TORIC VARIETIES, MORI DREAM SPACES AND COX RINGS

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INTRODUCTION

These are lecture notes for a curse I gave at the *IMPA-UFF Escola Transguanabara de Geometria Algébrica* in February 2020.

The goal of the minimal model program is to construct a birational model of any complex projective variety which is as simple as possible in a suitable sense. This subject has its origins in the classical birational geometry of surfaces studied by the Italian school. In 1988 S. Mori extended the concept of minimal model to 3-folds by allowing suitable singularities on them. In 2010 there was a great breakthrough in the minimal model theory when C. Birkar, P. Cascini, C. Hacon and J. McKernan proved the existence of minimal models for varieties of log general type.

Mori Dream Spaces, introduced by Y. Hu and S. Keel in 2002, form a class of algebraic varieties that behave very well from the point of view of Mori's minimal model program. They can be algebraically characterized as varieties whose total coordinate ring, called the *Cox ring*, is finitely generated. In addition to this algebraic characterization there are several algebraic varieties characterized by some positivity property of the anti-canonical divisor, such as *weak Fano* and *log Fano* varieties, that turn out to be Mori Dream Spaces.

Toric varieties are the easiest examples of Mori dream spaces, and provide an elementary way to see many examples and phenomena in algebraic geometry. The goal of the course is to introduce the notion of Cox ring with a particular attention to toric varieties and their birational geometry from the point of view of Mori theory.

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1. Quotient construction of toric varieties

Let X be a normal projective variety. We denote by $\mathrm{N}^1(X)$ the real vector space of Cartier divisors and by $\rho_X = \dim(\mathrm{N}^1(X))$ the Picard number of X.

- The *effective cone* $\mathrm{Eff}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of effective divisors. In general it is not a closed cone.
- The *nef cone* $\mathrm{Nef}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of divisors D such that $D \cdot C \geq 0$ for any curve $C \subset X$. It is closed, but in general it is neither polyhedral nor rational.
- A divisor $D \subset X$ is called *movable* if its stable base locus is in codimension greater or equal that two. The *movable cone* $\mathrm{Mov}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of movable divisors. In general, it is not closed.

A *small* $\mathbb{Q}$*-factorial transformation* of X is a birational map $f : X \dashrightarrow Y$ to another normal $\mathbb{Q}$-factorial projective variety Y, such that f is an isomorphism in codimension one.
The exponential exact sequence

$$0 \mapsto \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \mapsto 0$$

induces the following exact sequence in cohomology

$$0 \mapsto H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X)$$

The complex torus $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is the *Picard variety* of X. This variety $\mathrm{Pic}^0(X)$ is the connected component of the identity of $\mathrm{Pic}(X) \cong H^1(X,\mathcal{O}_X^*)$ and it is an abelian variety. The image of $\mathrm{Pic}(X)$ inside $H^2(X,\mathbb{Z})$ is isomorphic to $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. The group $\mathrm{NS}(X) \cong \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is a finitely generated abelian group called the *Néron-Severi group*. The group $\mathrm{NS}(X)$ parametrizes divisor on X modulo numerical equivalence.

Example 1.1. Let us consider a smooth projective curve X of genus g. That is X is a compact Riemann surface with g handles. Then $H^0(X,\mathbb{Z}) \cong H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ because X is connected, and $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Since $H^0(X,\mathcal{O}_X) \cong \mathbb{C}^g$ we have $\mathrm{Pic}^0(X) \cong \mathbb{C}^g/\mathbb{Z}^{2g} \cong \mathrm{Jac}(X)$, the Jacobian variety of X. In this case the degree gives an isomorphism $\mathrm{NS}(X) \cong \mathbb{Z}$.

Now, let X_Σ be an n-dimensional toric variety associated to a fan $\Sigma \subset N_\mathbb{R}$ with no torus factor that is such that $N_\mathbb{R}$ is generated by $\{u_\rho \mid \rho \in \Sigma(1)\}$. Applying $\mathrm{Hom}_\mathbb{Z}(-,\mathbb{C}^*)$ to the exact sequence

$$0 \to M \to \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \to \mathrm{Cl}(X_\Sigma) \to 0$$

we get

$$1 \to \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma),\mathbb{C}^*) \to \mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^{|\Sigma(1)|},\mathbb{C}^*) \to \mathrm{Hom}_\mathbb{Z}(M,\mathbb{C}^*) \to 1$$

Now, $T_N \cong \mathrm{Hom}_\mathbb{Z}(M,\mathbb{C}^*)$, $\mathrm{Hom}_\mathbb{Z}(\mathbb{Z}^{|\Sigma(1)|},\mathbb{C}^*) \cong (\mathbb{C}^*)^{|\Sigma(1)|}$, and set $G := \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X_\Sigma),\mathbb{C}^*)$.

