THE KODAIRA EMBEDDING THEOREM

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Contents

Introduction

In this work we prove a well known theorem due to Kodaira, which characterizes the compact complex manifolds that can be embedded in a projective space i.e. that are projective varieties.
Kodaira's theorem asserts that a compact complex manifold is projective if and only if it has a positive line bundle. Recall that by *Chow's theorem* every complex, projective manifold is indeed algebraic i.e. defined by the zeros of homogeneous polynomials.
So, thanks to these two theorems, a question of Complex Analysis can be translated in the language of Algebraic Geometry. This problem is developed by *Serre* in great detail in his famous paper *GAGA*.
We will use the theory of harmonic forms to prove the *Kodaira-Nakano vanishing theorem*, which is necessary in the proof of the *Kodaira embedding theorem*. The vanishing theorem is proved using methods of complex analytic differential geometry. If M is supposed to be a smooth, projective variety one can expect to find an algebraic proof of the vanishing theorem but at the present there is no purely algebraic proof.

1. Complex and Kähler Manifolds

Let M be a smooth manifold. An *almost complex structure* on M is an isomorphism $J : TM \to TM$ of the tangent bundle TM such that $J^2 = J \circ J = -1$. In other words it consists of a family of isomorphisms of vector spaces $J_x : T_xM \to$

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T_xM such that $J_x^2 = -1$ and the assignment $x \mapsto J_x$ is smooth. A smooth manifold equipped with an almost complex structure is called an *almost complex manifold*. In a certain sense the isomorphism J plays the role of the imaginary unit $i \in \mathbb{C}$, in fact if V is a $\mathbb{R}$-vector space with an almost complex structure J we can define the multiplication by a scalar $a+ib \in \mathbb{C}$ as $(a+ib)v := av + bJ(v)$ for any $v \in V$.
Let $dim(M) = n$ be the dimension of M, suppose $n = 2k+1$ odd. Then $P_n(\lambda) = det(J - \lambda I)$ is a polynomial of odd degree in the variable λ and it has a real root $\lambda_0 \in \mathbb{R}$. There exists an eigenvector $v \in TM$ such that $Jv = \lambda_0 v$ and $JJv = J(\lambda_0 v) = \lambda_0^2 v \neq -v$ since λ_0 is real. A contradiction since $J^2 = -1$. We conclude that if M admits an almost complex structure it has even dimension.
Indeed starting from a $\mathbb{R}$-vector space V of even dimension $2n$ one can define

$$J(v) = J(v_1, ..., v_n, v_{n+1}, ..., v_{2n}) = (-v_{n+1}, ..., -v_{2n}, v_1, ..., v_n), \ \forall \, v \in V.$$

Note that $J^2 = -Id_V$ and so on every $\mathbb{R}$-vector space of even dimension it is possible to define an almost complex structure. Then on any differentiable manifold M of even dimension it is possible to define point wise an almost complex structure $J_x : T_xM \to T_xM$ for any $x \in M$, but in general these isomorphism do not glue together to define a global isomorphism $J : TM \to TM$. Indeed this fact is equivalent to say that the structure group $GL(2n, \mathbb{R})$ of the tangent bundle TM is reducible to the group $GL(n, \mathbb{C})$.

Definition 1.1. A complex manifold M is a differentiable manifold admitting an open cover $\{\mathcal{U}_\alpha\}$ and coordinate maps $\phi_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^n$ such that $\phi_\alpha \circ \phi_\beta^{-1}$ is holomorphic on $\phi_\beta(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \subseteq \mathbb{C}^n$ for any α, β.

Let X be a complex manifold and let $X_\mathbb{R}$ be the underlying differentiable manifold. We consider a point $x \in X$, the complex tangent space T_xX and the real tangent space $T_xX_\mathbb{R}$. We claim that $T_xX_\mathbb{R}$ is canonically isomorphic to the underlying real vector space of T_xX, in this way T_xX induces a complex structure J_x on $T_xX_\mathbb{R}$. Let $(\phi, \mathcal{U})$ be a complex holomorphic chart for X near x. We have $\phi : \mathcal{U} \to \mathbb{C}^n$ and we get a differentiable chart for $X_\mathbb{R}$ near x defined by $\phi_\mathbb{R} : \mathcal{U} \to \mathbb{R}^{2n}$,

$$\phi_\mathbb{R}(x) = (Re\phi_1(x), Im\phi_1(x), ..., Re\phi_n(x), Im\phi_n(x)).$$

So it suffices to prove the claim above for $T_0\mathbb{C}^n$ and $T_0\mathbb{R}^{2n}$ in $0 \in \mathbb{C}^n$. Let $\{\frac{\partial}{\partial z_j}\}$ for $j = 1, ..., n$ be a basis for $T_0\mathbb{C}^n$, where $\frac{\partial}{\partial z_j} = \frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}$ and $z_j = x_j + iy_j$. The map

$$T_0\mathbb{R}^n \to T_0\mathbb{C}^n, \ (\frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}) \mapsto \frac{\partial}{\partial z_j},$$

defines an isomorphism between $T_0\mathbb{C}^n$ and $T_0\mathbb{R}^{2n}$. So $T_0\mathbb{C}^n$ induces a complex structure on $T_0\mathbb{R}^{2n}$ and more in general T_xX induces a complex structure J_x on $T_xX_\mathbb{R}$. It is easy to see that the complex structure does not depend from the complex chart chosen. We have see that any complex manifold is an almost complex manifold. An almost complex manifold (X, J) is a complex manifold if the almost complex structure J is of the form J_x.
Now we come to the definition of symplectic manifold. Let M be a differentiable manifold. A symplectic form on M is a non-degenerate, skew-symmetric, closed, differential two-form ω, where non-degenerate means that for any $x \in M$ does not exists a non zero $X \in T_xM$ such that $\omega(X, Y) = 0$ for any $Y \in T_xM$, the skew-symmetric condition means that for all $X, Y \in T_xM$ we have $\omega(X, Y) = -\omega(Y, X)$, the closed condition means $d\omega = 0$.

Definition 1.2. A symplectic manifold is a pair (M, ω), where M is a differentiable manifold and ω is a symplectic form on M.

A symplectic manifold has always even dimension. For instance $\mathbb{R}^{2n}$ with the coordinate system $\{x_1, y_1, ..., x_n, y_n\}$ and the two-form $\omega = \sum_{j=1}^{n} dx_j \wedge dy_j$ is a symplectic manifold.
To introduce the concept of Kähler Manifold we begin with the definition of Hermitian form. Let V be a $\mathbb{C}$-vector space of dimension n.

Definition 1.3. A Hermitian form on V is a map $h : V \times V \to \mathbb{C}$ such that

(1) $h(y, x) = \overline{h(x, y)}$, $\forall x, y \in V$,
(2) $h(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 h(x_1, y) + \alpha_2 h(x_2, y)$, $\forall x_1, x_2, y \in V$, $\alpha_1, \alpha_2 \in \mathbb{C}$.

