) Let $\mathcal{E}$ be a locally free sheaf on a smooth scheme X of dimension n, then*

$$\chi(\mathcal{E}) = deg(ch(\mathcal{E}) \cdot td(T_X))_n,$$

where the expression on the right denotes the component of degree n in $A(X) \otimes \mathbb{Q}$.

REMARK 3.11. (*Riemann-Roch for Curves*) Let X be curve and $\mathcal{L} = \mathcal{O}_X(D)$ be an invertible sheaf on X. Then $c_1(\mathcal{L}) = D$ and $ch(\mathcal{L}) = 1 + D$. Furthermore $c_1(T_X) = -c_1(K_X)$ and $td(T_X) = 1 + \frac{1}{2}c_1(T_X) = 1 - \frac{1}{2}K_X$. By 3.10 we have

$$\chi(\mathcal{O}_X(D)) = deg((1+D)(1-\frac{1}{2}K))_1 = deg(D - \frac{1}{2}K_X).$$

For $\mathcal{O}_X(D) = \mathcal{O}_X$ we get $1 - g = h^0(\mathcal{O}_X) - h^0(\mathcal{O}_X(K_X) = -\frac{1}{2}deg(K_X)$, and $deg(K_X) = 2g - 2$. We recover the Riemann-Roch theorem for smooth curves

$$\chi(\mathcal{O}_X(D)) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = deg(D) - g + 1.$$

REMARK 3.12. (*Riemann-Roch for Surfaces*) Let X be smooth surface and let $\mathcal{O}_X(D)$ be a line bundle on X. Then $c_1(\mathcal{O}_X(D)) = D$ and $ch(\mathcal{O}_X(D)) = 1 + D + \frac{1}{2}D^2$. The Chern class c_1 and c_2 of the tangent sheaf depends only on X and are called the Chern classes of X. Note that

$$c_1(T_X) = c_1(\bigwedge^2 T_X) = -c_1(\bigwedge^2 \Omega_X) = -c_1(K).$$

Then the Todd class is of the form $td(T_X) = 1 - \frac{1}{2}K + \frac{1}{12}(K^2 + c_2)$. By 3.10 we get

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D \cdot (D + K_X) + \frac{1}{12}(K^2 + c_2),$$

and for $\mathcal{O}_X(D) = \mathcal{O}_X$ we have $\frac{1}{12}(K^2 + c_2) = \chi(\mathcal{O}_X) = 1 + p_a$, where p_a is the arithmetic genus (on a scheme X of dimension n the arithmetic genus is defined as $p_a = (-1)^n(\chi(\mathcal{O}_X) - 1)$). Then $c_2 = 12(1 + p_a) - K^2$, and we recover the Riemann-Roch formula for surfaces

$$\chi(\mathcal{O}_X(D)) = \frac{1}{2}D \cdot (D - K) + p_a + 1.$$

EXAMPLE 3.13. Let $X \subseteq \mathbb{P}^4$ be a smooth surface of degree d, in $A(Proj^4)$ it is rationally equivalent to d times a plane, then $X \cdot X = d^2$. Consider the exact sequence

$$0 \mapsto T_X \to T_{\mathbb{P}^4|X} \to N \mapsto 0.$$

Note that $c_t(T_{\mathbb{P}^4|X}) = 1 + 5Ht + 10H^2t^2$, since $H^k = 0$ on X for any $h \geq 3$. We get $(1 + c_1t + c_2t^2)(1 + c_1(N)t + c_2(N)t^2) = 1 + 5Ht + 10H^2t^2$, and substituting $c_1 = c_1(T_{|X}) = -c_1(\bigwedge^2 \Omega_X) = -K$, we have

$$(1 - Kt + c_2t^2)(1 + c_1(N)t + c_2(N)t^2) = 1 + 5Ht + 10H^2t^2.$$

Comparing the coefficients we get $c_1(N) = 5H + K$ and $c_2(N) = 10H^2 - c_2 + 5H \cdot K + K^2$. We take the degrees and note that $deg(c_2(N)) = d^2$ and $deg(H^2) = d$. By 3.12 we know that $c_2 = 12(1 + p_a) - K^2$, then $c_2(N) = 10H^2 - 12 - 12p_a + 2K^2 + 5HK$, and taking degrees