Note that the map $M \to \mathbb{Z}^{|\Sigma(1)|}$ is defined by $m \mapsto (\langle m, u_{rho}\rangle)_{\rho\in\Sigma(1)}$, and hence

$$G = \left\{ (t_\rho) \in (\mathbb{C}^*)^{|\Sigma(1)|} \mid \prod_{\rho\in\Sigma(1)} t_\rho^{\langle e_1, u_\rho\rangle} = 1,\ i = 1,\dots,n \right\}$$

We want to construct X_Σ as a quotient of $\mathbb{C}^{|\Sigma(1)|}$ by G. In order to do this we have specify the exceptional set of non semi-stable points $Z \subset \mathbb{C}^{|\Sigma(1)|}$ that we must remove from $\mathbb{C}^{|\Sigma(1)|}$ before taking the quotient.

Note that G and $\mathbb{C}^{|\Sigma(1)|}$ depend only on $\Sigma(1)$, and to get back X_Σ we need to take into account also the rest of the fan Σ. In order to do this for each ray $\rho \in \Sigma(1)$ we introduce the variable x_ρ and consider the *total coordinate ring*

$$S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$$

Then $\mathrm{Spec}(S) = \mathbb{C}^{|\Sigma(1)|}$. Now, for each cone $\sigma \in \Sigma$ define the monomial

$$x^{\widehat{\sigma}} = \prod_{\rho \notin \sigma(1)} x_\rho$$

and defined the irrelevant ideal

$$B(\Sigma) = \left\langle x^{\widehat{\sigma}} \mid \sigma \in \Sigma \right\rangle = \left\langle x^{\widehat{\sigma}} \mid \sigma \in \Sigma(n) \right\rangle \subset S$$

Note that $\Sigma(1)$ and $B(\Sigma)$ uniquely determine Σ. The irrelevant locus is defined as

$$Z(\Sigma) = \mathbf{Z}(B(\Sigma)) \subset \mathbb{C}^{|\Sigma(1)|}$$

and hence $Z(\Sigma)$ is a union of coordinate subspaces of $\mathbb{C}^{|\Sigma(1)|}$.

With respect to a fixed basis $\{D_1, \dots, D_{\rho-n}\}$ of $\mathrm{Cl}(X_\Sigma)$ we may write

$$D_{x_1} = \alpha_1^1 D_1 + \dots + \alpha_1^{\rho-n} D_{\rho-n}, \dots, D_{x_\rho} = \alpha_\rho^1 D_1 + \dots + \alpha_\rho^{\rho-n} D_{\rho-n}$$

Then the grading matrix of $D_{x_1}, \dots, D_{x_\rho}$ with respect to the fixed basis is

$$\begin{pmatrix} \alpha_1^1 & \alpha_2^1 & \dots & \alpha_\rho^1 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{\rho-n} & \alpha_2^{\rho-n} & \dots & \alpha_\rho^{\rho-n} \end{pmatrix}$$

and hence the action of G on $\mathbb{C}^{|\Sigma(1)|} \cong (\mathbb{C}^*)^\rho$ is given by

$$(\lambda_1, \dots, \lambda_{\rho-n}) \cdot (x_1, \dots, x_\rho) = (\lambda_1^{\alpha_1^1} \dots \lambda_{\rho-n}^{\alpha_1^{\rho-n}} x_1, \dots, \lambda_1^{\alpha_\rho^1} \dots \lambda_{\rho-n}^{\alpha_\rho^{\rho-n}} x_\rho)$$

The toric variety X_Σ is isomorphic to the quotient

$$(\mathbb{C}^{|\Sigma(1)|} \setminus Z(\Sigma)) /\!/ G$$

We refer to [CLS11, Section 5.1] for details on this construction.