Choosing coordinates on V we can write h in the form $h(x, y) = \sum h_{j,k} x_j \overline{x_k}$, whit $h_{j,k} = \overline{h_{k,j}}$. Viewing V has a $\mathbb{R}$-vector space of dimension $2n$, and writing $h(x, y) = \alpha(x, y) + i\beta(x, y)$, we get two $\mathbb{R}$-bilinear forms with α symmetric and β skewsymmetric. Note that $h(ix, iy) = \alpha(ix, iy) + i\beta(ix, iy)$ and $h(ix, iy) = ih(x, iy) = -i^2 h(x, y) = h(x, y) = \alpha(x, y) + i\beta(x, y)$ and we have $\alpha(ix, iy) = \alpha(x, y)$, $\beta(ix, iy) = \beta(x, y)$. Moreover from $h(ix, y) = \alpha(ix, y) + i\beta(ix, y)$ and $ih(x, y) = i\alpha(x, y) - \beta(ix, y)$, we conclude $\alpha(x, y) = \beta(ix, y)$.
Conversely, if β is any skewsymmetric $\mathbb{R}$-bilinear form such that $\alpha(x, y) = \beta(ix, y)$, then this relation determines the form α uniquely and $h = \alpha + i\beta$ is a Hermitian form. The form $\omega = -\beta$ is called the *associated skewsymmetric bilinear form* of the skewsymmetric form ϕ. We consider ω as an element of $\bigwedge^2 V^*$. If $h(x, y) = \sum h_{j,k} x_j \overline{y_k}$ we have

$$\omega(x, y) = \frac{i}{2} \sum (h_{j,k} x_j \overline{y_k} - \overline{h_{j,k}} \overline{x_j} y_k) = \frac{i}{2} \sum h_{j,k} (x_j \overline{y_k} - y_j \overline{x_k}).$$

In other words , $\omega = \frac{i}{2} \sum h_{j,k} \xi_j \overline{\xi_k}$, where ξ_k is the basis of V^* dual to the chosen basis of V. If the Hermitian form is definite positive there exists a basis of V such that

$$h(x, y) = \sum x_j \overline{x_j}, \omega = \frac{i}{2} \sum \xi_j \wedge \overline{\xi_j}.$$

Let X be a complex manifold and let h be a Hermitian form on X that is a positive Hermitian form h_x on $T_x X$ for any $x \in X$ such that the assignment $x \mapsto h_x$ is holomorphic. In a neighborhood $\mathcal{U}$ with local coordinates $(z_1, ..., z_n)$ we can write $h = \sum h_{j,k} dz_j dz_k$, where the $h_{j,k}$ are complex analytic function of $z_1, ..., z_n, \overline{z_1}, ..., \overline{z_n}$. The form h defines a *Hermitian metric* on X.
It is well known that on a Riemannian Manifold (M, g) there exists a system of coordinates in a neighborhood of any point $p \in M$, called *normal* or *geodesic* coordinates, such that $g_{j,k}(p) = \delta_{j,k}$ and $(\frac{\partial g_{j,k}}{\partial x_i})(p) = 0$. The obstruction to make the metric g flat involves the second partial derivatives of the coefficients of g, this is the *curvature tensor*.
In the complex case the situation is more delicate. We can find a complex analytic change of coordinates such that $h_{j,k}(p) = \delta_{j,k}$. There is an obstruction to find a coordinate system such that $\frac{\partial h_{j,k}}{\partial z_t}(p) = \frac{\partial h_{j,k}}{\partial \overline{z_t}}(p) = 0$, that is already the first derivatives of the coefficients of the Hermitian metric distinguished it from a flat metric. Consider the skewsymmetric form ω associated to the Hermitian form h. This defines a differential 2-form $\omega = \frac{i}{2} \sum h_{j,k} dz_j \wedge \overline{dz_k}$ in local coordinates. The condition $d\omega = 0$ is independent from the local coordinate system chosen and

is a necessary condition for the metric to be flat. We can write explicitly these conditions as

$$\frac{\partial h_{j,k}}{\partial z_t} = \frac{\partial h_{t,k}}{\partial z_j}, \frac{\partial h_{j,k}}{\partial \overline{z_t}} = \frac{\partial h_{j,t}}{\partial \overline{z_k}}.$$

Definition 1.4. A Hermitian metric h on a complex manifold X such that the associated Hermitian form ω is closed i.e. $d\omega = 0$ is called a Kähler metric. A complex manifold with a given Kähler metric is called a *Kähler manifold.*

We have seen that $h = g - i\omega$. Note that on a Kähler manifold g defines a Riemannian metric and ω a symplectic structure. So a Kähler manifold is in particular a complex manifold, a Riemannian manifold and a symplectic manifold.

Example 1.5. *Let $x_0, ..., x_n$ be homogeneous coordinates on $\mathbb{P}^n$. We write $d = d^{'} + d^{''}$, where $d^{'}$ is the differential with respect z_j and $d^{''}$ is the differential with respect $\overline{z_j}$. We consider the affine chart $\mathcal{U}_0 = \{x_0 \neq 0\} \cong \mathbb{A}^n$. Set*

$$H = log\sum\nolimits_{j=1}^{n} |z_j|^2, \omega = id^{'}d^{''}H.$$

Since $d(d^{'}d''h) = 0$ for any function h, the form ω is clearly closed. We have the following coordinate expression

$$\omega = \frac{i}{2}d^{'}d^{''}log\sum\nolimits_{j=1}^{n} |z_j|^2 = \frac{i}{2}d^{'}\frac{\sum z_j d\overline{z_j}}{\sum |z_j|^2} = \frac{i}{2}\frac{\sum dz_j \wedge d\overline{z_j}}{\sum |z_j|^2} - \frac{i}{2}\frac{(\sum \overline{z_j}dz_j) \wedge (\sum z_j d\overline{z_j})}{(\sum |z_j|^2)^2}.$$

In $p = [1 : 0 : ... : 0]$ we have $\omega = \frac{i}{2}\sum_{j=1}^{n} dz_j \wedge \overline{dz_j}$ that is definite positive, by symmetry we see that ω is definite positive on $\mathcal{U}_0$. The associated Hermitian form is $\sum dz_j d\overline{z_j}$. The Kähler metric we have defined is called the Fubini-Study metric on $\mathbb{P}^n$. With this metric $\mathbb{P}^n$ is a Kähler manifold.
Let X be a Kähler manifold and let $Y \subseteq X$ be a complex submanifold. The restriction of differential forms in X to Y takes closed forms in closed forms, any definite positive Hermitian form on X restrict to a definite positive Hermitian form on Y, finally the relation with a Hermitian form and its associated 2-form is preserved by restriction. It follows that any complex submanifold of a Kähler manifold is a Kähler manifold with the restriction of the Kähler metric of X.
In particular if X is a projective variety then it is a Kähler manifold with the metric induced by the Fubini-Study metric on $\mathbb{P}^n$. Clearly the Kähler structure on X is not an intrinsic property but depends from its embedding in $\mathbb{P}^n$.

1.1. **Connections on Holomorphic Vector Bundles.** Let E be a complex vector bundle on a complex manifold M, an hermitian metric on E is an hermitian inner product on each fiber E_x that varies smoothly with $x \in M$. An holomorphic vector bundle equipped with an hermitian metric is called an hermitian vector bundle.

Definition 1.6. A connection on a complex vector bundle E on M is a map

$$\nabla : \Gamma(E) \to \Gamma(E \otimes TM^*),$$

that satisfies the Leibniz's rule

$$\nabla(f \cdot \sigma) = f \cdot \nabla(\sigma) + df \otimes \sigma$$

for any $f \in C^\infty(M)$, $\sigma \in \Gamma(E)$.

