

UNIRATIONAL FANO THREEFOLDS FROM THE CLOP CONSTRUCTION

ALEX MASSARENTI

ABSTRACT. We study unirational Fano threefolds through the relative Abel–Jacobi construction of Cheltsov–Loginov–Orlov–Prokhorov. We obtain a 50-dimensional family of smooth unirational degree-one del Pezzo threefolds, explicit one-parameter families of terminal factorial unirational sextic double solids and quartic threefolds of Picard rank one, and a relative Abel–Jacobi interpretation of the Roth–Iskovskikh construction for smooth double covers of a quadric threefold. We also describe the inverse degree-two construction, its quadratic Galois twist, and an explicit parametrization of a smooth degree-one example.

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1. BACKGROUND, STATE OF THE ART, AND MAIN RESULTS

A smooth complex Fano threefold X with $\rho(X) = 1$ belongs to one of the 17 deformation families in the Iskovskikh–Mori–Mukai classification. Rationality is rare, while unirationality is known for most families. The following table records the situation for smooth members before the results discussed here; “unknown” means that no smooth unirational member was known in the corresponding family. Here $G(h, n)$ is the Grassmannian of h -planes in $\mathbb{P}^n$, $S^{10} \subset \mathbb{P}^{15}$ is the spinor variety, $\mathrm{LG}(2, 5) \subset \mathbb{P}^{13}$ is the Lagrangian Grassmannian in the projective indexing convention, and $G_2G(1, 6) \subset \mathbb{P}^{13}$ is the G_2 -Grassmannian.

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X	Rational	Unirational
Degree-6 hypersurface in $\mathbb{P}(1, 1, 1, 1, 3)$, that is, a double cover of $\mathbb{P}^3$ branched along a sextic	no	unknown
Quartic threefold in $\mathbb{P}^4$; the same genus also contains hyperelliptic double covers of a quadric threefold	no	some are
Complete intersection of a quadric and a cubic in $\mathbb{P}^5$	no	yes
Complete intersection of three quadrics in $\mathbb{P}^6$	no	yes
Gushel–Mukai threefolds: linear sections of $G(1, 4)$ cut by a quadric, or the corresponding double-cover model	general one is not	yes
Codimension-7 linear section of $S^{10} \subset \mathbb{P}^{15}$	yes	yes
Codimension-5 linear section of $G(1, 5)$	no	yes
Codimension-3 linear section of $\mathrm{LG}(2, 5)$	yes	yes
Codimension-2 linear section of $G_2G(1, 6)$	yes	yes
Zero locus of a section of $\left(\wedge^2 U^\vee\right)^{\oplus 3}$ on $G(2, 6)$	yes	yes
Degree-6 hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$	no	unknown
Double cover of $\mathbb{P}^3$ branched along a smooth quartic surface	no	yes
Cubic threefold in $\mathbb{P}^4$	no	yes
Complete intersection of two quadrics in $\mathbb{P}^5$	yes	yes
Codimension-3 linear section of $G(1, 4)$	yes	yes
Quadric threefold in $\mathbb{P}^4$	yes	yes
Projective space $\mathbb{P}^3$	yes	yes

The cases relevant here are quartic threefolds, sextic double solids, and degree-one del Pezzo threefolds. Unirationality of a general smooth quartic threefold is open, although Segre constructed smooth unirational examples using quartic surfaces with separable asymptotic directions [Seg60]. Marchisio developed this construction and obtained a 54-dimensional locus in the projective space $\mathbb{P}^{69}$ of quartic equations, hence of codimension 15 [Mar06].

A del Pezzo threefold is a Fano threefold with $-K_X = 2H$ for an ample Cartier divisor H ; its degree is H^3 . A smooth degree-one del Pezzo threefold is a sextic in $\mathbb{P}(1, 1, 1, 2, 3)$. Before the present construction no smooth unirational example was known in this family [CTZ25].

Theorem A. *Let*

$$(1.1) \quad X = \{w^2 = z^3 + a_2(x_0, x_1, x_2)z^2 + a_4(x_0, x_1, x_2)z + h_6(x_0, x_1)\} \subset \mathbb{P}(1, 1, 1, 2, 3),$$

where a_i is homogeneous of degree i and h_6 is a binary sextic. If X is smooth, then X is unirational. These hypersurfaces sweep a 50-dimensional locus in $\mathbb{P}^{63}$, hence a locus of codimension 13.

Thus the new smooth V_1 locus is comparable with the Segre–Marchisio locus: 50 dimensions in $\mathbb{P}^{63}$ versus 54 dimensions in $\mathbb{P}^{69}$. The same method gives explicit one-parameter families of terminal factorial unirational sextic double solids and quartic threefolds of Picard rank one. The sextic double solids are dominated by $\mathbb{P}^3$ through maps of degree 9; distinguished members have baskets $cD_4 + 2cA_3 + 13cA_1$ and $cD_4 + 2cA_2$, respectively. Finally, the same relative Abel–Jacobi mechanism gives an interpretation of the classical Roth–Iskovskikh construction for smooth double covers of a quadric threefold.