Example 1.2. Consider $X_\Sigma = \mathbb{P}^2$. The rays of Σ are generated by $u_1 = (1,0)$, $u_2 = (0,1)$ and $u_0 = (-1,-1)$. Then $(t_0, t_1, t_2) \in (\mathbb{C}^*)^{|\Sigma(1)|} \cong (\mathbb{C}^*)^3$ lies in G if and only if $t_0^1 t_1 = t_0^1 t_2 = 1$ that is $t_0 = t_1 = t_2$. The irrelevant locus is given by $Z(\Sigma) = \{x = y = z = 0\}$, and therefore we get back the projective plane as the quotient of $\mathbb{C}^3 \setminus \{(0,0,0)\}$ by G acting via $\lambda \cdot (x,y,z) = (\lambda x, \lambda y, \lambda z)$.

Example 1.3. Consider $X_\Sigma = \mathbb{P}^1 \times \mathbb{P}^1$. Then $(t_1, t_2, t_3, t_4) \in (\mathbb{C}^*)^{|\Sigma(1)|} \cong (\mathbb{C}^*)^4$ lies in G if and only if $t_1 t_2^{-1} = t_3 t_4^{-1} = 1$ that is $t_1 = t_2$ and $t_3 = t_4$. So $G \cong \{(u, u, \lambda, \lambda) \mid u, \lambda \in \mathbb{C}^*\} \cong (\mathbb{C}^*)^2$. Consider Cox coordinates x_1, x_2, x_3, x_4. Their grading matrix, with respect to the standard basis of $\mathrm{Pic}(X_\Sigma)$ given by the two rulings, is

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and hence the action of G is given by $(u, \lambda) \cdot (x_1, x_2, x_3, x_4) = (u x_1, \lambda x_2, u x_3, \lambda x_4)$. Note that the irrelevant locus is defined by

$$Z(\Sigma) = \mathbf{Z}(x_1 x_2, x_3 x_4, x_2 x_3, x_1 x_4) = \{x_1 = x_3 = 0\} \cup \{x_2 = x_4 = 0\}$$

and $(\mathbb{C}^4 \setminus Z(\Sigma))//G \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Example 1.4. Consider the cone $\sigma = \langle 2e_1 - e_2, e_2 \rangle \subset \mathbb{R}^2$, corresponding to the affine X cone over a smooth conic. Then $\mathrm{Cl}(X) \cong \frac{\mathbb{Z}}{2\mathbb{Z}}$ and $G = \mathrm{Hom}_{\mathbb{Z}}\left(\frac{\mathbb{Z}}{2\mathbb{Z}}, \mathbb{C}^*\right) \cong \mu_2$. The grading matrix is simply $(1,1)$ and hence $X = \mathbb{C}^2//\mu_2$ where μ_2 acts via $\epsilon \cdot (x,y) = (\epsilon x, \epsilon y)$.

Note that the group G in Example 1.4 has torsion. In general G is the product of a torus and a finite abelian group. In particular, G is reductive.

Example 1.5. Let X_Σ be the weighted projective space with weights $a_0, \dots, a_n$. Then $Z(\Sigma) = \{(0,\dots,0)\}$, $\mathrm{Cl}(X_\Sigma) \cong \mathbb{Z}[H]$, and the grading matrix is $(a_0,\dots,a_n)$. So $G \cong \mathbb{C}^*$ acts on $\mathbb{C}^{n+1} \setminus \{(0,\dots,0)\}$ via $\lambda \cdot (x_0,\dots,x_n) = (\lambda^{a_0}x_0,\dots,\lambda^{a_n}x_n)$, and $X_\Sigma \cong (\mathbb{C}^{n+1} \setminus \{(0,\dots,0)\})//G$.

Example 1.6. Let X_Σ be the blow-up of $\mathbb{P}^2$ at a point. Then $\mathrm{Cl}(X_\Sigma) \cong \mathbb{Z}[H,E]$ and the torus invariant divisors are

$$D_1 \sim H - E, D_2 \sim E, D_3 \sim H - E, D_4 \sim H$$

with grading matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & -1 & 0 \end{pmatrix}$$

The group $G \cong (\mathbb{C}^*)^2$ is defined in $(\mathbb{C}^*)^4$ by $\{t_1 - t_3 = t_4 - t_1t_2 = 0\}$ and acts on $\mathbb{C}^4$ via

$$(\lambda, u) \cdot (x_1,x_2,x_3,x_4) = (\lambda u^{-1}x_1, ux_2, \lambda u^{-1}x_3, \lambda x_4)$$

The irrelevant locus is defined by

$$\{x_3x_4 = x_1x_4 = x_1x_2 = x_2x_3 = 0\} = \{x_1 = x_2 = 0\} \cup \{x_3 = x_4 = 0\}$$

Indeed, any divisor in $\langle H, H-E \rangle$ must be ample on X_Σ so the locus where all of its sections vanish needs to be removed.