Essentially a connection gives a way to differentiate sections of a vector bundle. Let $\mathcal{U}$ be an open subset of M on which E is trivial and let $\{e_1, ..., e_n\}$ be a frame for E on $\mathcal{U}$, we can think e_i as a section of E and write

$$\nabla(e_i) = \sum \theta_{i,j} e_j,$$

where the $\theta_{i,j}$ are section of the cotangent bundle TM^* i.e. 1-forms on M. The matrix $\theta = (\theta_{i,j})$ is called the connection matrix with respect to $\{e_1, ..., e_n\}$. Now if σ is a section of E on $\mathcal{U}$ we can write $\sigma = \sum \sigma_i e_i$, and we have

$$\nabla(\sigma) = \sum d\sigma_i e_i + \sum \sigma_i \nabla(e_i) = \sum\nolimits_j (d\sigma_j + \sum\nolimits_i \sigma_i \theta_{i,j}) e_j.$$

We see that the data $\{e_1, ..., e_n\}$ and $\theta_{i,j}$ determine uniquely the connection ∇. In general there is no canonical connection on a vector bundle E, but if M is complex and E is hermitian we can require two facts that dictate a natural connection.
We denote with TM^* the complex cotangent bundle and with TM^*_h and TM^*_{ah} respectively the holomorphic and antiholomorphic cotangent bundle. Using the decomposition $TM^* = TM^*_h \oplus TM^*_{ah}$ we can write $\nabla = \nabla_h + \nabla_{ah}$, with

$$\nabla_h : \Gamma(E) \to \Gamma^{1,0}(E \otimes TM^*),\ \nabla_{ah} : \Gamma(E) \to \Gamma^{0,1}(E \otimes TM^*).$$

We say that the connection is compatible with the complex structure if $\nabla_{ah} = \overline{\partial}$. If E is hermitian, we say that ∇ is compatible with the metric if $d(s, \sigma) = (\nabla(s), \sigma) + (s, \nabla(\sigma))$
One can prove that if E is an hermitian vector bundle then there exists a unique connection ∇ on E that is compatible with both the metric and the complex structure. This connection is called *metric connection*. In particular if $E = TM$ is the complex tangent bundle on M the unique connection on TM compatible with the metric and the complex structure is called the *Chern connection* and denoted by ∇_C.
Now given a connection ∇ on E we can define an operator

$$\nabla^{p+1} : \Gamma(E) \to \Gamma(\bigwedge\nolimits^p E \otimes TM^*)$$

forcing the Leibniz rule

$$\nabla^{p+1}(\psi \wedge f) = d\psi \otimes f + (-1)^p \psi \wedge \nabla f.$$

for $\psi \in \Gamma(\bigwedge^p E)$, $f \in \Gamma(E)$. In particular consider the operator

$$\nabla^2 : \Gamma(E) \to \Gamma(\bigwedge\nolimits^2 E \otimes TM^*),$$

note that is linear in fact for σ section of E and f smooth function,

$$\nabla^2(f\sigma) = \nabla(df \otimes \sigma + f \cdot \nabla(\sigma)) = -df \wedge \nabla(\sigma) + df \wedge \nabla(\sigma) + f \cdot \nabla^2(\sigma) = f \cdot \nabla^2(\sigma).$$

Again we can write

$$\nabla^2 e_i = \sum \Theta_{i,j} \otimes e_j,$$

where the $\Theta_{i,j}$ are 2-forms. The matrix $\Theta_e = (\Theta_{i,j})$ is called the curvature matrix of ∇ in terms of the frame $\{e_1, ..., e_n\}$. By definition

$$\nabla^2 e_i = \nabla(\sum \theta_{i,j} \otimes e_j) = \sum (d\theta_{i,j} - \sum \theta_{i,k} \wedge \theta_{k,j}) \otimes e_j,$$

and we have the so called *Cartan structure equation*

$$\Theta_e = d\theta_e - \theta_e \wedge \theta_e.$$

2. Line Bundle and Morphism in Projective Spaces

Let M be a compact complex manifold and let L be a holomorphic line bundle on M. Any subspace $W \subseteq \mathbb{P}(H^0(M,\mathcal{O}(L))$ determines a linear system $|W| = \{s\}_{s\in W} \subseteq Div(M)$. Since M is compact $(s) = (s^{'})$ if and only if $s = \lambda s^{'}$ for some $\lambda \in \mathbb{C}^*$, $|W|$ is parametrized by the projective space $\mathbb{P}(W)$.
If the linear system $|W|$ has no base point then for each $p \in W$ the set of sections $s \in W$ vanishing at p forms a Hyperplane $H_p \subseteq \mathbb{P}(W)$, and we have a map

$$\phi : M \rightarrow \mathbb{P}(W)^*,\ p \mapsto H_p.$$

Now we describe this map in a more explicit way. Let $\{s_0,...,s_N\}$ be a basis of W and let $\mathcal{U}$ be an open subset of M and φ a local trivialization of L on $\mathcal{U}$. Let $s_{i,f} = \varphi^*(s_i)$, then $[s_{0,f}(p) : ... : s_{N,f}(p)]$ is independent from the local chart and we denote it by $[s_0(p) : ... : s_N(p)]$. In this terms the map ϕ is given by

$$\phi(p) = [s_0(p) : ... : s_N(p)].$$

Clearly the map ϕ is holomorphic and $L = \phi^*_{|W}(\mathcal{O}_{\mathbb{P}^N}(1))$. If the linear system $|W|$ is very ample i.e. its elements separates points and tangent vectors then the morphism ϕ is an embedding.
A variety $X \subseteq \mathbb{P}^N$ is called *normal* if the linear system on X giving the embedding $X \hookrightarrow \mathbb{P}^N$ is complete i.e. if the restriction map

$$H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(1)) \rightarrow H^0(X,\mathcal{O}_X(1))$$

is surjective.
Let X be an hypersurface in $\mathbb{P}^N$. From the exact sequence of shaves

$$0 \mapsto \mathcal{O}_{\mathbb{P}^N}(H \setminus V) \rightarrow \mathcal{O}_{\mathbb{P}^N}(H) \rightarrow \mathcal{O}_V(H) \mapsto 0,$$

we have an exact sequence in cohomology

$$... \rightarrow H^0(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(H)) \rightarrow H^0(X,\mathcal{O}_X(H)) \rightarrow H^1(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(H \setminus X)) \rightarrow ...$$

but

$$H^1(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(H \setminus X)) = H^1(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(1)\otimes\mathcal{O}_{\mathbb{P}^N}(-d)) = H^1(\mathbb{P}^N,\mathcal{O}_{\mathbb{P}^N}(1-d)) = 0,$$

where $d = deg(X) > 1$, implies that X is normal.
We return to our problem. Let L be a holomorphic line bundle on M and let $\phi_L : M \rightarrow \mathbb{P}^N$ be the map induced by L. First of all L cannot have base points i.e. for any $x \in M$ the map

$$H^0(M,\mathcal{O}_M(L)) \rightarrow L_x,\ s \mapsto s(x)$$

must be surjective. The map ϕ_L will be an embedding if and only if the following two conditions are satisfied.

(1) ϕ_L is injective if and only if for any $x,y \in M$ with $x \neq y$ there exists a section $s \in H^0(M,\mathcal{O}_M(L))$ such that $s(x) = 0$ but $s(y) \neq 0$. In other words if and only if the restriction map

$$H^0(M,\mathcal{O}_M(L)) \rightarrow L_x \otimes L_y,\ s \mapsto s(x)\otimes s(y)$$

is surjective for all $x,y \in M$ with $x \neq y$. In particular if L satisfies this condition it must be base point free.