2. THE CLOP CONSTRUCTION AND ITS ABEL–JACOBI MAP

The CLOP construction starts from an anticanonical pencil on a del Pezzo surface, replaces the resulting genus-one fibration by its relative Jacobian, and contracts the zero section to a degree-one del Pezzo surface. The relative Abel–Jacobi map gives the required dominant map.

From an anticanonical pencil to a degree-one del Pezzo surface. Let Z be a smooth del Pezzo surface of degree at least two over a field k , and let $\mathcal{P} \subset |-K_Z|$ be an admissible pencil in the sense of [CLOP26]: its base scheme is reduced, its generic member is smooth, and every geometric member is irreducible. Blowing up the base scheme gives a genus-one fibration $\theta : Y \rightarrow \mathbb{P}^1$. Its relative Jacobian $\vartheta : J \rightarrow \mathbb{P}^1$ is an elliptic surface with zero section C ; in the cases considered here J is rational, $C^2 = -1$, and contracting C gives a smooth degree-one del Pezzo surface S . Thus $Z \dashleftarrow Y \xrightarrow{\theta} \mathbb{P}^1$, $J = \mathrm{Pic}^0(Y/\mathbb{P}^1) \xrightarrow{\vartheta} \mathbb{P}^1$, and $J \rightarrow S$ with $K_S^2 = 1$.

Let A be an ample divisor on Z and set $n = A \cdot (-K_Z)$. Since $\deg(A|_E) = n$ on a smooth fiber E , the relative Abel–Jacobi construction gives a dominant rational map $Z \dashrightarrow S$ of degree n^2 [CLOP26, Lemma 3.4].

The relative Abel–Jacobi map. Let $U \subset \mathbb{P}^1$ be the smooth locus of θ , set $Y_U = \theta^{-1}(U)$, and let $\mathcal{A}$ be the pullback of A . On $Y_U \times_U Y_U$, with projections p_1, p_2 and relative diagonal Δ , the line bundle $\mathcal{L}_A = \mathcal{O}(n\Delta) \otimes p_1^* \mathcal{A}^{-1}$ has degree zero on every fiber of p_2 . Hence the relative Picard scheme gives

$$(2.1) \quad \Phi_A : Y_U \rightarrow \mathrm{Pic}^0(Y_U/U), \quad y \mapsto [\mathcal{O}_{Y_b}(ny) \otimes \mathcal{A}_b^{-1}].$$

No section of $Y_U \rightarrow U$ is required. After choosing an origin O on a geometric smooth fiber E , formula (2.1) becomes $x \mapsto nx - q$, where q is the point corresponding to $A|_E \otimes \mathcal{O}_E(-nO) \in \mathrm{Pic}^0(E)$. Thus Φ_A is multiplication

by n followed by a translation and has degree n^2 . In particular, for a degree-two del Pezzo surface with $A = -K_Z$ the degree is 4, while for $Z = \mathbb{P}^2$ and $A = \mathcal{O}_{\mathbb{P}^2}(1)$ it is 9.

The degree-two fiberwise formula. Assume $K_Z^2 = 2$ and let Q_1, Q_2 be the geometric base points of the pencil. On the generic genus-one fiber Y_η one has $\pi^*(-K_Z)|_{Y_\eta} \simeq \mathcal{O}_{Y_\eta}(Q_1 + Q_2)$, so the Abel–Jacobi map is $Q \mapsto \mathcal{O}_{Y_\eta}(2Q - Q_1 - Q_2) \in \text{Pic}^0(Y_\eta)$. After a quadratic extension splitting Q_1, Q_2 and choosing Q_1 as origin, this becomes $Q \mapsto 2Q - (Q_2 - Q_1)$, that is, a translation composed with [2].

3. REVERSING THE DEGREE-TWO CONSTRUCTION

We now characterize the degree-one del Pezzo surfaces obtained from the degree-two CLOP construction.

Root lines. Let S be a smooth degree-one del Pezzo surface over a perfect field k of characteristic different from 2 and 3. The pencil $|-K_S|$ has a unique base point $O \in S(k)$. Set $J = \text{Bl}_O(S)$; then $J \rightarrow \mathbb{P}^1$ is a relatively minimal elliptic surface and the exceptional curve C is its zero section. A line $E \subset S_{\bar{k}}$ is a (-1) -curve with $(-K_S) \cdot E = 1$. Its strict transform is a section P_E of J , corresponding under the Shioda identification to the root $[E] + K_S \in K_S^\perp \simeq E_8(-1)$ [Man86, Shi90].

Definition 3.1. A *quadratic root line* is a line $E \subset S_L$, where L/k is separable of degree at most two, such that, if $[L : k] = 2$ with nontrivial involution σ , then $\sigma(P_E) = -P_E$.

If $E_- := \sigma(E_+)$, then $\sigma(P_{E_+}) = P_{E_-}$, while the quadratic-root condition is the additional identity $P_{E_-} = -P_{E_+}$. Under the root–Mordell–Weil correspondence this is equivalent to $[E_-] = -[E_+] - 2K_S$, that is, $E_+ + E_- \sim -2K_S$.