If instead we look at divisors in $\langle H, E \rangle$ then we must remove

$$\{x_2x_4 = x_1x_2 = x_2x_3 = 0\} = \{x_2 = 0\} \cup \{x_1 = x_3 = x_4 = 0\}$$

Since $x_2 \neq 0$ we may set $x_2 = 1$. Hence the action of u identifies all the 3-spaces parallel to the 3-space x_1, x_3, x_4, and we may set $u = 1$. Therefore, we are left with action of $\mathbb{C}^*$ on $\mathbb{C}^3 \setminus \{(0,0,0)\}$ given by $\lambda \cdot (x_1,x_3,x_4) = (\lambda x_1, \lambda x_3, \lambda x_4)$ so that the quotient is $\mathbb{P}^2$.

Note that in Example 1.6 varying the irrelevant locus we recover the birational models of X_Σ which in this case are just X_Σ itself and $\mathbb{P}^2$. We will see, in Section 4, that this phenomenon holds for toric varieties in general, and even for a larger class of varieties called Mori dream spaces.

2. Singularities

Canonical singularities appear as singularities of the canonical model of a projective variety, and terminal singularities are special cases that appear as singularities of minimal models. Terminal singularities are important in the minimal model program because smooth minimal models do not always exist, and thus one must allow certain singularities, namely the terminal singularities. For instance, two-dimensional terminal singularities are smooth. The singular locus of a variety with at most terminal singularities has codimension at least three. In particular for curves and surfaces all terminal singularities are smooth. For 3-folds terminal singularities are isolated and have been classified by *S. Mori*.

Surface canonical singularities are exactly the *du Val singularities*, and are analytically isomorphic to quotients of $\mathbb{C}^2$ by finite subgroups of $SL_2(\mathbb{C})$.

2.1. **Cyclic quotient singularities.** Any cyclic quotient singularity is of the form $\mathbb{A}^n/\mu_r$, where μ_r is the group of r-roots of unit. The action $\mu_r \curvearrowright \mathbb{A}^n$ can be diagonalized, and then written in the form

$$\begin{array}{ccc} \mu_r \times \mathbb{A}^n & \longrightarrow & \mathbb{A}^n \\ (\epsilon, x_1, ..., x_n) & \longmapsto & (\epsilon^{a_1}x_1, ..., \epsilon^{a_n}x_n) \end{array}$$

for some $a_1, ..., a_n \in \mathbb{Z}/\mathbb{Z}_r$. The singularity is thus determined by the numbers $r, a_1, ..., a_n$. Following the notation set by *M. Reid* in **[?]**, we denote by $\frac{1}{r}(a_1, ..., a_n)$ this type of singularity.

Example 2.1. Let us consider the action:

$$\begin{array}{ccc} \mu_2 \times \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \\ (\epsilon, x_0, x_1) & \longmapsto & (\epsilon x_0, \epsilon x_1) \end{array}$$

The ring of invariants is given by:

$$k[x_0^2, x_0x_1, x_1^2] \cong k[y_0, y_1, y_2]/(y_0y_2 - y_1^2)$$

and we see that the singularity $X = \mathbb{A}^2/\mu_2$ corresponds to the vertex v of the affine cone

$$X = \operatorname{Spec}(k[x_0^2, x_0x_1, x_1^2] \cong k[y_0, y_1, y_2]/(y_0y_2 - y_1^2))$$

that is the vertex of a quadric cone $Q \subset \mathbb{P}^2$ or equivalently the singularity $\frac{1}{2}(1,1)$ of the weighted projective plane $\mathbb{P}(1,1,2)$. Now, $dx_0 \wedge dx_1$ is a basis of $\bigwedge^2 \Omega_{\mathbb{A}^2}$, and $(dx_0 \wedge dx_1)^{\otimes 2}$ is invariant under the action. The form

$$\omega = \frac{(dy_0 \wedge dy_1)^{\otimes 2}}{y_0^2} \in (\bigwedge^2 \Omega_{k(X)})^{\otimes 2}$$