(2) The map ϕ_L has nonzero differential everywhere. Let ψ_x be a trivialization of L near x and let $v \in T_xM$ be a tangent vector, we require the existence of a section $s \in H^0(M, \mathcal{O}_M(L))$ such that $\overline{s}(x) = 0$ and $d\overline{s}(x) = v$, where $\overline{s} = \phi_L^*(s)$. More intrinsically let $\mathcal{I}_x$ the sheaf of holomorphic function on M vanishing at x and let $\mathcal{I}_x(L)$ the sheaf of section of L vanishing at x. If s is a such section, α, β are trivialization of L near x and $s_\alpha = \alpha^* s$, $s_\beta = \beta^*(s)$, $s_\alpha = g_{\alpha,\beta} s_\beta$, where $g_{\alpha,\beta}$ is the transition function, we have
$$ds_\alpha = ds_\beta \cdot g_{\alpha,\beta} + dg_{\alpha,\beta} \cdot s_\beta = ds_\beta \cdot g_{\alpha,\beta}$$
at x, since $s_\beta(x) = 0$. We have a map
$$d_x : H^0(M, \mathcal{I}_x(L)) \to T_x \otimes L_x.$$
Condition (2) can be stated requiring that the map d_x be surjective for all $x \in M$.

Note that (2) is the limit of (1) when $y \mapsto x$.

3. The Kodaira-Nakano Vanishing Theorem

In this section we denote with M a compact Kähler manifold.

Definition 3.1. A line bundle L on M is *positive* if there exists a metric on L with curvature form Θ such that $\frac{\sqrt{-1}}{2\pi}\Theta$ is a positive $(1,1)$-form; L is *negative* if L^* is positive.

The following proposition gives another characterization of the positivity of a line bundle. First we need a lemma which proof is omitted.

Lemma 3.2. *Let ξ be a (p,q)-form on a compact Kähler manifold such that ξ is d, ∂ or $\overline{\partial}$-exact. Then $\xi = \partial\overline{\partial}\rho$ for some $(p-1, q-1)$-form ρ. Furthermore in the case $p = q$ and ξ real we can take $\sqrt{-1}\cdot\rho$ to be real.*

Proposition 3.3. *Let ω be a real, closed, $(1,1)$-form on M such that $[\omega] = c_1(L) \in H^2_{dR}(M)$. Then there exists a metric connection on L with curvature form $\Theta = \frac{\sqrt{-1}}{2\pi}\omega$. The line bundle L is positive if and only if its Chern class $c_1(L)$ can be represented by a positive form in $H^2_{dR}(M)$.*

Proof. Let $|h|^2$ be a metric on L with curvature form Θ and let $\phi : L_{|\mathcal{U}} \to \mathcal{U} \times \mathbb{C}$ be a trivialization of L over $\mathcal{U}$. Let $\sigma : \mathcal{U} \to L_{|\mathcal{U}} \cong \mathcal{U} \times \mathbb{C}$, $z \mapsto (z, \sigma_{|\mathcal{U}}(z))$ be a section of L over $\mathcal{U}$.Then $|h|^2 = f(z)|\sigma_{|\mathcal{U}}|^2$ where f is a real valued, positive function.
Suppose that L is positive, then we can assume that $\frac{\sqrt{-1}}{2\pi}\Theta$ is a positive $(1,1)$-form. Now the curvature form is given by
$$\Theta = -\partial\overline{\partial} log f(z),$$
then Θ is a real, close $(1,1)$-form. Furthermore we have
$$c_1(L) = [\frac{\sqrt{-1}}{2\pi}\Theta] \in H^2_{dR}(M),$$
and $\frac{\sqrt{-1}}{2\pi}\Theta$ is positive by hypothesis.
Now let $|h'|^2$ another metric on L and let Θ' be its curvature form. Again we have

$|h^{'}|^2 = f^{'}(z)|\sigma_{|\mathcal{U}}|^2$, and locally we can write $f^{'}(z) = e^{\alpha(z)}f(z)$, where α is a real C^{∞} function on M. Then we have $\frac{|h^{'}|^2}{|h|^2} = e^{\alpha}$. For the curvature form $\Theta^{'}$ we have

$$\Theta^{'} = -\partial\overline{\partial}logf^{'} = -\partial\overline{\partial}loge^{\alpha}f = -\partial\overline{\partial}(loge^{\alpha} + logf) = -\partial\overline{\partial}\alpha + \Theta.$$

Then $\Theta = \partial\overline{\partial}\alpha + \Theta^{'}$, so $[\frac{\sqrt{-1}}{2\pi}\Theta] = [\frac{\sqrt{-1}}{2\pi}\Theta^{'}]$.
Conversely suppose that $c_1(L)$ is represented in $H^2_{dR}(M)$ by the real, closed $(1,1)$-form $\frac{\sqrt{-1}}{2\pi}\gamma$. By the preceding argument if we can solve the equation $\Theta = \partial\overline{\partial}\beta + \gamma$ for a real, smooth function β, then the metric $e^{\beta}|\sigma|^2$ on L will have curvature form γ. But this come from lemma 3.2. □

Thanks to this proposition we can give an equivalent definition of positive line bundle.

Definition 3.4. Let L be a holomorphic line bundle on M and let $c_1(L)$ be its Chern class. Then L is said to be positive if there exists a real, closed $(1,1)$-form η such that $\eta \in c_1(L)$ and η is positive.

Let $\Omega^{p,q}(M)$ be the space of (p,q)-forms on M and let ω the Kähler form on M. We define the Lefschetz operator

$$L : \Omega^{p,q}(M) \rightarrow \Omega^{p+1,q+1}(M),\ \xi \mapsto \xi \wedge \omega,$$

and let

$$\Lambda : \Omega^{p,q} \rightarrow \Omega^{p-1,q-1}$$

its adjoint operator.
If $[A,B] = AB - BA$ denotes the commutator of two operators the following identities hold

$$[\Lambda,\overline{\partial}] = -\sqrt{-1}\cdot\partial^*,\ [\Lambda,\partial] = \sqrt{-1}\cdot\overline{\partial}^*,\ [L,\Lambda] = p+q-n,$$

where $n = dim(M)$.
Let E be a vector bundle on M and let $\Omega^{p,q}(E)$ the space of E-valued (p,q)-forms. Consider the operator

$$\Delta = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial} : \Omega^{p,q}(E) \rightarrow \Omega^{p,q}(E).$$

We let

$$\mathcal{H}^{p,q}(E) = Ker(\Delta)$$

be the space of E-valued *harmonic forms*. By harmonic theory we know that

$$H^q(M,\Omega^p(E)) \cong \mathcal{H}^{p,q}(E).$$

Now let $D = D^{'} + D''$, with $D^{''} = \overline{\partial}$, be the metric connection on E. By $[\Lambda,\overline{\partial}] = -\sqrt{-1}\cdot\partial^*$ we get the formula

$$[\Lambda,\overline{\partial}] = -\sqrt{-1}\cdot D^{'*}.$$

Now we come the principal result of this section that will we be fundamental in the proof of Kodaira embedding theorem.