Theorem 3.2 (Inverse degree-two construction). *A smooth degree-one del Pezzo surface S arises from the relative Jacobian of an admissible anticanonical pencil on a smooth degree-two del Pezzo surface if and only if S carries a quadratic root line. In this case there is a dominant rational map $Z \dashrightarrow S$ of degree 4 from a smooth degree-two del Pezzo surface Z .*

The split case. Assume that $P = P_E$ is defined over k . The disjoint (-1) -curves $C, P \subset J$ can be contracted simultaneously by $q : J \rightarrow Z$. If F is the fiber class, then $q^*(-K_Z) = F + C + P$; this divisor has square 2, is trivial exactly on C and P , and is positive on every other irreducible curve. Hence Z is a degree-two del Pezzo surface. The fibers of $J \rightarrow \mathbb{P}^1$ descend to an admissible anticanonical pencil on Z with base points $q(C), q(P)$.

Let $\pi_S : J \rightarrow S$ contract C . The common resolution gives a k -birational map $\beta = \pi_S \circ q^{-1} : Z \dashrightarrow S$. This map is distinct from the degree-4 Abel–Jacobi map. Indeed, choosing the exceptional section over one base point as origin, the latter restricts to $Q \mapsto 2Q - (Q_2 - Q_1)$ on the generic fiber. Thus in the split case both a birational map $Z \dashrightarrow S$ and the degree-4 Abel–Jacobi map are defined over k .

The non-split case and the quadratic twist. Assume that L/k is quadratic and write $P = P_E$. For the original k -form J/k , Galois acts by $\sigma(C) = C$ and $\sigma(P) = -P$, so the pair $\{C, P\}$ is not stable. Set $\tau = t_P \circ \sigma$, where t_P denotes translation by P on J_L . Formally, τ is the L -isomorphism ${}^\sigma J_L \rightarrow J_L$ defining the twisted descent datum. Since $\sigma(P) = -P$, one has $\tau^2 = \text{id}$, $\tau(C) = P$, and $\tau(P) = C$. Thus τ satisfies the cocycle condition and exchanges exactly the two curves to be contracted. Projective descent is effective, for instance after replacing an ample line bundle M by the τ -invariant ample bundle $M \otimes \tau^* M$. The resulting genus-one fibration Y/k has $Y_L \simeq J_L$; since translations act trivially on Pic^0 , its relative Jacobian is the original elliptic surface J/k .

Let $q_L : J_L \rightarrow Z_L$ be the simultaneous contraction of C and P . Since τ exchanges them, uniqueness of the contraction gives a unique semilinear isomorphism $\tau_Z : {}^\sigma Z_L \rightarrow Z_L$ such that

$$\begin{array}{ccc} {}^\sigma J_L & \xrightarrow{{}^\sigma q_L} & {}^\sigma Z_L \\ \tau \downarrow & & \downarrow \tau_Z \\ J_L & \xrightarrow{q_L} & Z_L. \end{array}$$

The cocycle condition for τ_Z follows from uniqueness. Since $-K_{Z_L}$ is ample and invariant, projective descent gives a smooth degree-two del Pezzo surface Z/k . In the non-split case $q_L(C)$ and $q_L(P)$ descend to one closed point of degree two.

For later use, note that the conjugate root lines themselves cannot be contracted. Indeed, if $E_+ + E_- \sim -2K_S$, then $E_+ \cdot E_- = 3$, so their intersection matrix is not negative definite. The contractible pair is $C \cup P$ on J_L .

The birational map over the splitting field. Over L the contractions $\pi_S : J_L \rightarrow S_L$ and $q_L : J_L \rightarrow Z_L$ give a birational map $\beta_L = \pi_S \circ q_L^{-1} : Z_L \dashrightarrow S_L$. In general it does not descend to k , since S_L carries the original descent datum while Z_L carries the twist $t_P\sigma$; descent of β_L would force $\pi_S \circ t_P = \pi_S$ generically. The degree-4 relative Abel–Jacobi map, on the other hand, is defined over k .

The converse. Conversely, let S come from an admissible anticanonical pencil on a degree-two del Pezzo surface Z . After an extension L/k of degree at most two the base points Q_1, Q_2 split. Blowing them up gives exceptional sections C_1, C_2 . Taking C_1 as zero identifies the genus-one fibration with its Jacobian; C_1 becomes the zero section C and C_2 a section P . Contracting C gives S_L , while the image of P is a line $E \subset S_L$. If Q_1, Q_2 are conjugate, Galois exchanges C_1, C_2 , hence $\sigma(P_E) = -P_E$. Thus E is a quadratic root line.

4. THE CRITERION OVER $\mathbb{C}(t)$

Set $K = \mathbb{C}(t)$.

