Some Connections between Representation Theory and Complex Geometry

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Introduction

We show how to associate a flat connection over a vector bundle E on a connected complex manifold X to a local system on X. Furthermore we associate to a representation of the fundamental group $\pi_1(X, x_0)$ a local system. Then we close the circle showing a bijective correspondence between local system, representations of the fundamental group and flat connections. Finally we show that there is a bijective correspondence between Hodge structure on a $\mathbb{Q}$-vector space H and representations of $\mathbb{C}^*$ over $H_{\mathbb{R}} = H \otimes_{\mathbb{Q}} \mathbb{R}$.

CHAPTER 1

Local Systems, Representations and Flat Vector Bundles

Let X be a connected complex manifold.

DEFINITION 1.1. A local system on X is a locally constant sheaf $\mathcal{F}$ on X, i.e. for any $x \in X$ there is an open neighborhood $\mathcal{U}_x$ of x such that $\mathcal{F}_{|\mathcal{U}_x}$ is the constant sheaf.

Recall that a sheaf $\mathcal{E}$ on X is constant if for any $x \in X$ the stalk $\mathcal{F}_x$ is isomorphic to $\mathbb{C}^n$ for a fixed n.
Let $\mathcal{L} \to X$ be a local system on X, and fix a base point $x_0 \in X$. Let

$$\gamma : [0,1] \to X,\ \gamma(0) = x_0,\ \gamma(1) = x_1,$$

be a curve in X. The pull-back $\gamma^*\mathcal{L}$ to $[0,1]$ is locally constant and hence constant. In this way we get a $\mathbb{C}$-vector space isomorphism

$$f_\gamma : \mathcal{L}_{x_0} \to \mathcal{L}_{x_1},$$

which depends only on the homotopy class of γ. If we take a loop based at x_0 we get a map

$$\rho : \pi_1(X, x_0) \to GL(\mathcal{L}_{x_0}) \cong GL(n, \mathbb{C}),$$

that is a group homomorphism and hence defines a representation of $\pi_1(X, x_0)$ on $\mathcal{L}_{x_0}$.

1.1. Flat Bundles

Let $E \to X$ be a holomorphic vector bundle. A holomorphic connection on E is a $\mathbb{C}$-linear map

$$\nabla : \mathcal{O}_E(U) \to \Omega^1(U) \otimes \mathcal{O}_E(U) = \Omega^1_E(U),$$

where U is an open subset of X, and such that

$$\nabla(f \cdot \sigma) = df \otimes \sigma + f \cdot \nabla\sigma, \quad f \in \mathcal{O}(U),\ \sigma \in \mathcal{O}_E(U).$$

If $\sigma_1, ..., \sigma_d$ is a local holomorphic coframe of $\mathcal{O}_E(U)$, we have

$$\nabla\sigma_j = \sum_{i=1}^{d} \theta_{i,j}\sigma_i,$$

the holomorphic forms $\theta_{i,j} \in \Omega^1(U)$ are called connection forms.

DEFINITION 1.2. Let $E \to X$ be a holomorphic vector bundle with a holomorphic connection ∇. A section $\sigma \in \mathcal{O}_E(U)$ is said to be flat if $\nabla\sigma = 0$. The connection ∇ is said to be flat if there exists a trivializing cover for E such that the corresponding coframe consists of flat section.

1.1.1. Locally free Shaves and Vector Bundles. Let $E \to X$ be a holomorphic vector bundle of rank n on a complex manifold X. For any open subset $U \subseteq X$ we can consider the $\mathcal{O}_X(U)$-module $\Gamma(U, E)$ of holomorphic section of E on U. The assignment $U \mapsto \Gamma(U, E)$ gives a sheaf of $\mathcal{O}_X$-module on X that is locally free of rank n.
Conversely let $\mathcal{F}$ be a locally free sheaf of rank n on X. Let $\{U_i\}$ be an open cover of X on which $\mathcal{F}$ trivializes. For each non trivial intersection $U_i \cap U_j = U_{i,j}$ we have two isomorphisms $\mathcal{F}_{U_i} \to \mathcal{O}_X^{\oplus n}$ and $\mathcal{F}_{U_i} \to \mathcal{O}_X^{\oplus n}$ and by restriction two different isomorphisms $g_i : \mathcal{F}_{U_i} \to \mathcal{O}_X^{\oplus n}$ and $g_j : \mathcal{F}_{U_j} \to \mathcal{O}_X^{\oplus n}$. Then we have an automorphism $g_{i,j} = g_j g_i^{-1}$ of $\mathcal{F}_{U_{i,j}} \cong \mathcal{O}_{X|U_{i,j}}^{\oplus n}$. Then we can identify $g_{i,j}$ with an $n \times n$ matrix with regular functions as entries. We construct a vector bundle