Theorem 3.5. *(Kodaira-Nakano Vanishing Theorem) Let L be a positive line bundle on a compact Kähler manifold M of dimension n. Then*

$$H^q(M,\Omega^p(L)) = 0 \quad \forall\, p+q > n.$$

Proof. By harmonic theory we know that $H^q(M,\Omega^p(L)) \cong \mathcal{H}^{p,q}(L)$, our aim is to show that on M there are not L-valued harmonic forms of degree larger than $n = dim(M)$.
By hypothesis there exists a metric on L such that $\omega = \frac{\sqrt{-1}}{2\pi}\Theta$, where Θ is the curvature form associated to the metric and ω is the Kähler form on M. Now let $\xi \in \mathcal{H}^{p,q}(L)$ an harmonic form, we have $\Theta = D^2 = (D^{'} + \overline{\partial}) \cdot (D^{'} + \overline{\partial}) = \overline{\partial} D^{'} + D^{'}\overline{\partial}$ and since $\overline{\partial}\xi = 0$ we get $\Theta\xi = \overline{\partial} D^{'}\xi$. Now we compute
$2\sqrt{-1}(\Lambda\Theta\xi,\xi) = 2\sqrt{-1}(\Lambda\overline{\partial} D^{'}\xi,\xi) = 2\sqrt{-1}((\overline{\partial}\Lambda - \frac{\sqrt{-1}}{2}D^{'*})D^{'}\xi,\xi) = (D^{'*}D^{'}\xi,\xi) = (D^{'}\xi, D^{'}\xi) \geq 0$, since $(\overline{\partial}\Lambda D^{'}\xi,\xi) = (\Lambda D^{'}\xi, \overline{\partial}^{*}\xi) = 0$.
On the other hand we have
$2\sqrt{-1}(\Theta\Lambda\xi,\xi) = 2\sqrt{-1}(D^{'}\overline{\partial}\Lambda\xi,\xi) = 2\sqrt{-1}(D^{'}(\Lambda\overline{\partial} + \frac{\sqrt{-1}}{2}D^{'*})\xi,\xi) = -(D^{'}D^{'*}\xi,\xi) = -(D^{'*}\xi, D^{'*}\xi) \leq 0$.
Subtracting the two inequalities we get

$$2\sqrt{-1}([\Lambda,\Theta]\xi,\xi) \geq 0.$$

So far we have interpreted the curvature operator $\Theta\eta = \Theta \wedge \eta$ as $D^2\eta$, now we reinterpret it as $\frac{2\pi}{\sqrt{-1}}L(\eta)$. Since $\Theta = \frac{2\pi}{\sqrt{-1}}L$ we have

$$2\sqrt{-1}([\Lambda,\Theta]\xi,\xi) = 4\pi([\Lambda, L]\xi,\xi) = 4\pi(n-p-q)|\xi|^2 \geq 0,$$

and this implies $n - p - q \geq 0$ i.e. $n \geq p + q$. Then $p + q > n$ implies $\xi = 0$. □

Note that dualizing the vanishing theorem, we obtain:

$$H^q(M,\Omega^p(L)) = 0 \quad \forall\, p+q < n$$

if $L \to M$ is a negative line bundle.

4. The Kodaira Embedding Theorem

The embedding theorem was proved by *Kunihiko Kodaira* in 1954 and appeared in his article *"On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)"*, published on *Annals of Mathematics.*
We begin this section recalling briefly some results on functions of several complex variables.

4.1. **Hartog's Theorem.**
Many results in several complex variables come directly from the theory of one complex variable. As instance

- *The Identity Theorem*: If f, g are two holomorphic functions on a connected open set $\mathcal{U} \subseteq \mathbb{C}^n$ such that $f = g$ on a nonempty open subset of $\mathcal{U}$, then $f = g$ on $\mathcal{U}$.
- *The Maximum Principle*: The absolute value $|f|$ of a holomorphic function f on a open set $\mathcal{U}$ has no maximum in $\mathcal{U}$.

However there are some differences the one and many complex variables theory. As instance let

$$\Delta(R) = \{z \in \mathbb{C}^n \mid |z| < R\}$$

be the polydisc of radius R and let $\Delta(R^{'})$ be a smaller polydisc, $R^{'} < R$.

Theorem 4.1. *(Hartog's Theorem) Let f be a holomorphic function in a neighborhood of $\Delta(R) \setminus \Delta(R^{'})$. Then f extends to a holomorphic function on $\Delta(R)$.*

Proof. For simplicity of notation we prove the assertion in the case $n = 2$, the same argument also works in the general case. In a slice $z_1 =$ constant the set $\Delta(R) \setminus \Delta(R^{'})$ looks either like a disk $|z_2| < R$ or like a annulus $R^{'} < |z_2| < R$. Consider the Cauchy formula and set

$$F(z_1, z_2) = \frac{1}{2\pi\sqrt{-1}} \int_{|w_2|=R} \frac{f(z_1, w_2)}{w_2 - z_2} dw_2.$$

The function F is defined on $\Delta(R)$ and since $|w_2| = R$ and $|z_2| < R$ it is holomorphic in z_2. Moreover $\frac{\partial f}{\partial \bar{z}_1} = 0$ implies that F is holomorphic in z_1. Finally by Cauchy formula in the open subset $|z_1| > R^{'}$ of $\Delta(R) \setminus \Delta(R^{'})$ we have $F(z_1, z_2) = f(z_1, z_2)$, so $F_{|\Delta(R)\setminus\Delta(R')} = f$. □

Remark 4.2. Often Hartog's Theorem is applied in the following form:
A holomorphic function on the complement of a finite number of points in an open set $\mathcal{U} \subseteq \mathbb{C}^n$, with $n > 1$, extends to a holomorphic function in all of $\mathcal{U}$.

Theorem 4.3. *(Kodaira Embedding Theorem) Let M be a compact complex manifold and $L \rightarrow M$ be a positive line bundle on M. There exists $\overline{k}$ such that for $k \geq \overline{k}$ the map*

$$\phi_{L^k} : M \rightarrow \mathbb{P}(H^0(M, L^k)^*)$$

defined by L^k is an embedding.

Proof. Our aim is to prove that there exists an integer $\overline{k}$ such that

(1) The restriction map

$$R_{x,y} : H^0(M, \mathcal{O}_M(L^k)) \rightarrow L^k_x \otimes L^k_y,$$

is surjective for all $x \neq y$ in M, $k \geq \overline{k}$.

(2) The differential map

$$d_x : H^0(M, \mathcal{I}_x(L^k)) \rightarrow T_x \otimes L^k_x,$$

is surjective for all $x \in M$, $k \geq \overline{k}$.

Let $\tilde{M} \xrightarrow{\pi} M$ be the blow-up of M at the points $x, y \in M$ and let $E_x = \pi^{-1}(x), E_y = \pi^{-1}(y)$ be the exceptional divisors. We denote $E = E_x + E_y$ and $\tilde{L} = \pi^*(L)$. We assume that $n = dim(M) \geq 2$; in the case $dim(M) = 1$, i.e. M is a Riemann surface, the following argument will work for $\tilde{M} = M$ and $\pi = Id$.
The blow-up map π induces the pullback map

$$\pi^* : H^0(M, \mathcal{O}_M(L^k)) \rightarrow H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k})).$$