Proposition 4.1. *Every smooth degree-two del Pezzo surface over K is K -unirational.*

Sketch of proof. A degree-two del Pezzo surface is geometrically rationally connected, hence has a K -point by Graber–Harris–Starr [GHS03]. Weak approximation at places of good reduction [HT06] allows one to choose such a point away from the ramification curve and the generalized Eckardt locus; the criterion of Salgado–Testa–Várilly-Alvarado then gives K -unirationality [STVA14, Corollary 3.3]. $\square$

If a complex threefold fibers over $\mathbb{P}^1$ and its generic fiber over K is unirational, then the threefold is unirational by spreading a dominant map $\mathbb{P}_K^2 \dashrightarrow Y_\eta$.

Theorem 4.2 (Criterion for V_1). *Let X be a smooth complex del Pezzo threefold of degree one and let $\Lambda \subset |H|$ be a pencil with smooth generic member S/K . If S carries a quadratic root line, then X is unirational.*

Sketch of proof. The quadratic root line gives a smooth degree-two del Pezzo surface Z/K and a degree-4 map $Z \dashrightarrow S$. Proposition 4.1 makes Z , hence S , unirational over K . Spreading a parametrization of S over the base gives a dominant rational map $\mathbb{P}^2 \times \mathbb{P}^1 \dashrightarrow X$. $\square$

5. A 50-DIMENSIONAL FAMILY OF SMOOTH UNIRATIONAL SEXTICS

Consider the family in (1.1) and the pencil generated by x_0, x_1 . On the generic member set $x_0 = u$, $x_1 = tu$, $x_2 = v$, and $K = \mathbb{C}(t)$. The generic surface is $S = \{w^2 = z^3 + A_2(u, v)z^2 + A_4(u, v)z + h(t)u^6\} \subset \mathbb{P}_K(1_u, 1_v, 2_z, 3_w)$, where $A_i(u, v) = a_i(u, tu, v)$ and $h(t) = h_6(1, t)$. Smoothness of X implies $h_6 \neq 0$: otherwise $\{z = w = 0\} \subset X$, and a zero of a_4 on this plane is singular. Hence $h(t) \in K^*$.

Set $L = K(\sqrt{h})$. Over L the curves $E_+ = \{z = 0, w = \sqrt{h}u^3\}$ and $E_- = \{z = 0, w = -\sqrt{h}u^3\}$ are lines. On the elliptic surface obtained by blowing up the anticanonical base point, the involution $(z, w) \mapsto (z, -w)$ is inversion, so $P_{E_-} = -P_{E_+}$. If L/K is quadratic, then $\sigma(E_+) = E_-$ and $\sigma(P_{E_+}) = -P_{E_+}$; if $L = K$, E_+ is already defined over K . Thus S carries a quadratic root line in either case. Moreover, $E_+ + E_- = \{z = 0\} \sim -2K_S$, in accordance with the criterion above. The generic member is smooth by Bertini and a direct check along the base curve.

Theorem 5.1. *Every smooth hypersurface of the form (1.1) is unirational.*

Sketch of proof. The generic member of the pencil generated by x_0, x_1 is a smooth degree-one del Pezzo surface over K carrying a quadratic root line, so Theorem 4.2 applies. $\square$

Dimension of the locus. The coefficient space has dimension $6 + 15 + 7 = 28$, while $\text{Aut } \mathbb{P}(1, 1, 1, 2, 3)$ has dimension 29. The subgroup preserving the special form has dimension 7, hence the swept locus has dimension at most $28 + 29 - 7 = 50$. At one explicit point the 28 coefficient directions and the infinitesimal graded automorphisms span a 51-dimensional affine subspace of the 64-dimensional space of weighted sextics. After removing the radial direction, the projective tangent space has dimension 50. Semicontinuity gives equality on a dense open set. Thus the locus has dimension 50, codimension 13 in $\mathbb{P}^{63}$, and 21-dimensional image in moduli.

6. A SMOOTH EXPLICIT EXAMPLE AND AN EXPLICIT PARAMETRIZATION

Consider

$$(6.1) \quad X = \{w^2 = z^3 - (x_0^2 x_1^2 + x_2^4)z + x_0^6 + x_1^6\} \subset \mathbb{P}(1, 1, 1, 2, 3).$$

The threefold X is smooth. Indeed, it avoids the singular points of the ambient weighted projective space. If all partial derivatives of the affine equation vanish at a nonzero point, then $w = 0$. For $z = 0$ the remaining derivatives force $x_0 = x_1 = x_2 = 0$. For $z \neq 0$ they give $x_2 = 0$, $x_0 x_1 \neq 0$, and simultaneously $(x_1/x_0)^6 = 1$ and $(x_1/x_0)^6 = 27$, a contradiction.

The generic fiber and its degree-two source. Set $K = \mathbb{C}(t)$, $t = x_1/x_0$, and $h = 1 + t^6$. The generic member is $S_t = \{w^2 = z^3 - (t^2 u^4 + v^4)z + hu^6\} \subset \mathbb{P}_K(1_u, 1_v, 2_z, 3_w)$. Since h has simple zeros, it is not a square in K . Hence $L = K(\sqrt{h})$ is quadratic and the lines $E_{\pm} = \{z = 0, w = \pm\sqrt{h}u^3\}$ are conjugate; their strict transforms are inverse sections. Thus this example is genuinely non-split over the generic field K .