$$E = \bigcup U_i \times \mathbb{C}^n / \sim,$$

where $(x, v) \sim (y, w) \iff x = y \in U_{i,j}$, $w = g_{i,j}(x)(v)$. Then E is a vector bundle of rank n on X. Note that the transition functions of E arise from a coframe of $\mathcal{F}$. We have established the following bijective correspondence that is actually an equivalence of categories.

$$\begin{pmatrix} Rank\, n\, Vector\, Bundle\, on\, X \\ up\, to\, isomorphism \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} Locally\, free\, of rank\, n\, on\, X \\ up\, to\, isomorphism \end{pmatrix}.$$

CHAPTER 2

The Equivalence Theorem

Our aim is to prove that there is a bijective correspondence

$$\left(\begin{array}{c} \textit{Local Systems on X} \\ \textit{up to isomorphism} \end{array} \right) \leftrightsquigarrow \left(\begin{array}{c} \textit{Representations of } \pi_1(X) \\ \textit{up to isomorphism} \end{array} \right).$$

A subset K of X is said to be good for the sheaf $\mathcal{L}$ if it is connected and there is an open subset $\mathcal{U}$ containing K such that $\mathcal{F}_{\mathcal{U}}$ is constant.

LEMMA 2.1. *Let K be a good set for $\mathcal{L}$, and let $x_1, ..., x_n \in K$ be points. Then there is a natural isomorphism $\mathcal{L}_{x_i} \to \mathcal{L}_{x_j}$ for any $i, j = 1, ..., n$. Furthermore for any i, j, k the composition $\mathcal{L}_{x_i} \to \mathcal{L}_{x_j} \to \mathcal{L}_{x_k}$ coincides with $\mathcal{L}_{x_i} \to \mathcal{L}_{x_k}$.*

PROOF. Let $\mathcal{U}$ be an open subset containing K on which $\mathcal{L}$ is constant. Then the natural maps $\mathcal{L}(\mathcal{U}) \to \mathcal{L}_{x_i}$, $s \to s_{x_i}$ are all isomorphisms. So we get isomorphisms $\mathcal{L}_{x_i} \to \mathcal{L}(\mathcal{U}) \to \mathcal{L}_{x_j}$ that clearly are compatible. ♠

THEOREM 2.2. *There is a bijective correspondence between the set of local systems on X up to isomorphism and the set of representations of $\pi_1(X)$ up to isomorphism.*

PROOF. We dived the proof in two steps.