Clearly this map is injective, if $s \in H^0(M, \mathcal{O}_M(L^k))$ is such that $\pi^*(s) = \tilde{s} = 0$ then it must be $s = 0$ since M and $\tilde{M}$ are birational via π. Furthermore if $\tilde{s} \in H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k}))$ then the corresponding section s defined a priori on $M \setminus \{x, y\}$ extends by Hartog's theorem to a global section on M. So the morphism π^* is an isomorphism.
By definition $\tilde{L^k}_{|E_x} = E_x \times L^k_x$, $\tilde{L^k}_{|E_y} = E_y \times L^k_y$, and $\tilde{L^k}$ is trivial along E_x and E_y, so $H^0(E, \mathcal{O}_E(\tilde{L^k})) \cong L^k_x \oplus L^k_y$. Consider the restriction map

$$H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k})) \xrightarrow{R_E} H^0(E, \mathcal{O}_E(\tilde{L^k})),$$

and suppose that it be surjective. This means that any section in $H^0(E, \mathcal{O}_E(\tilde{L^k}))$ extends to a section in $H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k}))$, in particular a section that vanishes on E_x

and is everywhere nonzero on E_y extends to a global section on $\tilde{M}$ that determines a global section of L^k on M that vanishes at x but is different from zero at y. The last assertion means that the restriction map $R_{x,y}$ is surjective.
On $\tilde{M}$ we have the following exact sequence of shaves

$$0 \mapsto \mathcal{O}_{\tilde{M}}(\tilde{L^k}) \otimes \mathcal{O}_{\tilde{M}}(-E) \to \mathcal{O}_{\tilde{M}}(\tilde{L^k}) \xrightarrow{R_E} \mathcal{O}_E(\tilde{L^k}) \mapsto 0.$$

We can choose k_1, k_2 such that $L^{k_1} + K_M^*$ is positive on M and $\tilde{L^k} - nE$ is positive on $\tilde{M}$ for $k \geq k_2$, where $n = dim(M) = dim(\tilde{M})$. We denote $\tilde{K}_M = \pi^*(K_M)$ and we have

$$K_{\tilde{M}} = \tilde{K}_M + (n-1)E.$$

For $k \geq \overline{k} = k_1 + k_2$ we have,

$$\mathcal{O}_{\tilde{M}}(\tilde{L^k}) \otimes \mathcal{O}_{\tilde{M}}(-E) = \mathcal{O}_{\tilde{M}}(\tilde{L^k} - E) = \omega_{\tilde{M}}(\tilde{L^k} - E + K_{\tilde{M}}^*) = \omega_{\tilde{M}}((\tilde{L^{k_1}} + \tilde{K_M^*}) + (\tilde{L^h} - nE)),$$

where $h \geq k_2$.
Now $L^{k_1} + K_M^*$ is positive on M and $\tilde{L^h} - nE$ is positive on $\tilde{M}$. Then $\tilde{L^{k_1}} + K_{\tilde{M}}^* + \tilde{L^h} - nE$ is positive on $\tilde{M}$ and by Kodaira vanishing theorem we have,

$$H^1(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k} - E)) = H^1(\tilde{M}, \Omega^n_{\tilde{M}}(\tilde{L^{k_1}} + \tilde{K_M^*} + \tilde{L^h} - nE)) = 0.$$

From the above exact sequence we obtain the following sequence in cohomology,

$$0 \mapsto H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k} - E)) \to H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k})) \xrightarrow{R_E} H^0(E, \mathcal{O}_E(\tilde{L^k})) \mapsto 0,$$

and the map $H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k})) \xrightarrow{R_E} H^0(E, \mathcal{O}_E(\tilde{L^k}))$ is surjective.
Now we have to prove (2), let $\tilde{M} \xrightarrow{\pi} M$ be the blow-up of M at the point $x \in M$ and let $E = E_x = \pi^{-1}(x)$ be the exceptional divisor. Again we have an isomorphism given by the pullback map

$$\pi^* : H^0(M, \mathcal{O}_M(L^k)) \to H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k})).$$

If $s \in H^0(M, \mathcal{O}_M(L^k))$ then $s(x) = 0$ if and only if $\tilde{s} = \pi^* s$ vanishes on E, and π^* restricts to an isomorphism

$$\pi^* : H^0(M, \mathcal{I}_x(L^k)) \to H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k} - E)).$$

Again we have

$$H^0(E, \mathcal{O}_E(\tilde{L^k} - E)) = L_x^k \otimes H^0(E, \mathcal{O}_E(-E)) \cong L_x^k \otimes T_x.$$

So to prove that $d_x : H^0(M, \mathcal{I}_x(L^k)) \to T_x \otimes L_x^k$, is surjective is equivalent to prove that $H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k} - E)) \xrightarrow{R_E} H^0(E, \mathcal{O}_E(\tilde{L^k} - E))$ is surjective.
Note that on $\tilde{M}$ we have the exact sequence of shaves

$$0 \mapsto \mathcal{O}_{\tilde{M}}(\tilde{L^k} - 2E) \to \mathcal{O}_{\tilde{M}}(\tilde{L^k} - E) \xrightarrow{R_E} \mathcal{O}_E(\tilde{L^k} - E) \mapsto 0.$$

Again We can choose k_1, k_2 such that $L^{k_1} + K_M^*$ is positive on M and $\tilde{L^k} - (n+1)E$ is positive on $\tilde{M}$ for $k \geq k_2$. So for $k \geq k_1 + k_2$ we have

$$\mathcal{O}_{\tilde{M}}(\tilde{L^k} - 2E) = \omega_{\tilde{M}}((\tilde{L^{k_1}} + \tilde{K}_M^*) + (\tilde{L^k} - (n+1)E)),$$

and by Kodaira vanishing theorem we have $H^1(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k} - 2E)) = 0$. Considering the cohomology sequence we see that the map $H^0(\tilde{M}, \mathcal{O}_{\tilde{M}}(\tilde{L^k} - E)) \xrightarrow{R_E} H^0(E, \mathcal{O}_E(\tilde{L^k} - E))$ is surjective.
Clearly if ϕ_L is defined at $x, y \in M$ with $\phi_L(x) \neq \phi_L(y)$ the same holds for any $\overline{x}$

in a neighborhood of x and $\overline{y}$ in a neighborhood of y. Furthermore if ϕ_L is smooth at x then it is smooth in any $\overline{x}$ near x and separate points $\overline{x} \neq \tilde{x}$. Then there is an open covering $\{\mathcal{U}_x\}$ of M such that ϕ_L is an embedding on $\mathcal{U}_x$ for any x, since M is compact we can find a finite open subcovering and the theorem is proved. □

An immediate consequence of the Kodaira Theorem is the following proposition.

Proposition 4.4. *A compact complex manifold M is embeddable in a projective space if and only if M is a Hodge manifold i.e. there exists a close, positive $(1,1)$-form ω on M whose cohomology class $[\omega]$ is rational.*

Proof. We have $[\omega] \in H^2(M,\mathbb{Q})$, then $[k\omega] \in H^2(M,\mathbb{Z})$ for some k. Consider the exponential exact sequence

$$0 \mapsto \mathbb{Z} \xrightarrow{i} \mathcal{O}_M \xrightarrow{exp} \mathcal{O}_M^* \mapsto 0.$$

In cohomology we have

$$... \rightarrow H^1(M,\mathcal{O}_M^*) \xrightarrow{f} H^2(M,\mathbb{Z}) \xrightarrow{i_*} H^2(M,\mathcal{O}_M) \rightarrow ...$$

and $i_*([k\omega]) = 0$. So $[k\omega] \in Ker(i_*) = Im(f)$ and since $H^1(M,\mathcal{O}_M^*) = Pic(M)$ there exists a line bundle L on M whose first Chern class is $c_1(L) = [k\omega]$, then by proposition 3.3 the line bundle L is positive and by Kodaira embedding theorem we conclude.
Conversely if $M \subseteq \mathbb{P}^n$ is embedded in a projective space then the restriction of the Fubini-Study metric of $\mathbb{P}^n$ on M gives a Kähler metric on M whose associated Kähler form is an integral, positive and closed $(1,1)$-form on M. □

Remark 4.5. By Kodaira's Theorem any compact Hodge manifold X admits an embedding in a projective space $X \subseteq \mathbb{P}^N$. If $n = dim(X)$, the dimension of the first secant variety of X is

$$expdim(Sec(X)) = 2n+1, \quad dim(Sec(X)) \leq 2n+1.$$

Then by iterated projections we can embed any Hodge manifold of dimension n in $\mathbb{P}^{2n+1}$.