The inverse CLOP surface is

$$(6.2) \quad Z_t = \{\Omega^2 = hc^4 + 4h^3(t^2 u^4 + v^4 - 2cu^3)\} \subset \mathbb{P}_K(1_u, 1_v, 1_c, 2\Omega).$$

It is a smooth degree-two del Pezzo surface. The pencil generated by u, v is admissible; its base scheme is $u = v = 0$, $\Omega^2 = hc^4$, hence consists of two conjugate geometric points over $K(\sqrt{h})$.

For the member $v = \lambda u$, set $M = t^2 + \lambda^4$. The corresponding genus-one curve is $\omega^2 = hc^4 - 8h^3cu^3 + 4h^3Mu^4$. Its binary-quartic invariants are $\mathcal{I} = 48h^4M$ and $\mathcal{J} = -1728h^7$. Hence its Jacobian is $Y^2 = X^3 - 27\mathcal{I}X - 27\mathcal{J}$, which after $X = 36h^2x$ and $Y = 216h^3y$ becomes $y^2 = x^3 - Mx + h$. This is the Weierstrass equation of the anticanonical pencil on S_t . Therefore the relative Jacobian of the pencil on Z_t contracts to S_t , and the relative Abel–Jacobi map $Z_t \dashrightarrow S_t$ has degree 4.

Two Manin steps on the auxiliary surface. Work on the chart $u = 1$ of (6.2) and set $f(v, c) = hc^4 + 4h^3(t^2 + v^4 - 2c)$. The surface contains $p_0 = (t^2, 0, 2h^2t)$. Along $(v, c) = (t^2 + s, ms)$, the quadratic Taylor approximation to $\sqrt{f}$ at p_0 is $T_0(s) = 2h^2t + \frac{2h(2t^6 - m)}{t}s + \frac{6ht^6 - (2t^6 - m)^2}{t^3}s^2$. Set

$$\begin{aligned} A_0 &= m^4(1 - t^6 - t^{12}) - 8t^6m^3 + 12(t^{12} - t^6)m^2 + 16(t^{18} + 3t^{12})m \\ &\quad + 12t^{18} + 24t^{12} - 4t^6, \\ B_0 &= 4t^2h(m^3 - 6t^6m^2 + 6(t^{12} - t^6)m + 4(t^{12} - t^6)). \end{aligned}$$

A direct expansion gives $T_0(s)^2 - f(t^2 + s, ms) = t^{-6}s^3(B_0 + A_0s)$. Hence the residual point is obtained for $s_0 = -B_0/A_0$, with $v_0 = t^2 + s_0$ and $c_0 = ms_0$. Set

$$\begin{aligned} Q_0 &= c_0^2 - 4t^6v_0c_0 + 2t^2(3t^6 + 1)c_0 - 2t^6(t^6 + 3)v_0^2 \\ &\quad + 8t^8v_0 - 2t^4(3t^6 + 1), \quad \omega_0 = -\frac{Q_0}{t^3}. \end{aligned}$$

Then $\omega_0 = T_0(s_0)$ and $\omega_0^2 = f(v_0, c_0)$.

For the second Manin step take the direction $(1, r)$ through (v_0, c_0) and set

$$\begin{aligned} f_v &= 16h^3v_0^3, & f_c &= 4hc_0^3 - 8h^3, & f_{vv} &= 48h^3v_0^2, \\ f_{cc} &= 12hc_0^2, & \omega_v &= \frac{f_v}{2\omega_0}, & \omega_c &= \frac{f_c}{2\omega_0}, \\ \omega_{vv} &= \frac{f_{vv}}{2\omega_0} - \frac{f_v^2}{4\omega_0^3}, & \omega_{vc} &= -\frac{f_v f_c}{4\omega_0^3}, & \omega_{cc} &= \frac{f_{cc}}{2\omega_0} - \frac{f_c^2}{4\omega_0^3}. \end{aligned}$$

Set

$$\begin{aligned} a &= \omega_v + r\omega_c, & b &= \frac{\omega_{vv} + 2r\omega_{vc} + r^2\omega_{cc}}{2}, \\ F_3 &= 24hc_0r^3 + 96h^3v_0, & F_4 &= 24hr^4 + 96h^3, \\ C_3 &= 2ab - \frac{F_3}{6}, & C_4 &= b^2 - \frac{F_4}{24}, & s_1 &= -\frac{C_3}{C_4}, \\ v_1 &= v_0 + s_1, & c_1 &= c_0 + rs_1, & \omega_1 &= \omega_0 + as_1 + bs_1^2. \end{aligned}$$

Then

$$(6.3) \quad \omega_1^2 = hc_1^4 + 4h^3(t^2 + v_1^4 - 2c_1).$$

Indeed, for $T(s) = \omega_0 + as + bs^2$, the difference between $T(s)^2$ and the right-hand side of (6.3) along $(v, c) = (v_0, c_0) + s(1, r)$ is $s^3(C_3 + C_4s)$.