(1) (*Local System $\to$ Representation*) Let $\gamma : [0,1] \to X$ be a curve, $\gamma(0) = x_0$, $\gamma(1) = x_1$. We can cover X by good open sets $\mathcal{U}_0, ..., \mathcal{U}_n$ such that each $\mathcal{U}_i \cap \mathcal{U}_| \neq \emptyset$ for any i, j. Furthermore we can find a partition $0 \leq a_0 < a_1 < ... < a_n \leq 1$ such that $\gamma([a_i, a_{i+1}]) \subseteq \mathcal{U}_i$. Then for any i the pullback of $\mathcal{L}$ to $[a_i, a_{i+1}]$ is locally constant and hence constant. So we get an isomorphism $\mathcal{L}_{\gamma(a_i)} \to \mathcal{L}_{\gamma(a_{i+1})}$. By composing these isomorphisms we get an ρ_γ isomorphism $\mathcal{L}_{\gamma(0)} \to \mathcal{L}_{\gamma(1)}$.
The isomorphism ρ_γ appears to depend from the partition $0 \leq a_0 < a_1 < ... < a_n \leq 1$. But if we add a new point b say between a_i and a_{i+1}, by lemma 2.1 the composition $\mathcal{L}_{a_i} \to \mathcal{L}_b \to \mathcal{L}_{a_{i+1}}$ coincides with $\mathcal{L}_{a_i} \to \mathcal{L}_{a_{i+1}}$. So our construction does not change.
Suppose now that γ and $\gamma^{'}$ are homotopic curves in X, and let $H : [0,1] \times [0,1] \to X$ be an homotopy. Then $H(t,0) = \gamma(t)$ and $H(t,1) = \gamma^{'}(t)$. Let $0 = a_1 < ... < a_n = 1$ and $0 = b_1, ..., b_m = 1$ be partitions of $[0,1]$ such that $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$ is good. Let $\gamma_j : [0,1] \to X$ be the curve $\gamma_j(t) = H(t, b_j)$, in particular $\gamma_0 = \gamma$ and $\gamma_m = \gamma^{'}$. To prove that $\rho(\gamma) = \rho(\gamma^{'})$ it suffices to prove that $\rho(\gamma_j) = \rho(\gamma_{j+1})$ for any j. Consider the following diagram

$$\begin{array}{ccccccccc}
\mathcal{L}_{|H(0,b_{j+1})} & \to & \mathcal{L}_{|H(a_1,b_{j+1})} & \to & \mathcal{L}_{|H(a_2,b_{j+1})} & \to & \cdots\cdots & \to & \mathcal{L}_{|H(1,b_{j+1})} \\
\| & & \updownarrow & & \updownarrow & & & & \| \\
\mathcal{L}_{|H(0,b_j)} & \to & \mathcal{L}_{|H(a_1,b_j)} & \to & \mathcal{L}_{|H(a_2,b_j)} & \to & \cdots\cdots & \to & \mathcal{L}_{|H(1,b_j)}
\end{array}$$

Since $H([a_i, a_{i+1}] \times [b_j, b_{j+1}])$ is good for any i, j, any small square diagram commutes. Furthermore $\rho(\gamma_j)$ is the composition of all the maps along the bottom and $\rho(\gamma_{j+1})$ is the composition of all the maps along the top, we conclude that $\rho(\gamma_j) = \rho(\gamma_{j+1})$.

In particular for $x_0 = x_1$ for any loop γ bases on x_0 we get a well defined linear map $\rho(\gamma) : \mathcal{L}_{x_0} \to \mathcal{L}_{x_0}$. Clearly the map

$$\rho : \pi(X, x_0) \to GL(\mathcal{L}_{x_0}) \cong GL(n, \mathbb{C}),\ \gamma \mapsto \rho(\gamma),$$

is a morphism of groups, and thus defines a representation of the fundamental group $\pi_1(X, x_0)$ with representation space $\mathcal{L}_{x_0}$.

(2) (*Representation $\to$ Local System*) Let $\rho : \pi_1(X, x_0) \to GL(n, \mathbb{C})$ be a representation of the fundamental group. Let $F : \tilde{X} \to X$ be the universal covering of X. The fundamental group acts on $\tilde{X}$ by deck transformations. We can define a holomorphic vector bundle by

$$E = \tilde{X} \times \mathbb{C}^n / \sim,$$

where the relation $\sim$ is given by

$$(\tilde{b}_1, v_1) \sim (\tilde{b}_2, v_2) \Longleftrightarrow b_2 = \sigma(b_1),\ v_2 = \rho(\sigma^{-1})(v_1), \quad \sigma \in \pi_1(X, x_0).$$

In other words, E is the vector bundle associated the the principal bundle over X with structure group $\pi_1(X, x_0)$ by the representation ρ. Let $\mathcal{U} \subseteq X$ be an open subset such that $F^{-1}(\mathcal{U})$ is a disjoint union of open sets $W_j \subseteq \tilde{X}$ biholomorphic to $\mathcal{U}$. Let us denote by $F_j = F_{|W_j}$. Given any vector $v \in \mathbb{C}^n$, for any choice of j we have a local section

$$\bar{v}(z) = (F_j^{-1}(z), v),\ z \in \mathcal{U},$$

on $\mathcal{U}$. We call $\bar{v}$ a constant section of the bundle E. We denote by $\mathcal{L}$ the sheaf of locally constant section E. Clearly $\mathcal{L}$ is a locally constant sheaf i.e. a local system.