Corollary 4.6. *Any compact Riemann Surfaces M is embeddable in $\mathbb{P}^3$ as a projective curve.*

Proof. Let ω be any positive $(1,1)$-form on M, then $d\omega = 0$ since it is a form of degree 3 on a manifold of real dimension 2. So ω is a Kähler form. We multiply ω by a constant so that its volume is normalized $\int_M \omega = 1$. Then $[\omega] \in H^2_{dR}(M)$ is an integral cohomology class and by Kodaira's theorem M can be embedded in $\mathbb{P}^n$. By remark 4.5 we conclude. □

Remark 4.7. Let ω, η be closed, integral, positive $(1,1)$-forms on the compact complex manifolds M,N respectively and let $p_M : M \times N \rightarrow M$, $p_N : M \times N \rightarrow N$ be the projections. Then $p_M^*\omega + p_N^*\eta$ is a closed, integral, positive $(1,1)$-form on $M \times N$. We conclude that if M,N are embeddable in a projective space then also $M \times N$ is.
Let M be an algebraic variety and let $\tilde{M} \xrightarrow{\pi} M$ be the blowup of M at a point $x \in M$. In the proof of the embedding theorem we have seen that if L is a positive line bundle on M and $E = \pi^{-1}(x)$ is the exceptional divisor then $\pi^*L^k - E$ is a positive line bundle on $\tilde{M}$ for $k \gg 0$. We conclude that also $\tilde{M}$ is an algebraic variety.

4.2. **Riemann Surfaces.** Let M be a connected, compact Riemann surface i.e. a compact complex manifold of complex dimension one. By corollary 4.6 M can be embedded in $\mathbb{P}^3$. In the one dimensional case there is a simpler proof that does not use the Kodaira embedding theorem.

Lemma 4.8. *Let L be a line bundle on M and D the divisor associated to L. Then*

- *the complete linear system $|D|$ induces a morphism of M in a projective space if and only if $dim|D-P| = dim|D| - 1$ for any point $P \in M$;*
- *the linear system D induces an embedding of M in a projective space if and only if $dim|D-P-Q| = dim|D| - 2$ for any $P, Q \in M$ including the case $P = Q$.*

Proof. Consider the sheaves $\mathcal{O}_M(D-P)$, $\mathcal{O}_M(D)$ and $k(P)$, where $k(P)$ is the skyscraper sheaf on P. We have the exact sequence

$$0 \mapsto \mathcal{O}_M(D-P) \to \mathcal{O}_M(D) \to k(P) \mapsto 0.$$

This sequence induces the following sequence in cohomology

$$0 \mapsto H^0(M, \mathcal{O}_M(D-P)) \to H^0(M, \mathcal{O}_M(D)) \to k,$$

and we have only two possibilities $dim|D-P| = dim|D|$ or $dim|D-P| = dim|D|-1$. The map

$$\phi : |D-P| \to |D|, \ E \mapsto E + P$$

is injective. Then $dim|D-P| = dim|D|$ if and only if ϕ is surjective if and only if P is a base point of $|D|$. This prove the first assertion.

To prove the second assertion we have to verify that $|D|$ separates point and tangent vectors. Now $|D|$ separates points is equivalent to say that for any $P, Q \in M$, Q is not a base point of $|D-P|$ and by the first part of the proof this is equivalent to say that

$$dim|D-P-Q| = dim|D| - 2.$$

The fact that $|D|$ separates tangent vectors means that for any point $P \in M$ there exists a divisor $E \in |D|$ such that P occurs with multiplicity one in E, in fact $dimT_PE = 0$ if P has multiplicity one in E and $dimT_PE = 1$ if $m_PE \geq 2$. This is equivalent to say that P is not a base point of $|D-P|$ and again by the first part we have

$$dim|D-2P| = dim|D| - 2.$$

□

Proposition 4.9. *Let L be a line bundle on a Riemann Surface M of genus g. If $deg|L| \geq 2g+1$ then L induces an embedding of M in $\mathbb{P}(H^0(M, \mathcal{O}_M(L))^*)$.*

Proof. Let D be the divisor associated to L. Since $deg(K-D) \leq 2g-2-2g-1 = -3 < 0$ and $deg(K-(D-P-Q)) \leq 2g-2-(2g+1-2) = -1 < 0$ we have $h^0(K-D) = h^0(K-(D-P-Q)) = 0$. By Riemann-Roch theorem on D and $D-P-Q$ we have

$$h^0(D) = deg(D) - g + 1, \ h^0(D-P-Q) = deg(D-P-Q) - g + 1 = deg(D) - g - 1.$$

Comparing the two equalities we get $h^0(D) - h^0(D-P-Q) = 2$ that is

$$dim|D-P-Q| = dim|D| - 2,$$

the assertion follows from the preceding lemma. □

We have seen that every compact Riemann Surface is projective, in fact in order to embed M in a projective space we can choose any divisor D on M of degree greater or equal to $2g+1$, where g is the genus of M. As instance $D = \sum_{i=1}^{2g+1} P_i$, where P_i are $2g+1$ distinct points on M.
Now we can see a Riemann Surface as a curve $X \subseteq \mathbb{P}^n$. Now the Secant Variety of X has expected dimension

$$expdim(Sec(X)) = 2 \cdot dim(X) + 1 = 3,$$

so its effective dimension is $dim(Sec(X)) < 3$. This means that through the general point of $\mathbb{P}^n$ with $n > 3$ there are no secant lines to X, and so the projection of X centered in the general point of $\mathbb{P}^n$ is an isomorphism.
Let P_h be a general point of $\mathbb{P}^h$, we have a sequence of projections

$$X \subseteq \mathbb{P}^n \xrightarrow{\pi_{P_n}} X_{n-1} \subseteq \mathbb{P}^{n-1} \xrightarrow{\pi_{P_{n-1}}} ... \xrightarrow{\pi_{P_4}} X_3 \subseteq \mathbb{P}^3.$$

By a sequence of projections we can embed any Riemann Surfaces in $\mathbb{P}^3$.

4.3. **Hopf Manifolds.** We conclude giving two examples of non-projective complex manifolds.
Let $M = \mathbb{C}^n \setminus \{0\}$ and let $\lambda \in \mathbb{R}$, $\lambda > 0$. Let G be the group of transformations of the form

$$M \to M,\ (z_1, ..., z_n) \mapsto (\lambda^h z_1, ..., \lambda^h z_n),\ h \in \mathbb{Z}.$$