$$d^2 - 10d + 12p_a + 12 - 2K^2 - 5HK = 0.$$

- Consider the rational cubic scroll $X \subseteq \mathbb{P}^4$. We can interpret X in many ways: the projection of the Veronese surface $V \subseteq \mathbb{P}^5$ from a point $p \in V$, the join in $\mathbb{P}^4$ of a conic and a line, the blow up of $\mathbb{P}^2$ at a point. We consider this latter interpretation. If $\pi : X \to \mathbb{P}^2$ is the blow up, the $K_X = \pi^*K_{\mathbb{P}^2} + E$, where E is the exceptional divisor. So $HK = H\pi^*K_{\mathbb{P}^2} + HE = -6 + 1 = -5$, because $K_{\mathbb{P}^2} = \mathcal{O}_{\mathbb{P}^2}(-3)$. Furthermore $K_X^2 = K_{\mathbb{P}^2}^2 - 1 = 8$, in particular we see that the self-intersection of the canonical divisor is not a birational invariant, in fact $K^2_{\mathbb{P}^2} = 9$. Now we verify the above formula $d^2 - 10d + 12p_a + 12 - 2K^2 - 5HK = 9 - 30 + 12 - 16 + 25 = 0$.

- Let $X \subseteq \mathbb{P}^4$ be a $K3$ surface. Then $K_X = 0$ and $q = 0$ implies that $p_a = p_g = 1$. Substituting in the above formula we get $d^2 - 10d + 24 = 0$. Then if X is a $K3$ surface in $\mathbb{P}^4$, then $deg(X) = 4$ or $deg(X) = 6$. As instance if X is a quartic surface in $\mathbb{P}^3$ the $K_X = \mathcal{O}_X(4-3-1) = \mathcal{O}_X$ and X is $K3$ of degree 4, if X is a complete intersection of a quadric and a cubic hypersurface in $\mathbb{P}^4$, the $deg(X) = 3 \cdot 2 = 6$, and $K_X = \mathcal{O}_X(2+3-4-1) = \mathcal{O}_X$, so X is $K3$ of degree 6.
- Let X be an abelian surface in $\mathbb{P}^4$. Then $p_a = -1$, $p_g = 1$ and $K_X = 0$. Its irregularity is $q = p_g - p_a = 2$. Then $d^2 - 10d - 12 + 12 = 0$ implies $d = 10$. An abelian surface in $\mathbb{P}^4$ must have degree 10. Horrocks and Mumford have shown that such abelian surfaces exist.

REMARK 3.14. (*Riemann-Roch for 3-folds*) Let X be a smooth projective 3-fold with Chern classes c_1, c_2, c_3, and let $\mathcal{O}_X(D)$ be an invertible sheaf on X. We have $ch(\mathcal{O}_X(D)) = 1 + c_1 + \frac{1}{2}c_1^2 + \frac{1}{6}c_1^3 = 1 + D + \frac{1}{2}D^2 + \frac{1}{6}D^3$. Then $td(T_X) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 + c_2) + \frac{1}{24}c_1c_2$, and since $c_1(T_X) = c_1(\bigwedge^3 T_X) = -c_1(K_X) = -K$ we get $td(T_X) = 1 - \frac{1}{2}K + \frac{1}{12}(K^2 + c_2) + \frac{1}{24}c_1c_2$. By 3.10 we get

$$\chi(\mathcal{O}_X(D)) = \frac{1}{6}D^3 - \frac{1}{4}KD^2 + \frac{1}{12}D(K^2 + c_2) + \frac{1}{24}c_1c_2,$$

in particular for $\mathcal{O}_X(D) = \mathcal{O}_X$, the formula yields $\chi(\mathcal{O}_X) = \frac{1}{24}c_1c_2$, so

$$1 - p_a = \frac{1}{24}c_1c_2.$$

Substituting the latest expression one has $\chi(\mathcal{O}_X(D)) = \frac{1}{6}D^3 - \frac{1}{4}KD^2 + \frac{1}{12}DK^2 + \frac{1}{12}Dc_2 + 1 - p_a$, so

$$\chi(\mathcal{O}_X(D)) = \frac{1}{12}D \cdot (D - K) \cdot (2D - K) + \frac{1}{12}D \cdot c_2 + 1 - p_a.$$