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EXAMPLE 2.3. Let $X = \Delta^* = \{z \in \mathbb{C} \mid 0 < |z| < r\}$, and suppose, for simplicity that we have scaled the variable so that $r > 1$. For $t_0 = 1 \in \Delta^*$ we have $\pi_1(\Delta^*, t_0) \cong \mathbb{Z}$, after choosing as generator a loop around the origin and oriented clockwise. Consider the representation

$$\rho : \pi_1(\Delta^*, t_0) \to GL(2, \mathbb{C}), \quad \rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}.$$

Recalling the upper half plane H is the universal covering of Δ^*, we have $E \cong H \times \mathbb{C}^2 / \sim$.

2.1. Local Systems and Flat Bundles

In this section we establish a bijective correspondence

$$\begin{pmatrix} \text{Local Systems on } X \\ \text{up to isomorphism} \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} \text{Flat Bundles on } X \\ \text{up to isomorphism} \end{pmatrix}.$$

Suppose that $E \to X$ is a vector bundle associated to a local system $\mathcal{L}$. Then E admits a trivializing covering relative to which the transition function are constant. Let $\sigma_1, ..., \sigma_d$ be the coframe arising from the trivializing cover, since $\mathcal{L}$ is a local system we can define a connection with connection forms $\theta_{i,j} = 0$, i.e.

$$\nabla(\sum f_i \sigma_i) = \sum df_i \otimes \sigma_i + \sum f_i \nabla \sigma_i = \sum df_i \otimes \sigma_i.$$

By definition there exists a trivializing cover for which the corresponding coframe consists of flat sections i.e. $\nabla \sigma_j = \sum \theta_{i,j} \otimes \sigma_j = 0$ for any $j = 1, ..., d$. In this way we associate a flat connection to a local system.
Conversely if $E \to X$ is a vector bundle with a flat connection ∇. Then the transition function corresponding to covering by open sets with flat coframes must be constant. Indeed from $\sigma_j = \sum g_{i,j} \sigma_i$ we get $0 = \nabla \sigma_j = \sum (dg_{i,j} \otimes \sigma_i + g_{i,j} \nabla \sigma_i) = \sum dg_{i,j} \otimes \sigma_i$, so $dg_{i,j} = 0$ and $g_{i,j}$ is constant. Consequently we can define a local system of constant sections i.e. the flat sections.

2.2. Conclusions

Summarizing the results we have bijective correspondences

$$\begin{pmatrix} \text{Flat Bundles on } X \\ \text{up to isomorphism} \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} \text{Local Systems on } X \\ \text{up to isomorphism} \end{pmatrix} \leftrightsquigarrow \begin{pmatrix} \text{Representations of } \pi_1(X) \\ \text{up to isomorphism} \end{pmatrix},$$

that actually are equivalences between the following three categories

1. Local Systems over a connected, complex manifold X,
2. Finite dimensional representations of the fundamental group $\pi_1(X, x_0)$,
3. Holomorphic bundles $E \to X$ with a flat connection ∇.

2.2.1. The Gauss-Manin Connection.

Let $\varphi : \chi \to B$ be an analytic family of compact complex manifolds, where φ is a proper holomorphic submersion. By Erhesmann theorem χ is locally trivial as smooth manifold, and it is trivial if B is simply connected. Let $b_0 \in B$ be a point, and let $X_{b_0} = X = \varphi^{-1}(b_0)$ be the corresponding fiber. Consider the diffeomorphism

$$F : \chi \to B \times X,$$

and let $G = F^{-1}$ be its inverse. For any curve $\gamma : [0,1] \to B$ such that $\gamma(0) = b_0$ and $\gamma(1) = b_1$ we get a diffeomorphism $f_\gamma : X_{b_0} \to X_{b_1}$. This give rise to an isomorphism

$$f_\gamma^* : H^j(X_{b_1}, k) \to H^j(X_{b_0}, k),$$

where $k = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Since these isomorphisms depends only on the homotopy class of γ we get a representation $\pi_1(B, b_0) \to End(H^j(X_{b_0}, k))$. We denote by $\mathbb{H}^j \to B$ the holomorphic vector bundle assiociated to this representation. The fiber of $\mathbb{H}^j \to B$ over $b \in B$ is isomorphic to $H^j(X_b, \mathbb{C})$, and the corresponding connection is called the *Gauss-Manin Connection*.