The relative Abel–Jacobi map and the final parametrization. Set $M = t^2 + v_1^4$ and $U(c, u) = hc^4 - 8h^3cu^3 + 4h^3Mu^4$. For the binary quartic U , let $H_U = \frac{1}{3} \det \text{Hess}(U)$ and $G_U = \frac{1}{12} \frac{\partial(U, H_U)}{\partial(c, u)}$. The classical covariant 2-covering map, which realizes the degree-4 relative Abel–Jacobi map fiberwise, is $[\xi : \eta : \zeta] = [-3\omega H_U : 27G_U : \omega U]$ [Fis08, Fis12]. After the normalization above its target is $y^2 = x^3 - Mx + h$. Evaluating at $(c, u, \omega) = (c_1, 1, \omega_1)$ gives

$$\begin{aligned} x_E &= \frac{4h^2(c_1^3 + h^2 - Mc_1^2)}{\omega_1^2}, \\ G_{\text{red}} &= -8M^2c_1h^2 + 2Mc_1^5 + 20Mc_1^2h^2 - c_1^6 - 20c_1^3h^2 + 8h^4, \\ y_E &= \frac{h^2G_{\text{red}}}{\omega_1^3}. \end{aligned}$$

A direct computation gives $y_E^2 = x_E^3 - Mx_E + h$. Therefore

$$(6.4) \quad \phi : \mathbb{A}_{t,m,r}^3 \dashrightarrow X, \quad (t, m, r) \mapsto [1 : t : v_1 : x_E : y_E]$$

is a rational map to (6.1).

The map (6.4) is dominant. It is enough to check dominance of $(t, m, r) \mapsto (t, v_1, c_1)$, since the projection of the total auxiliary surface to (t, v, c) is generically finite and the 2-covering map is finite. Modulo 101, at $(t, m, r) = (1, 1, 0)$ all denominators are nonzero, $(A_0, \omega_0, C_4) = (87, 74, 70)$ and $(v_1, c_1, \omega_1) = (56, 26, 79)$, while $\det \frac{\partial(v_1, c_1)}{\partial(m, r)} = 45 \neq 0$. Hence the Jacobian determinant is not identically zero over $\mathbb{C}$, and (6.4) is an explicit unirational parametrization of (6.1).

7. SEXTIC DOUBLE SOLIDS

Set $K = \mathbb{C}(t)$ and consider the admissible cubic pencil on $\mathbb{P}_K^2$ generated by $f_0 = tyz^2 - (t^2 - 1)x^3$ and $f_1 = y^3 + (2t + 1)y^2z - t(t^2 - 1)xz^2 - t^2(t + 2)z^3$. Its relative Jacobian is a degree-one del Pezzo surface S/K . Since $Z = \mathbb{P}^2$ and $A = \mathcal{O}_{\mathbb{P}^2}(1)$, one has $A \cdot (-K_{\mathbb{P}^2}) = 3$, so the relative Abel–Jacobi map $\mathbb{P}_K^2 \dashrightarrow S$ has degree 9.

Globalizing S gives a threefold $\pi : \mathcal{Y} \rightarrow \mathbb{P}^1$ with $\text{Cl}(\mathcal{Y}) = \mathbb{Z}H \oplus \mathbb{Z}F$ and $-K_{\mathcal{Y}} = F + H$, where F is the fiber class and H the relative anticanonical class. The two-ray game crosses the wall generated by $F + H = -K_{\mathcal{Y}}$, hence the corresponding small modification $\mathcal{Y} \dashrightarrow \mathcal{Y}^+$ is a flop. It exchanges the rational curves $C_- = \{p = z = v = 0\} \subset \mathcal{Y}$ and $C_+ = \{r = s = v = 0\} \subset \mathcal{Y}^+$.

The next extremal contraction is the restriction of the toric morphism $[x_0 : x_1 : x_2 : x_3 : w] = [rp^2 : sp^2 : qp : z : vp^3]$. It contracts $E = (p = 0)$ to $P = [0 : 0 : 0 : 1 : 0]$; on the chart $x_3 = 1$ the valuation of $p = 0$ on (x_0, x_1, x_2, w) has weights $(2, 2, 1, 3)$. Thus the Sarkisov link is

$$\boxed{X \xleftarrow{(2,2,1,3) \text{ weighted extraction}} \mathcal{Y}^+ \dashrightarrow_{\text{flop}} \mathcal{Y} \xrightarrow{dP_1} \mathbb{P}^1.}$$

Theorem 7.1. *There is an explicit one-parameter family of sextic double solids X_λ and a nonempty Zariski open subset of the parameter line such that every member is a terminal factorial Fano threefold with $\rho(X_\lambda) = 1$, birational to a degree-one del Pezzo fibration and dominated by $\mathbb{P}^3$ through a rational map of degree 9. The distinguished member X_2 has basket $cD_4 + 2cA_3 + 13cA_1$.*

The degree 9 comes entirely from the cubic pencil. The two-ray game only identifies a Fano model birational to the resulting del Pezzo fibration; spreading the degree-9 map on the generic fiber gives $\mathbb{P}^3 \dashrightarrow X_\lambda$.