Clearly any point $z \in M$ can be written as

$$z = R^z v^z,\ R^z > 0,\ v^z = (v_1^z, ..., v_n^z) \mid |v_1^z|^2 + ... + |v_n^z|^2 = 1,$$

and this representation is unique, we get an isomorphism

$$M \to \mathbb{R}_+ \times S^{2n-1},\ z \mapsto (R^z, (v_1^z, ..., v_n^z)).$$

In this representation G acts trivially on S^{2n-1} and on $\mathbb{R}_+$ multiplying by a power of λ. Consider the function $log : \mathbb{R}_+ \to \mathbb{R}$, since $log(r\lambda^h) = log(r) + h \cdot log(\lambda)$ the action of G becomes the translation by vectors of the lattice $\mathbb{Z}_\alpha$ where $\alpha = log(\lambda)$. From this it is clear that G acts freely and discretely on M and so M/G is a complex manifold homeomorphic to $\mathbb{R}_+/\mathbb{Z} \times S^{2n-1} \cong S^1 \times S^{2n-1}$, called Hopf Manifold.
Recall that on a Kähler manifold X we have the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

it follows that $b_k = \sum_{p+q=k} h^{p,q}(X)$, where b_k is the $k-th$ Betti number of X. Since the complex structure is compatible with the differentials we have $H^{p,q}(X) \cong \overline{H^{q,p}(X)}$ and $h^{p,q} = h^{q,p}$. If $k = 2m+1$ is odd it follows that

$$b_k = \sum_{p+q=k} h^{p,q}(X) = 2\sum_{p \leq m} h^{p,2m+1-p}(X).$$

Then the odd dimensional Betti number of a Kähler manifold are even.
Return to our Hopf Manifold $M/G \cong S^1 \times S^{2n-1}$ and consider as instance the case $n = 2$. Recall the Künneth theorem

Theorem 4.10. *(Künneth) Let X, Y be topological spaces and let $\mathcal{F}$ be a sheaf of abelian groups defined on X and Y. Then for any integer k we have*

$$H^k(X \times Y, \mathcal{F}) \cong \bigoplus_{i+j=k} H^i(X, \mathcal{F}) \otimes H^j(Y, \mathcal{F}).$$

By Künneth formula on $b_3(M/G) = b_3(S^1 \times S^3)$ we have

$$b_3(M/G) = b_0(S^1) \cdot b_3(S^3) = 1.$$

So for $X = (\mathbb{C}^2 \setminus \{0\})/G$ the Betti number b_3 is odd and X can not be a Kähler manifold, in particular it can not be projective. More generally any Hopf Manifold is a non-projective complex manifold.

4.4. **K3 Surfaces.** Let S be a $K3$ surface i.e. a complex compact surface with irregularity $q(S) = 0$ and trivial canonical bundle $\omega_S = \mathcal{O}_S$, so its geometric genus and its arithmetic genus are equal $p_a = p_g = 1$.
Andre Weil named them in honor of three Algebraic Geometers, Kummer, Kähler and Kodaira, and "La belle Montagne K2 au Cachemire" (The beautiful Mountain K2 in Kashmir).
By Riemann-Roch theorem we know that if D is any divisor on a surface X then

$$h^0(D) - h^1(D) + h^0(K - D) = \frac{1}{2} D \cdot (D - K) + 1 + p_a.$$

On the structure sheaf $\mathcal{O}_S$ of S we have

$$h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^0(\mathcal{O}_S^*) = 2,$$

now $h^0(\mathcal{O}_S) = h^2(\mathcal{O}_S) = 1$ since S is compact, so $h^1(\mathcal{O}_S) = 0$. We conclude that the Euler holomorphic characteristic of S is

$$\chi_{hol}(S) = h^0(\mathcal{O}_S) - h^1(\mathcal{O}_S) + h^2(\mathcal{O}_S) = 2.$$

Every compact analytic manifold X has a moduli space parametrizing small deformation of X. By infinitesimal deformation theory we know that if $H^2(X, T_X) = 0$ then the moduli space is smooth and of dimension $h^1(T_X)$. Our aim is to compute $h^1(\mathcal{O}_S)$ for a $K3$ surface S.
Note that for a $K3$ surface $H^2(S, T_S) = H^2(S, \Omega_S^*) \cong H^0(S, \Omega_S)^* = 0$ since $H^1(S, \Omega_S) \cong H^1(S, \mathcal{O}_S) = 0$ and $h^{1,0} = h^{0,1}$.
By Dolbeault theorem we have $H^1(S, T_S) = H^1(S, \Omega_S^*) \cong H^{1,1}(S)$. We want to compute $H^{1,1}(S)$. By Hodge decomposition

$$H^2(S, \mathbb{C}) = H^{2,0}(S, \mathbb{C}) \oplus H^{1,1}(S, \mathbb{C}) \oplus H^{0,2}(S, \mathbb{C}),$$

furthermore $H^{2,0}(S) \cong H^0(S, \Omega_S^2) \cong \mathbb{C}$ implies $h^{2,0}(S) = h^{0,2}(S) = 1$. By Noether's formula we know that

$$\chi_{hol}(\mathcal{O}_S) = \frac{1}{12}(c_1(K)^2 + \chi_{top}(S)),$$

in our case the canonical bundle is trivial, so $2 = \chi_{hol}(\mathcal{O}_S) = \frac{1}{12}\chi_{top}(S)$ implies $\chi_{top}(S) = 24$. On the other hand

$$\chi_{top}(S) = b_0(S) - b_1(S) + b_2(S) - b_3(S) + b_4(S).$$

Since S is compact and simply connected for the homology $H_1(S, \mathbb{Z}) = Ab(\Pi_1(S)) = 0$, and by duality $H^1(S, \mathbb{C}) = 0$. By Poincarè duality $H^3(S, \mathbb{C}) = 0$. Since S is compact $H^0(S, \mathbb{C}) = H^2(S, \mathbb{C}) = \mathbb{C}$. Then

$$24 = \chi_{top}(S) = 1 + b_2(S) + 1,$$

implies $b_2(S) = dim H^2(S, \mathbb{C}) = 22$. We conclude that $h^{1,1}(S) = 20$ and the $K3$ surfaces are parametrized by a moduli space of dimension 20.

Now suppose that S has a positive line bundle L, the by Kodaira embedding theorem it is projective. Let C be the divisor associated to L. By adjunction formula

$$2g-2 = C\cdot(C+K)$$

where g is the genus of the curve C and K is the canonical divisor of S. So $2g-2=C^2$, we fix $g=3$ and we have $2g-2=C^2=4$. By Riemann-Roch theorem

$$h^0(C) = \frac{1}{2}C^2+1+p_a = 3+1 = 4.$$

This means that L induces an embedding of S in $\mathbb{P}^3$ as a surface of degree 4 and that any projective $K3$ surface that admits a line bundle L whose associated divisor is a curve of genus $g=3$ can be realized as a smooth quartic in $\mathbb{P}^3$. We estimate the dimension of the moduli space of $K3$ surface of this type doing some heuristic calculation.

The projective space $\mathbb{P}(H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(4))$ parametrizing homogeneous polynomials of degree 4 in 4 variables, has dimension $\binom{4+3}{3}-1=34$. We subtract the automorphism of $\mathbb{P}^3$, the space $PGL(3)$ has dimension $4\cdot4-1=15$. We conclude that our moduli space has dimension $34-15=19$.

We conclude that the projective $K3$ surface are a subvariety of dimension 19 of the moduli space parametrizing the $K3$ surface which has dimension 20 and the generic $K3$ surface in non-projective. Indeed *Siu* in 1983 showed that all complex $K3$ surfaces are Kähler manifolds.

References

[Be] A. Beauville, *Complex Algebraic Surfaces.* London Mathematical Society Texts 34.
[BT] R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology.* Springer.
[De] J.P. Demailly, *Complex Analytic and Differential Geometry.*
[GH] P. Griffiths, J. Harris, *Principles of Algebraic Geometry.* Wiley Classics Library.
[Ha] R. Hartshorne, *Algebraic Geometry.* Springer.
[RW] O. Raymond, J. Wells, *Differential Analysis on Complex Manifolds.* Springer.

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