is a basis of $(\bigwedge^2 \Omega_X)^{\otimes 2}$ because the quotient map $\pi : \mathbb{A}^2 \to X$ is étale on $X \setminus \{v\}$, and $\pi^*\omega = 4(dx_0 \wedge dx_1)^{\otimes 2}$.
Blowing-up the vertex v we get a resolution $f : Y \to X$. If $[\lambda_0 : \lambda_1 : \lambda_2]$ are homogeneous coordinates on $\mathbb{P}^2$ then the equations of Y in $\mathbb{A}^3 \times \mathbb{P}^2$ are:

$$\begin{cases} y_0\lambda_1 - y_1\lambda_0 = 0 \\ y_0\lambda_2 - y_2\lambda_0 = 0 \\ y_1\lambda_2 - y_2\lambda_1 = 0 \\ y_0y_2 - y_1^2 \end{cases}$$

Therefore, $y_1 = \frac{\lambda_1}{\lambda_0}y_0$, and $\frac{\lambda_2}{\lambda_1} = \frac{\lambda_1}{\lambda_0}$ yields $y_2 = \frac{\lambda_1}{\lambda_0}y_1 = (\frac{\lambda_1}{\lambda_0})^2 y_0$. Then, in Y we have an affine chart isomorphic to $\mathbb{A}^2$ with coordinates (y_0, t) where the resolution is given by $(y_0, t) \mapsto (y_0, y_0t, y_0t^2)$, with $t = \frac{\lambda_1}{\lambda_0}$, and the exceptional divisor E over v is given by $\{y_0 = 0\}$. We have

$$f^*\omega = (dy_0 \wedge dt)^{\otimes 2}$$

Therefore, $f^*\omega$ has neither a pole nor a zero along E, and we may write $K_Y = f^*K_X$.

Example 2.2. Let us consider the action:

$$\begin{array}{ccc} \mu_3 \times \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \\ (\epsilon, x_0, x_1) & \longmapsto & (\epsilon x_0, \epsilon x_1) \end{array}$$

The ring of invariants is given by:

$$k[x_0^3, x_0^2x_1, x_0x_1^2, x_1^3] \cong k[y_0, y_1, y_2, y_3]/(y_0y_3 - y_1y_2, y_0y_2 - y_1^2, y_1y_3 - y_2^2)$$

and we see that the singularity $X = \mathbb{A}^2/\mu_3$ corresponds to the vertex v of the affine cone

$$X = \mathrm{Spec}(k[y_0, y_1, y_2, y_3]/(y_0y_3 - y_1y_2, y_0y_2 - y_1^2, y_1y_3 - y_2^2))$$

over a twisted cubic $C \subset \mathbb{P}^3$. Now, $dx_0 \wedge dx_1$ is a basis of $\bigwedge^2 \Omega_{\mathbb{A}^2}$, and $(dx_0 \wedge dx_1)^{\otimes 3}$ is invariant under the action. The form

$$\omega = \frac{(dy_0 \wedge dy_1)^{\otimes 3}}{y_0^4} \in (\bigwedge^2 \Omega_{k(X)})^{\otimes 3}$$

is a basis of $(\bigwedge^2 \Omega_X)^{\otimes 3}$ because the quotient map $\pi : \mathbb{A}^2 \to X$ is étale on $X \setminus \{v\}$, and

$$\pi^*\omega = \frac{(3x_0^4(dx_0 \wedge dx_1))^{\otimes 3}}{x_0^{12}} = 27(dx_0 \wedge dx_1)^{\otimes 3}$$

Blowing-up the vertex v we get a resolution $f : Y \to X$, and we have an affine chart isomorphic to $\mathbb{A}^2$ with coordinates (y_0, t) where the resolution is given by $(y_0, t) \mapsto (y_0, y_0t, y_0t^2, y_0t^3)$, and the exceptional divisor E over v is given by $\{y_0 = 0\}$. We have

$$f^*\omega = \frac{(dy_0 \wedge (y_0dt + tdy_0))^{\otimes 3}}{y_0^4} = \frac{(dy_0 \wedge dt)^{\otimes 3}}{y_0}$$

Therefore, $f^*\omega$ has a pole along E, and we may write $K_Y = f^*K_X - \frac{1}{3}E$.