8. QUARTIC THREEFOLDS

Set $K = \mathbb{C}(t)$ and consider the smooth degree-two del Pezzo surface

$$Z = \left\{ \omega^2 = ty^4 + \frac{t}{3}x^2y^2 - \frac{1}{t}x^3y - y^3z + (t^2y^2 - x^2 - xy)z^2 + tz^4 \right\} \subset \mathbb{P}_K(1_x, 1_y, 1_z, 2_\omega).$$

The anticanonical pencil induced by the lines through $P_0 = [0 : 0 : 1]$ has base scheme given by $\omega^2 = t$. It is admissible, and its relative Jacobian is a degree-one del Pezzo surface S/K . Since $A = -K_Z$, the Abel–Jacobi map $Z \dashrightarrow S$ has degree 4. Hence S is unirational.

Globalize S to $\mathcal{V} \rightarrow \mathbb{P}^1$ and set $A = F + H$, $B = F + 2H$, and $M = F + 3H$. The relevant chambers are $\langle F, A \rangle$, $\langle A, B \rangle$, and $\langle B, M \rangle$, with models $\mathcal{V}$, $\mathcal{V}_A$, and $\mathcal{V}_B = \mathcal{V}^+$. By adjunction, $-K_{\mathcal{V}} = A$.

The two small modifications. The crossing $\mathcal{V} \dashrightarrow \mathcal{V}_A$ is K -trivial, hence a flop. It exchanges $C_- = \{p = z = w = 0\} \subset \mathcal{V}$ and $C_+ = \{r = s = 0\} \subset \mathcal{V}_A$. On the common resolution $\mathcal{W} = \text{Bl}_{C_+}(\mathcal{V}_A) = \text{Bl}_{C_-}(\mathcal{V})$, the rational map $[r : s]$ becomes a morphism to $\mathbb{P}^1$ with generic fiber $\tilde{S} = \text{Bl}_{P_1}(S)$, where $P_1 = [0 : 1 : 0 : 0] \in S(K)$.

The second crossing $\mathcal{V}_A \dashrightarrow \mathcal{V}_B$ exchanges the rational curves $D_- = \{p = w = 0, rz + 9q^2 = 0\} \subset \mathcal{V}_A$ and $D_+ = \{r = s = q = 0\} \subset \mathcal{V}_B$. In this direction the contracted ray is K -negative, so the crossing is a flip; read from the quartic toward the del Pezzo fibration, $\mathcal{V}_B \dashrightarrow \mathcal{V}_A$ is an antiflip.

The extraction from the quartic. The contraction $\varphi : \mathcal{V}_B \rightarrow X_4 \subset \mathbb{P}^4$ is given by $[x_0 : x_1 : x_2 : x_3 : x_4] = [rp^3 : sp^3 : qp^2 : zp : w]$. It contracts $E = (p = 0)$ to $P = [0 : 0 : 0 : 0 : 1]$ and has valuation weights $(3, 3, 2, 1)$ on (x_0, x_1, x_2, x_3) in the chart $x_4 = 1$. Thus

$$X_4 \xleftarrow{(3,3,2,1) \text{ weighted extraction}} \mathcal{V}_B \dashrightarrow_{\text{antiflip}} \mathcal{V}_A \dashrightarrow_{\text{flop}} \mathcal{V} \xrightarrow{dP_1} \mathbb{P}^1.$$

Theorem 8.1. *There is an explicit one-parameter family of quartic threefolds $X_{4,\kappa} \subset \mathbb{P}^4$ and a nonempty Zariski open subset of the parameter line such that every member is a terminal factorial Fano threefold of Picard rank one and is unirational. The distinguished member $X_{4,1}$ has basket $cD_4 + 2cA_2$.*

Indeed, Z/K is unirational, the degree-4 Abel–Jacobi map dominates S/K , and spreading over the base dominates $\mathcal{V}$; the birational factorization above transfers unirationality to $X_{4,\kappa}$.

9. SMOOTH DOUBLE COVERS OF QUADRICS

Let $\pi : X \rightarrow Q^3$ be a smooth double cover of a smooth quadric threefold branched along a smooth divisor in $|\mathcal{O}_Q(4)|$. Set $H = -K_X = \pi^*\mathcal{O}_Q(1)$ and $M = F_1(Q) \simeq \mathbb{P}^3$. Consider $W = \{(z, \ell) \in X \times M \mid \pi(z) \in \ell\}$ with projections $p : W \rightarrow X$ and $f : W \rightarrow M$. The map p is a $\mathbb{P}^1$ -bundle. For a general $\ell \in M$, the fiber $E_\ell = f^{-1}(\ell) = \pi^{-1}(\ell)$ is a smooth double cover of $\ell \simeq \mathbb{P}^1$ branched in four points, hence a genus-one curve.

