NAKAYAMA'S LEMMA

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Contents

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1. Nakayama's Lemma

We begin with some notations. Let R be a ring.

- $\Omega_l(R) = \{I \mid I \text{ is a left maximal ideal of } R\}$.
- $\Omega_r(R) = \{I \mid I \text{ is a right maximal ideal of } R\}$.
- $\Omega(R) = \{I \mid I \text{ is a twosided maximal ideal of } R\}$.

Let M be a left R-module. If $x \in M$ the submodule generated by x is the set

$$\{rx \mid r \in R\}$$

and is denoted by *(x)* or by Rx. If $M = \sum_{i\in I} Rx_i$ with $x_i \in M$ then the x_i are said to be a set of generators of M. A left R-module is said to be finitely generated if it has a finite set of generators.

A free left R-module is one which is isomorphic to a left R-module of the form $\bigoplus_{i\in I} M_i$ with $M_i \cong R$. A finitely generated left R-module is one which is isomorphic to $R^n = R\oplus \ldots \oplus R$, for $n \in \mathbb{N}$, $n \geq 1$.

Definition 1. *Let R be a ring. The Jacobson radical of R is defined by*

$$Jac(R) = \bigcap_{J\in\Omega_l(R)} J = \bigcap_{J\in\Omega_r(R)} J.$$

Note that if $R \neq 0$, maximal left ideals always exist by Zorn's Lemma. If $R = 0$, then there are no maximal left ideals; in this case, we define the Jacobson radical to be zero.

Lemma 1. *(Nakayama) Let R be a ring. For any left ideal I of R the following facts are equivalent:*

1. $I \subseteq Jac(R)$.
2. *For any finitely generated left R-module M, $I{\cdot}M = M$ implies that $M = \{0\}$.*
3. *For any left R-modules $N \subseteq M$ such that $\frac{M}{N}$ is finitely generated, $N + I{\cdot}M = M$ implies that $N = M$.*

Proof. *(1)⇒(2)* Suppose $M \neq \{0\}$ and consider the set

$$X = \{N \mid N \text{ is a submodule of } M \text{ and } N \neq M\}.$$

We note that $\{0\} \in X$ and so $X \neq \varnothing$. Since M is finitely generated every chain of elements of X has a upper bound. Let $N_1 \subseteq N_2 \subseteq \ldots \subseteq N_r \subseteq \ldots$ be a chain of elements in X, then $\bigcup N_i$ is an upper bound for the chain. If $\bigcup N_i = M$, since M is finitely generated we have $M = Rx_1 + \ldots + Rx_n$. Then there exists an N_j such that $x_i \in N_j$ for any $i = 1, \ldots, n$ and so $M = N_j$, a contradiction. We conclude that $\bigcup N_i \in X$.

By Zorn's lemma the set X contains a maximal element S. Since the submodules of M/S are of the form N/S where N is a submodule of M such that $N \supseteq S$, the left R-module M/S is simple. There exists an element $x \in M/S$, $x \neq 0$. We consider the submodule Rx, since M/S is simple we have $Rx = M/S$ and $R/Ann_R(x) \cong Rx = M/S$. We know that $R/Ann_R(x)$ is simple if and only if $Ann_R(x)$ is a maximal ideal. Then $Jac(R) \subseteq Ann_R(x)$ and $I \subseteq Ann_R(x)$. We have that $ax = 0$ for any $a \in I$ for a generic $x \in M/S$, i.e. $I \cdot \frac{M}{S} = \{0\}$. Finally $I \cdot \frac{M}{S} = \frac{I \cdot M + S}{S} = \{0\}$ implies $I \cdot M + S = S$ and so $I \cdot M \subseteq S$, in particular $I \cdot M \neq M$.

(2)⇒(3) We consider the finitely generated left R-module M/N. By *(2)* we have that $I \cdot \frac{M}{N} = \frac{M}{N}$ implies $M/N = \{0\}$. We have that

$$I \cdot \frac{M}{N} \cong \frac{N + I \cdot M}{N} \cong \frac{M}{N}$$

implies $M/N = \{0\}$, so $M = N$.

(3)⇒(1) Suppose to have $x \in I$ such that $x \notin Jac(R)$. Then there exists a maximal left ideal $\mathfrak{m}$ of R such that $x \notin \mathfrak{m}$. Now $\mathfrak{m} \subset \mathfrak{m}+I$ and $\mathfrak{m} \neq \mathfrak{m}+I$ implies $\mathfrak{m}+I = R$, so a fortiori, we have $\mathfrak{m}+I \cdot R = R$ and by *(3)* we get $\mathfrak{m} = R$, a contradiction. □

In the following proposition we state a direct consequence of Nakayama's lemma that is very useful in Algebraic Geometry.