In particular if X is a Calabi-Yau 3-fold, since its canonical sheaf is trivial we get a simplified formula

$$\chi(\mathcal{O}_X(D)) = \frac{1}{6}D^3 + \frac{1}{12}Dc_2 + 1 - p_a.$$

3.4. The Atiyah-Singer Index Theorem

Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator between vector bundles E and F on a compact oriented differentiable manifold X of dimension n.

(1) The *topological index* $\gamma(D)$ of the operator D is $ch(D)Td(X)[X]$, where
  - $Td(X)$ is the Todd class of X,
  - $ch(D)$ is equal to $\varphi^{-1}(ch(d(p^*E, p^*F, \sigma(D)))$,
  - φ is the Thom isomorphism from $H^k(X, \mathbb{Q})$ to $H^{n+k}(B(X)/S(X), \mathbb{Q})$,
  - $B(X)$ is the unit ball bundle of the cotangent bundle of X, and $S(X)$ is its boundary, and p is the projection to X.
  - ch is the Chern character from K-theory $K(X)$ to the rational cohomology ring $H(X, \mathbb{Q})$,

- $d(p^*E, p^*F, \sigma(D))$ is the "difference element" of $K(B(X)/S(X))$ associated to two vector bundles p^*E and p^*F on $B(X)$ and an isomorphism $\sigma(D)$ between them on the subspace $S(X)$,
- $\sigma(D)$ is the symbol of D.

(2) The elliptic differential operator D has a pseudoinverse, it is a Fredholm operator. Its *analytic index* is defined as the difference between the finite dimension of $Ker(D)$ and the finite dimension $Coker(D)$ i.e.

$$Index(D) = dimKer(D)dimCoker(D) = dimKer(D)dimKer(D^*).$$

THEOREM 3.15. *(Atiyah-Singer) Let $D : \Gamma(E) \to \Gamma(F)$ be an elliptic differential operator between vector bundles E and F on a compact oriented differentiable manifold X of dimension n. Then the analytic index and the topological index of D are equal,*

$$Index(D) = \gamma(D).$$

REMARK 3.16. The Grothendieck-Riemann-Roch theorem and the Hirzebruch-Riemann-Roch theorem can be recovered by the Atiyah-Singer theorem by considering the appropriate elliptic differential operators. As instance for the Hirzebruch-Riemann-Roch theorem one has to consider the Laplacian $\Delta_{\bar{\partial}_E}$.

3.5. Grassmannians and The Universal Bundle

Let $f : M \to N$ be a map between manifolds. If E is a vector bundle on N, then the pullback $f^{-1}E$ is a vector bundle on M less twisted than E. We want to find a vector bundle so twisted that any bundle is a pullback of this universal bundle. Let V be a $\mathbb{C}$-vector space of dimension n, and let $G_k(V)$ be the Grassmannian of codimension k subspaces of V. Recall that on $G_k(V)$ we have the universal bundle S, the tautological bundle $\hat{V}$, and the quotient bundle defined by the sequence

$$0 \mapsto S \to \hat{V} \to Q \mapsto 0.$$

One can prove that

- As a ring $H^*(G_k(V)) = \frac{\mathbb{R}[c_1(S),...,c_{n-k}(S),c_1(Q),...,c_k(Q)]}{(c(S)c(Q)=1)}$,
- the Chern classes $c_1(Q), ..., c_k(Q)$ of the quotient bundle generate the cohomology ring $H^*(G_k(V))$,
- for a fixed k and a fixed j there are no polynomial relations of degree j among $c_1(Q), ..., c_k(Q)$ if $dim(V)$ is large enough.