The relative Jacobian. Over the smooth locus $M^\circ \subset M$ of f , set $\mathcal{J} = \text{Pic}^0(W^\circ/M^\circ)$. Fiberwise, $E_\ell \simeq \text{Pic}^1(E_\ell)$ via $z \mapsto \mathcal{O}_{E_\ell}(z)$, and $\text{Pic}^1(W^\circ/M^\circ)$ is a torsor under $\mathcal{J}$. Unlike the CLOP surface construction, here the base is $M \simeq \mathbb{P}^3$, so $\mathcal{J}$ is a fourfold and no contraction to a degree-one del Pezzo surface occurs. Only the relative Abel–Jacobi mechanism is used.

The degree-nine correspondence. For a general ℓ , $\deg H|_{E_\ell} = 2$. Set $A = 2H|_{E_\ell}$. The fiberwise Abel–Jacobi map is $\Phi_A(z) = \mathcal{O}_{E_\ell}(4z) \otimes A^{-1}$. Using the action of $\text{Pic}^0(E_\ell)$ on $\text{Pic}^1(E_\ell) \simeq E_\ell$, define $\mu(z) = (-\Phi_A(z)) \cdot z$. Then

$$(9.1) \quad 3z + \mu(z) \sim 2H|_{E_\ell}.$$

After choosing an origin, μ is multiplication by -3 followed by a translation, hence $\deg \mu = 9$. These maps globalize to a rational self-map $\mu : W \dashrightarrow W$ over M ; set $r = p \circ \mu$.

Formula (9.1) has a projective interpretation. The degree-4 linear system $|2H|_{E_\ell}|$ embeds E_ℓ as an elliptic normal quartic in $\mathbb{P}^3$, and its osculating plane at z cuts the divisor $3z + \mu(z)$. Thus $\mu(z)$ is the residual fourth point. More generally, $A = mH|_{E_\ell}$ gives $(2m-1)z + \mu_m(z) \sim mH|_{E_\ell}$ and a map of degree $(2m-1)^2$; the Roth–Iskovskikh construction is the case $m = 2$ [Rot50, Isk80].

The rational curve associated with a point. Fix a general $x \in X$ and set $q = \pi(x)$. The lines on Q through q form a $\mathbb{P}^1$, and $C_x = \overline{r(p^{-1}(x))} \subset X$ is rational once it is nonconstant. Resolve the tangent quadric cone $Q \cap T_q Q$ by $\mathbb{F}_2$. The corresponding double cover $T_x \rightarrow \mathbb{F}_2$ is a K3 surface fibered by the curves E_ℓ . The inverse image of the negative section splits as $O \sqcup P$, where O, P correspond to x and its conjugate point; use O as zero section. For general q the elliptic fibers are irreducible, and Shioda’s height formula gives $\langle P, P \rangle = 4$, $\langle 2P, 2P \rangle = 16$, and $2P \cdot O = 6$. Since $H|_{E_\ell} \sim O + P$, setting $z = O$ in (9.1) gives $\mu(O) = 2P$. Hence C_x is the image of the section $2P$. If a section R were contracted to a point $y \in X$, then $\pi(y)$ would lie on every line through q , so $\pi(y) = q$ and R would be either O or P . Since $2P$ is neither, C_x is nonconstant; moreover $2P \cdot O = 6$, so C_x contains x .

The correspondence (p, r) has no nonconstant rational invariant. Indeed, if $g \in \mathbb{C}(X)$ satisfies $p^*g = r^*g$ and $h = p^*g$, then $\mu^*h = h$. On a general elliptic fiber, nonconstancy of h would give $\deg(h \circ \mu) = 9 \deg h$, a contradiction. Hence h comes from a rational function on M . This function is constant on every pencil of lines through a general point of Q , and therefore constant on M . Thus g is constant.

For a general $x \in X$ set $V_0(x) = \{x\}$ and $V_{i+1}(x) = \overline{r(p^{-1}(V_i(x)))}$. The dimensions cannot stabilize below 3, otherwise the resulting family of cycles would yield a nonconstant rational invariant of (p, r) . Hence $\dim V_1(x) = 1$, $\dim V_2(x) = 2$, and $V_3(x) = X$. Now $V_1(x) = C_x$ is rational; since p is a $\mathbb{P}^1$ -bundle, $p^{-1}(V_1(x))$

is rational, so $V_2(x)$ is unirational and therefore rational over $\mathbb{C}$. Thus $p^{-1}(V_2(x))$ is a rational threefold and $r : p^{-1}(V_2(x)) \dashrightarrow X$ is dominant.

Theorem 9.2 (Roth–Iskovskikh). *A smooth double cover of a smooth quadric threefold branched along a smooth quartic section is unirational. The classical point correspondence is the degree-nine relative Abel–Jacobi correspondence above.*

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ALEX MASSARENTI, DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI FERRARA, VIA MACHIAVELLI 30, 44121 FERRARA, ITALY

Email address: msslxa@unife.it