CHAPTER 3

Hodge Structures and Representations

In this chapter we state a bijective correspondence between

$$\left(\begin{array}{c} \textit{Rational Hodge Structures of weight } k \\ \textit{over a } \mathbb{Q}-\textit{vector space } H \end{array} \right) \leftrightsquigarrow \left(\begin{array}{c} \textit{Algebraic Representations on the} \\ \mathbb{R}-\textit{vector space } H_{\mathbb{R}} = H \otimes_{\mathbb{Q}} \mathbb{R} \end{array} \right).$$

DEFINITION 3.1. A *rational Hodge structure* of weight k is a $\mathbb{Q}$-vector space H with a direct sum decomposition

$$H_{\mathbb{C}} = H \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{p+q=k} H^{p,q},$$

such that $\overline{H^{p,q}} = H^{q,p}$.

PROPOSITION 3.2. *There is a bijective correspondence between rational Hodge structures of weight k on a rational vector space H and algebraic representations $\rho : \mathbb{C}^* \to GL(H_{\mathbb{R}})$, where $H_{\mathbb{R}} = H \otimes_{\mathbb{Q}} \mathbb{R}$.*

PROOF. Let $H_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$ be a rational Hodge structure. For $\alpha = \sum \alpha^{p,q} \in H_{\mathbb{R}} \subseteq H_{\mathbb{C}}$ we define

$$\rho(z)(\alpha) = \sum (z^p \overline{z}^q) \alpha^{p,q}.$$

Note that $\rho(zw)(\alpha^{p,q}) = ((zw)^p \overline{zw}^q)\alpha^{p,q} = (z^p\overline{z}^q)(w^p\overline{w}^q)\alpha^{p,q} = (\rho(z) \circ \rho(w))\alpha^{p,q}$. Furthermore if $\alpha \in H_{\mathbb{R}}$ then $\rho(z)\alpha$ is still real. So the map $\rho : \mathbb{C}^* \to GL(H_{\mathbb{R}})$ is well define and it is a representation. Clearly it is algebraic.
Conversely suppose to have a representation $\rho : \mathbb{C}^* \to GL(H_{\mathbb{R}})$. Let $\rho_{\mathbb{C}} : \mathbb{C}^* \to GL(H_{\mathbb{C}})$ be the $\mathbb{C}$-linear extension of ρ, and consider the spaces

$$H^{p,q} = \{v \in H_{\mathbb{C}} \mid \rho_{\mathbb{C}}(z)(v) = (z^p\overline{z}^q)v, \forall z \in \mathbb{C}^*\}.$$

Since $\mathbb{C}^*$ is abelian, the representation $\rho_{\mathbb{C}}$ splits into a direct sum of one-dimensional representations $r_i : \mathbb{C}^* \to \mathbb{C}^*$. To show that $H_{\mathbb{C}} = \bigoplus H^{p,q}$, we have to argue that every one-dimensional representation r_i that might occur is of the form $r_i(z) = z^p\overline{z}^q$ with $p + q = k$. Now the hyphotesis that ρ is algebraic comes in. We can write $z = x + iy$, one can identify $\mathbb{C}^*$ with the subgroup of $GL(2, \mathbb{R})$ of all matrices of the form $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$. A representation $\rho : \mathbb{C}^* \to GL(H_{\mathbb{R}})$ is algebraic if $\rho \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ is a matrix whose entries are polynomials in x, y, and the inverse of the determinant $\frac{1}{x^2+y^2}$. Hence $r_i(z)$ must be a polynomial in $z, \overline{z}$, and $z\overline{z}$, henceforth of the form $z^p\overline{z}^q$ for some p, q with $p + q = k$. ♠