Let R be a commutative local ring and let $\mathfrak{m}$ be its maximal ideal. Let $k = R/\mathfrak{m}$ be the residue field. We consider a finitely generated R-module M. Then $M/\mathfrak{m}M$ is annihilated by $\mathfrak{m}$ and then it has a structure of $R/\mathfrak{m}$-module i.e. $M/\mathfrak{m}M$ is a k-vector space of finite dimension.

PROPOSITION 1. *Let x_i, $1 \leq i \leq n$, be elements of M, and let*

$$\pi\colon M \to M/\mathfrak{m}M, \text{ defined by } x \mapsto x+\mathfrak{m}M$$

be the projection map. If the elements $\pi(x_i)$, $1 \leq i \leq n$, form a basis of the k-vector space $M/\mathfrak{m}M$, then the x_i generate M.

Proof. We consider the submodule N of M generated by the x_i, and the composition map

$$N \xrightarrow{i} M \xrightarrow{\pi} M/\mathfrak{m}M.$$

Since the $\pi(x_i)$, $1 \leq i \leq n$, form a basis of the k-vector space $M/\mathfrak{m}M$, the composition map is surjective. Let $x \in M$ and let $x+\mathfrak{m}M$ its class in $M/\mathfrak{m}M$, then there exists $y \in N$ such that $y+\mathfrak{m}M = x+\mathfrak{m}M$. So $x\text{-}y \in \mathfrak{m}M$ and there exists $z \in \mathfrak{m}M$ such that $x\text{-}y = z$, so $x = y+z \in N+\mathfrak{m}M$. We conclude that $N+\mathfrak{m}M = M$ and hence by *(3)* of lemma 1 we get $N = M$. □

Lemma 2. *Let R be a ring, let I be an ideal of R and let M be an R-module. Then $\frac{R}{I} \otimes_R M$ is isomorphic to $\frac{M}{IM}$.*

Proof. We consider the exact sequence

$$0 \mapsto I \to R \to R/I \mapsto 0$$

tensorizing by M we obtain a exact sequence

$$0 \mapsto I \otimes M \to R \otimes M \to R/I \otimes M \mapsto 0$$

since the map $i \otimes M : I \otimes M \to R \otimes M$ is injective. Now $I \otimes M \cong IM$ and $R \otimes M \cong M$. Then we have the exact sequence

$$0 \mapsto IM \to M \to R/I \otimes M \mapsto 0$$

and so $R/I \otimes M \cong M/IM$. $\square$

Lemma 3. *Let R be a local ring and let M,N be finitely generated R-modules. Then $M \otimes_R N = 0$ implies $M = 0$ or $N = 0$.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of R and let $k = R/\mathfrak{m}$ be the residue field. We denote $M_k = k \otimes_R M \cong M/\mathfrak{m}M$ by lemma 2. Now $M \otimes_R N = 0$ implies $(M \otimes_R N)_k = 0$ then $M_k \otimes_k N_k = 0$ and since M_k, N_k are vector spaces, we have $M_k = 0$ or $N_k = 0$. By Nakayama's lemma we conclude that $M = 0$ or $N = 0$. $\square$

1.1. **Applications in Commutative Algebra.** In this section we state some proposition which proves need Nakayma's lemma. In the following all rings are commutative.

Definition 2. *Let A and B be local rings with maximal ideals $\mathfrak{m}_A$ and $\mathfrak{m}_B$ respectively. A local morphism of local rings is a morphism of rings $\varphi{:}A \to B$ such that $\varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.*

Let A and B be local rings and let $\mathfrak{m}_A$ and $\mathfrak{m}_B$ their maximal ideals. A local morphism $\varphi{:}A \to B$ induces two morphisms

- $\alpha{:}A/\mathfrak{m}_A \to B/\mathfrak{m}_B,\ a{+}\mathfrak{m}_A \mapsto \varphi(a){+}\mathfrak{m}_B$.
- $\beta{:}\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2,\ a{\mapsto}\ \varphi(a){+}\mathfrak{m}_B^2$.

Suppose $x{+}\mathfrak{m}_A = y{+}\mathfrak{m}_A$ then x-$y \in \mathfrak{m}_A$. Since φ is a local morphism $\varphi(x\text{-}y) = \varphi(x)\text{-}\varphi(y) \in \mathfrak{m}_B$ and $\alpha(x{+}\mathfrak{m}_A) = \alpha(y{+}\mathfrak{m}_B)$.
Similarly if $x \in \mathfrak{m}_A$ then $\varphi(x) \in \mathfrak{m}_B$ and β is well defined.

