Minimal Surfaces and Curvature

Detailed examples, soap films, and real-world applications

Alex Massarenti

Abstract

A minimal surface is a surface whose mean curvature vanishes everywhere; equivalently, it is a critical point of the area functional. Soap films provide a physical model: they minimize area subject to a boundary wire frame. We introduce curvature (principal, mean, Gaussian), define minimal surfaces, derive the minimal surface equation for graphs, and work through classic examples in detail: planes, catenoids, and helicoids (plus a quick look at Scherk's surface). We then connect the math to soap films, bubbles, and applications in architecture, materials science, and biology.

1 Motivation: soap films and "least area"

Dip a bent wire loop into soapy water: the film that forms spans the loop and (approximately) minimizes surface area. This is a real-world optimization problem: *among all surfaces with the same boundary, the soap film chooses one with the smallest area.*

The mathematics of minimal surfaces connects *geometry (curvature)* and *optimization (area minimization).*

2 Curvature of a surface: k_1, k_2, mean curvature H, Gaussian curvature K

2.1 Surfaces and tangent planes

A smooth surface in $\mathbb{R}^3$ can be given parametrically by a map

$$X(u,v) \in \mathbb{R}^3, \qquad (u,v) \in U \subset \mathbb{R}^2,$$

with X_u and X_v linearly independent. The tangent plane is spanned by X_u and X_v.

2.2 Normal vector and normal sections

At each point, a unit normal vector is

$$N = \frac{X_u \times X_v}{\|X_u \times X_v\|}.$$

Intersect the surface with planes containing the normal N; each intersection gives a curve on the surface called a *normal section.* Different normal sections curve by different amounts.

2.3 Principal curvatures

Among all normal sections, the maximum and minimum curvatures are called the *principal curvatures* k_1 and k_2.

Definition 1 (Mean and Gaussian curvature)**.**

$$H = \frac{k_1 + k_2}{2} \qquad \text{(mean curvature)}, \qquad K = k_1 k_2 \qquad \text{(Gaussian curvature)}.$$

Remark 1. Intuition: H measures *how the surface bends on average*, while K measures *how saddle-like vs. dome-like* it is. Many minimal surfaces have $K < 0$ (saddle behavior), except the plane where $K = 0$.

3 What is a minimal surface? Two equivalent definitions

3.1 Geometric definition: zero mean curvature

Definition 2 (Minimal surface)**.** A smooth surface in $\mathbb{R}^3$ is *minimal* if its mean curvature satisfies

$$H \equiv 0 \quad \text{everywhere on the surface.}$$

3.2 Variational definition: critical points of area

If a surface is given as a graph $z = u(x, y)$ over a domain $\Omega \subset \mathbb{R}^2$, its area is

$$\text{Area}(u) = \iint_\Omega \sqrt{1 + u_x^2 + u_y^2}\, \mathrm{d}x\, \mathrm{d}y.$$

A minimal surface is a *critical point* of this functional.

3.3 First variation and the minimal surface equation for graphs

Let $u_t = u + t\varphi$ where φ is a smooth test function vanishing on the boundary (so the boundary stays fixed). Differentiating the area functional (details omitted in a first talk, but doable in multivariable calculus) gives the Euler–Lagrange equation:

$$\frac{\partial}{\partial x}\left(\frac{u_x}{\sqrt{1 + u_x^2 + u_y^2}}\right) + \frac{\partial}{\partial y}\left(\frac{u_y}{\sqrt{1 + u_x^2 + u_y^2}}\right) = 0.$$

Equivalently,

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0, \tag{1}$$

called the *minimal surface equation.*

Remark 2 (Curvature connection)**.** For a graph $z = u(x, y)$, one can show

$$2H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1 + \|\nabla u\|^2}}\right).$$

So the PDE above is exactly $H = 0$ written in coordinates.

4 Worked examples in detail

4.1 Example 1: planes are minimal

Let $u(x, y) = ax + by + c$. Then $u_{xx} = u_{yy} = u_{xy} = 0$, so (1) holds. Thus every plane is a minimal surface.

4.2 Example 2: why cylinders are *not* minimal

A circular cylinder of radius R has one principal curvature $k_1 = 1/R$ (bending around the circle) and the other $k_2 = 0$ (straight along the axis). Hence

$$H = \frac{1/R + 0}{2} = \frac{1}{2R} \neq 0.$$

So cylinders are *not* minimal.

4.3 Example 3: spheres and soap *bubbles*

A sphere of radius R has $k_1 = k_2 = 1/R$, so $H = 1/R$. A soap *bubble* (a closed surface enclosing air) has a pressure difference across the film. The Young–Laplace law says: pressure difference ΔP is proportional to H. So bubbles prefer *constant mean curvature* surfaces, and the sphere is the simplest one.

4.4 Example 4: the catenoid, derived by calculus of variations

The *catenoid* is the minimal surface obtained by rotating a catenary curve. It is the classic soap film between two parallel circular rings.