LEMMA 3.17. *Let E be a rank r complex vector bundle over a differentiable manifold M, suppose that M admits a finite good cover. Then there exists on M finitely many smooth sections of E which span the fiber at every point.*

PROPOSITION 3.18. *Let E be a rank r complex vector bundle over a differentiable manifold M of finite type. Suppose there are r global sections of E which spans the fiber at every point. Then there is a map $f : M \to G_r(\mathbb{C}^n)$ to some Grassmannian $G_r(\mathbb{C}^n)$ such that $E = f^{-1}Q$, where Q is the universal quotient bundle of $G_r(\mathbb{C}^n)$.*

PROOF. We consider $s_1, ..., s_r$ spanning sections of E, and let V be the $\mathbb{C}$ vector space generated by $s_1, ..., s_r$. Clearly the evaluation map

$$ev_p : V \to E_p$$

is surjective for any $p \in M$. So $Ker(ev_p)$ is a codimension r subspace of V i.e. a point of the Grassmannian $G_r(V)$. The fiber of the quotient bundle $Q \to G_r(V)$ over the point $Ker(ev_p)$ is the space $V/Ker(ev_p) = E_p$. We define a map

$$f_E : M \to G_r(V),\ p \mapsto Ker_{ev_p},$$

then the quotient bundle Q pulls back to E via f_E. ♠

REMARK 3.19. The map f_E is called a *classifying map* for the bundle E. One can show that the homotopy class of the map f_E is uniquely determined by the vector bundle E.

We denote by $Vect_r(M, \mathbb{C})$ the set of isomorphism classes of rank r complex vector bundles on M, and by $[X, Y]$ the set of homotopy classes of maps between X and Y. *Then for n sufficiently large there is a well defined map*

$$\rho : Vect_r(M, \mathbb{C}) \to [M, G_r(\mathbb{C}^n)],\ E \mapsto f_E.$$

Furthermore this is a bijective correspondence. In fact for n sufficiently large the map

$$\psi : [M, G_r(\mathbb{C}^n)] \to Vect_r(M, \mathbb{C}),\ f \mapsto f^{-1}E$$

is the inverse of ρ.
Now we will show that the Chern classes are the only cohomological invariants of a smooth complex vector bundles. We think

$$Vect_r(-, \mathbb{C}),\ H^*(-) : (\mathfrak{Manifolds}) \to (\mathfrak{Sets}),$$

as functors from the category of manifolds to the category of sets.

THEOREM 3.20. *Every natural transformation from the isomorphism classes of complex vector bundles over a manifold of finite type to the de Rham cohomology can be given as a polynomial in the Chern classes.*

PROOF. Let φ be a natural transformation between $Vect_r(-, \mathbb{C})$ and $H^*(-)$ in the category of manifolds of finite type. By the naturality of φ, if E is a rank r complex vector bundle over M and $f_E : M \to G_r(\mathbb{C}^n)$ is a classifying map for E, then $\varphi(E) = \varphi(f_E^{-1}Q) = f_E^*\varphi(Q)$. Since the cohomology of the Grassmannian $G_r(\mathbb{C}^n)$ is generated by the Chern classes of Q, then $\varphi(Q)$ can be written as $\varphi(Q) = P_\varphi(c_1(Q), ..., c_r(Q))$, for some polynomial P_φ depending on φ. Finally we get

$$\varphi(E) = f_E^*\varphi(Q) = P_\varphi(f_E^* c_1(Q), ..., f_E^* c_r(Q)) = P_\varphi(c_1(E), ..., c_r(E)).$$

♠

REMARK 3.21. Similar results hold for real vector bundles. Note that one can define the Chern class $c_j(E)$ as $f_E^* c_j(Q)$. Finally if M does not have a finite good cover one can repeat the preceding argument considering the infinite Grassmannian $G_r(V^\infty)$, the evaluation map $ev_p : V^\infty \to E_p \mapsto 0$. One gets a bijective correspondence

$$Vect_r(M, \mathbb{C}) \to [M, G_r(\mathbb{C}^\infty)],$$

using the fact that the cohomology ring of the infinite Grassmannian is the free polynomial algebra

$$H^*(G_r(\mathbb{C}^\infty)) = \mathbb{R}[c_1(Q), ..., c_r(Q)].$$

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