PROPOSITION 2. *Let $\varphi{:}A \to B$ be a local morphism of local noetherian rings such that*

(1) $\alpha{:}A/\mathfrak{m}_A \to B/\mathfrak{m}_B$ *is an isomorphism,*
(2) $\beta{:}\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ *is surjective,*
(3) B *is a finitely generated A-module.*

Then φ is surjective.

Proof. The ring B becomes an A-module defining $ab = \varphi(a)b$. We consider in B the ideal $\mathcal{I} = \mathfrak{m}_A B$, then $\mathcal{I} \subseteq \mathfrak{m}_B$. By (2) $\mathcal{I}$ contains a set of generators for $\mathfrak{m}_B/\mathfrak{m}_B^2$. We note that since B is noetherian the ideal $\mathfrak{m}_B$ is finitely generated. We apply Nakayama's lemma to the local ring B and to the finitely generated B-module $\mathfrak{m}_B$, we conclude that $\mathcal{I} = \mathfrak{m}_B$. Now we apply Nakayama's lemma to the A-module B. By (*3*) B is finitely generated as A-module and the element $1 \in B$ is a generator for $B/\mathfrak{m}_A B = B/\mathfrak{m}_A = A/\mathfrak{m}_B$ by (*1*). Then *1* is also a generator for B as an A-module and for any $b \in B$ there exists $a \in A$ such that $b = 1\varphi(a) = \varphi(a)$, i.e, φ is surjective. □

PROPOSITION 3. *Let A be a noetherian local integral domain with residue field k and quotient field K. If M is a finitely generated A-module and if $dim_k M \otimes_A k = dim_K M \otimes_A K = n$, then M is free of rank n.*

Proof. Let $\mathfrak{m}$ be the maximal ideal of A, we have $M \otimes k = M \otimes A/\mathfrak{m} \cong M/\mathfrak{m}M$. Since $\dim_k M \otimes_A k = n$ by Nakayama's lemma M can be generated by n elements and we have a surjective morphism $A^n \to M \mapsto 0$. If R is its kernel, we have an exact sequence

$$0 \mapsto R \to A^n \to M \mapsto 0.$$

Tensorizing by the quotient field K which is flat we obtain the sequence

$$0 \mapsto R \otimes K \to A^n \otimes K \to M \otimes K \mapsto 0.$$

We recall that $A^n \otimes K \cong K^n$ and we have

$$0 \mapsto R \otimes K \to K^n \to M \otimes K \mapsto 0.$$

Now $\dim_K M \otimes_A K = \dim_K K^n = n$ implies $R \otimes K = 0$. Let $(r_1, ..., r_n)$ be a non zero element of R, with $r_i \in A$. If $a \in A$ is such that $a(r_1, ..., r_n) = (0, ..., 0)$, then there exist an $r_j \neq 0$ such that $ar_j = 0$. Since A is an integral domain we have $a = 0$. We have proved that R is torsion free and so $R \otimes K = 0$ implies $R = 0$. We conclude that $M \cong A^n$. □

1.2. **Applications in Algebraic Geometry.** Let $(X, \mathcal{O}_X)$ be a noetherian scheme. For any $x \in X$ the ring $\mathcal{O}_x$, whose elements are the germs of regular function in a neighborhood of x is a local ring. Its maximal ideal is the ideal $\mathfrak{m}_x$ of regular functions vanishing in x. We denote by $k(x) = \mathcal{O}_x/\mathfrak{m}_x$ the residue field. A sheaf of $\mathcal{O}_X$-modules on X is a sheaf of abelian groups $\mathcal{F}$ on X, such that for any open subset $\mathcal{U}$ of X, $\mathcal{F}(\mathcal{U})$ is an $\mathcal{O}_X(\mathcal{U})$-module. The stalk $\mathcal{F}_x$ of $\mathcal{F}$ in x has a structure of $\mathcal{O}_x$-module.

Definition 3. *Let $x \in X$ be a point and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The sheaf $\mathcal{F}$ is of finite type in x if there exist an open subset $\mathcal{U}$ of X and a surjective morphism $\mathcal{O}^n_{X|\mathcal{U}} \to \mathcal{F}_{|\mathcal{U}} \mapsto 0$, for some integer n. The sheaf $\mathcal{F}$ is of finite type on X if it is of finite type in any $x \in X$.*

Definition 4. *Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The sheaf $\mathcal{F}$ is coherent if:*

- *The sheaf $\mathcal{F}$ is of finite type.*
- *For any open subset $\mathcal{U}$ of X and for any morphism $\varphi : \mathcal{O}^n_{X|\mathcal{U}} \to \mathcal{F}_{|\mathcal{U}}$, the sheaf $ker(\varphi)$ is of finite type.*

One can prove that if $\mathcal{F}$ is a coherent sheaf and if $s_1, ..., s_n$ generate the stalk $\mathcal{F}_x$, then there exists an open neighborhood $\mathcal{U}$ of x in X such that $s_1, ..., s_n$ generates $\mathcal{F}$ on $\mathcal{U}$.