Set-up: surfaces of revolution

Consider a surface of revolution around the z-axis given by a profile curve $r \mapsto (r, f(r))$ rotated:

$$X(r, \theta) = \big(r\cos\theta,\ r\sin\theta,\ f(r)\big), \qquad r \in [r_0, r_1],\ \theta \in [0, 2\pi].$$

Compute the area element:

$$X_r = (\cos\theta, \sin\theta, f'(r)), \quad X_\theta = (-r\sin\theta, r\cos\theta, 0),$$

$$\|X_r \times X_\theta\| = r\sqrt{1 + f'(r)^2}.$$

Hence the area is

$$\text{Area}(f) = \int_0^{2\pi}\int_{r_0}^{r_1} r\sqrt{1 + f'(r)^2}\ \mathrm{d}r\ \mathrm{d}\theta = 2\pi\int_{r_0}^{r_1} r\sqrt{1 + f'(r)^2}\ \mathrm{d}r.$$

So we minimize the 1D functional

$$J(f) = \int_{r_0}^{r_1} r\sqrt{1 + f'(r)^2}\ \mathrm{d}r.$$

Euler–Lagrange and the first integral

The integrand depends on r and $f'(r)$ but not on $f(r)$ itself. Let $L(r, f') = r\sqrt{1+f'^2}$. A standard trick ("conservation law" for autonomous f) gives:

$$\frac{\partial L}{\partial f'} = \text{constant}.$$

Compute

$$\frac{\partial L}{\partial f'} = r \cdot \frac{f'}{\sqrt{1+f'^2}} = a,$$

for some constant $a > 0$. Solve for f':

$$\frac{f'}{\sqrt{1+f'^2}} = \frac{a}{r} \quad \Longrightarrow \quad f'^2 = \frac{a^2}{r^2 - a^2} \quad \Longrightarrow \quad f'(r) = \pm\frac{a}{\sqrt{r^2-a^2}}.$$

Integrate:

$$f(r) = \pm a \operatorname{arcosh}\left(\frac{r}{a}\right) + C.$$

Equivalently, solving for r gives the catenary form

$$r = a\cosh\left(\frac{z - C}{a}\right).$$

Rotating this curve produces the *catenoid*. It satisfies $H \equiv 0$, hence is minimal.

4.5 Example 5: the helicoid

The *helicoid* is a ruled surface generated by a line rotating and translating upward. A standard parametrization is

$$X(u, v) = \big(u\cos v,\ u\sin v,\ c\,v\big), \qquad u \in \mathbb{R},\ v \in \mathbb{R},$$

where $c \neq 0$ controls the pitch of the helix.

Step 1: first fundamental form

Compute

$$X_u = (\cos v, \sin v, 0), \qquad X_v = (-u\sin v,\ u\cos v,\ c).$$

Then

$$E = \langle X_u, X_u\rangle = 1, \quad F = \langle X_u, X_v\rangle = 0, \quad G = \langle X_v, X_v\rangle = u^2 + c^2.$$

Step 2: a unit normal

$$X_u \times X_v = \big(c\sin v,\ -c\cos v,\ u\big), \qquad \|X_u \times X_v\| = \sqrt{u^2 + c^2}.$$

So

$$N = \frac{1}{\sqrt{u^2+c^2}}(c\sin v,\ -c\cos v,\ u).$$

Step 3: second fundamental form and mean curvature

Second derivatives:

$$X_{uu} = (0, 0, 0), \qquad X_{uv} = (-\sin v, \cos v, 0), \qquad X_{vv} = (-u\cos v, -u\sin v, 0).$$

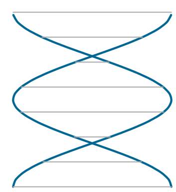
Coefficients of the second fundamental form:

$$e = \langle X_{uu}, N\rangle = 0, \qquad f = \langle X_{uv}, N\rangle = -\frac{c}{\sqrt{u^2+c^2}}, \qquad g = \langle X_{vv}, N\rangle = 0.$$

Mean curvature for a parametrized surface satisfies

$$2H = \frac{eG - 2fF + gE}{EG - F^2}.$$

Since $e = g = 0$ and $F = 0$, we get $H = 0$ everywhere. Therefore the helicoid is minimal.



Helicoid
a minimal *ruled* surface

4.6 Example 6: a nontrivial minimal graph (Scherk's surface)

A famous explicit minimal graph is

$$u(x, y) = \log\left(\frac{\cos y}{\cos x}\right) \quad \text{(defined where } \cos x, \cos y > 0\text{)}.$$

One can compute $u_x = \tan x$, $u_y = -\tan y$, and check that (1) holds. This gives a periodic "saddle" surface with alternating asymptotic planes.

5 Soap films vs. soap bubbles

- **Soap film (spanning a wire frame):** typically has (approximately) *zero* pressure jump across it, so it aims for $H = 0$.
- **Soap bubble (closed surface enclosing air):** has nonzero pressure jump, so it aims for *constant mean curvature* $H = \text{const}$.

Remark 3 (Plateau's problem)**.** Given a closed curve in space, does there exist a surface of least area spanning it? This is Plateau's problem; soap films give a physical existence experiment, while mathematics provides rigorous existence theorems.

6 Applications in the real world

Minimal surfaces show up wherever *area minimization* and *efficient distribution of stress* appear.

- **Architecture and structural design:** tensile membranes and lightweight roofs often follow minimal (or near-minimal) shapes. Classic "soap-film models" (physical form-finding) inspire efficient structures.
- **Materials science:** triply periodic minimal surfaces (TPMS) appear in self-assembled materials (e.g. certain block copolymers), and in porous scaffolds where high surface area and uniform channels are desired.
- **Biology:** cell membranes and biological interfaces can adopt shapes balancing curvature energies; minimal/CMC models appear as idealizations.
- **Engineering and manufacturing:** lattice infill patterns and 3D-printed structures can use minimal-surface-like geometries to achieve high stiffness-to-weight ratios.
- **Computer graphics / geometry processing:** minimal surfaces arise in smoothing, mesh fairing, and surface reconstruction algorithms.