Definition 5. *Let X be a topological space. A map $\varphi : X \to \mathbb{Z}$ is uppersemicontinuous if for each $x \in X$ there is an open neighborhood $\mathcal{U}$ of x in X, such that $\varphi(u) \leqslant \varphi(x)$ for any $u \in \mathcal{U}$.*

Proposition 4. *Let X be a noetherian scheme and let $\mathcal{F}$ be a coherent sheaf on X. We consider the map*

$$\varphi : X \to \mathbb{Z}, \text{ defined by } \varphi(x) = dim_{k(x)} \mathcal{F}_x \otimes_{\mathcal{O}_x} k(x).$$

Then we have the following results.

1. *The map φ is uppersemicontinuous.*
2. *If X is irreducible, and φ is constant, then $\mathcal{F}$ is locally free.*
3. *If $\mathcal{F}$ is locally free, and X is connected, then φ is a constant map.*

Proof. (1) By lemma 2 we have that $\mathcal{F}_x \otimes_{\mathcal{O}_x} k(x) = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_x/\mathfrak{m}_x \cong \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$. Then $\varphi(x)$ is equal to the number of elements of a basis of the $k(x)$-vector space $\mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x$, by Nakayama's lemma $\varphi(x)$ is equal to the minimal number of generators of the $\mathcal{O}_x$-module $\mathcal{F}_x$. If $s_1, ..., s_n \in \mathcal{F}_x$ are germs that form a minimal set of generators, then they extend to sections of $\mathcal{F}$ is some neighborhood $\mathcal{U}$ of x in X, and they generate $\mathcal{F}$ is some neighborhood, because $\mathcal{F}$ is coherent. If y is in that neighborhood, then $\varphi(y)$, which is the minimal number of generators of $\mathcal{F}_y$ by Nakayama's lemma, is $r \leqslant \varphi(x)$. Then for any $x \in X$ there exists a neighborhood $\mathcal{V}$ of x in X such that $\varphi(y) \leqslant \varphi(x)$ for any $y \in \mathcal{V}$, i.e. the map φ is uppersemicontinuous.

(2) We have $\varphi(x) = n$ for any $x \in X$. Let $y \in X$ be a point, then $\varphi(y) = n$ and by Nakayama's lemma n is the minimal number of generators of $\mathcal{F}_y$. Since $\mathcal{F}$ is coherent there exists an open neighborhood $\mathcal{U}$ of y in X, such that $\mathcal{F}_{|\mathcal{U}}$ is generated by n sections and we have a surjective morphism $\mathcal{O}^n_{X|\mathcal{U}} \rightarrow \mathcal{F}_{|\mathcal{U}}$. If $\mathcal{R}$ is the kernel of that morphism we have the exact sequence

$$0 \mapsto \mathcal{R} \rightarrow \mathcal{O}^n_{X|\mathcal{U}} \rightarrow \mathcal{F}_{|\mathcal{U}} \mapsto 0.$$

Let $\mathcal{V} \subseteq \mathcal{U}$ be an open subset, and let $(r_1, ..., r_n) \in \mathcal{R}_{|\mathcal{V}}$ then the r_i are regular function on $\mathcal{V}$ such that $r_i(x) = 0$ for any $x \in \mathcal{V}$. Then $r_i = 0$ and $\mathcal{R} = 0$. We conclude that the morphism $\mathcal{O}^n_{X|\mathcal{U}} \rightarrow \mathcal{F}_{|\mathcal{U}}$ is an isomorphism and $\mathcal{F}$ is free of rank n on $\mathcal{U}$. Since φ is a constant map the sheaf $\mathcal{F}$ have to be locally free.

(3) Suppose that there exist $x, y \in X$ such that $\varphi(x) \neq \varphi(y)$. Then by (2) there exist $\mathcal{U}$, $\mathcal{V}$ open neighborhood of x and y respectively, such that $\mathcal{F}$ is locally free of rank n on $\mathcal{U}$ and is locally free of rank m on $\mathcal{V}$, with $n \neq m$. A contradiction because $\mathcal{F}$ is locally free and X is connected. $\square$