Example 2.3. Now, let us consider the action:

$$\begin{array}{ccc} \mu_2 \times \mathbb{A}^3 & \longrightarrow & \mathbb{A}^3 \\ (\epsilon, x_0, x_1, x_2) & \longmapsto & (\epsilon x_0, \epsilon x_1, \epsilon x_2) \end{array}$$

The ring of invariants is given by:

$$k[x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2] \cong \frac{k[y_0, y_1, y_2, y_3, y_4, y_5]}{(y_0y_3 - y_1^2, y_0y_4 - y_1y_2, y_0y_5 - y_2^2, y_1y_4 - y_2y_3, y_1y_5 - y_2y_4, y_3y_5 - y_4^2)}$$

The singularity $X = \mathbb{A}^3/\mu_2$ corresponds to the vertex v of the affine cone X over a Veronese surface $V \subset \mathbb{P}^5$. The differential form $dx_0 \wedge dx_1 \wedge dx_2$ is a basis of $\bigwedge^3 \Omega_{\mathbb{A}^3}$, and $(dx_0 \wedge dx_1 \wedge dx_2)^{\otimes 2}$ is invariant under the action. The form

$$\omega = \frac{(dy_0 \wedge dy_1 \wedge dy_2)^{\otimes 2}}{y_0^3} \in (\bigwedge^3 \Omega_{k(X)})^{\otimes 2}$$

is a basis of $(\bigwedge^3 \Omega_X)^{\otimes 2}$ because the quotient map $\pi : \mathbb{A}^3 \to X$ is étale on $X \setminus \{v\}$, and

$$\pi^*\omega = \frac{(4x_0^6(dx_0 \wedge dx_1 \wedge dx_2))^{\otimes 2}}{x_0^6} = 4(dx_0 \wedge dx_1 \wedge dx_2)^{\otimes 2}$$

Blowing-up the vertex v we get a resolution $f : Y \to X$, and we have an affine chart isomorphic to $\mathbb{A}^3$ with coordinates (y_0, s, t) where the resolution is given by $(y_0, s, t) \mapsto (y_0, y_0s, y_0t, y_0s^2, y_0st, y_0t^2)$, and the exceptional divisor E over v is given by $\{y_0 = 0\}$. We have

$$f^*\omega = y_0(dy_0 \wedge ds \wedge dt)^{\otimes 2}$$

Therefore, $f^*\omega$ has a zero along E, and we may write $K_Y = f^*K_X + \frac{1}{2}E$.

Definition 2.4. *A normal variety X is terminal (canonical) if K_X is $\mathbb{Q}$-Cartier and there exists a resolution $f : Y \rightarrow X$ such that*

$$K_Y = f^*K_X + \sum_i a_i E_i$$

with $a_i > 0$ ($a_i \geq 0$). The rational numbers a_i are called discrepancies.

For instance, the quadric cone in Example 2.1 is canonical but not terminal, the cone over the twisted cubic in Example 2.2 is not even canonical, and the cone over the Veronese surface in Example 2.3 is terminal.
A projective variety X has canonical singularities if it is normal, some power of the canonical bundle of the smooth locus of X extends to a line bundle on V, and X has the same plurigenera as any resolution of its singularities.
A normal projective variety X has terminal singularities, if some power of the canonical line bundle of the smooth locus of X extends to a line bundle on X, and the pullback of any section of $\omega_X^{\otimes m}$ vanishes along any codimension one component of the exceptional locus of a resolution of the singularities of X.

Example 2.5. Let S be a terminal projective surface, and let $f : Y \rightarrow S$ be a resolution of S. Then

$$K_Y = f^*K_S + \sum_i a_i E_i$$

with $a_i > 0$. By Grauert-Mumford theorem the intersection matrix of the E_i is negative definite. Therefore, there exists an E_j such that

$$E_j \cdot (\sum_i a_i E_i) < 0.$$

Let us check this in the case of two components E_1, E_2. The general case will be clear. The intersection matrix

$$I = \left(\begin{array}{cc} E_1^2 & E_1E_2 \\ E_1E_2 & E_2^2 \end{array}\right)$$

is negative definite. In particular, if for the vector $a = (a_1, a_2)$ we have

$$a \cdot I \cdot a^t = a_1^2E_1^2 + 2a_1a_2E_1E_2 + a_2^2E_2^2 < 0$$

On the other hand

$$a_1^2E_1^2 + 2a_1a_2E_1E_2 + a_2^2E_2^2 = a_1E_1(a_1E_1 + a_2E_2) + a_2E_2(a_1E_1 + a_2E_2) < 0.$$

Since $a_1, a_2 > 0$ the last inequality yields either $E_1(a_1E_1 + a_2E_2) < 0$ or $E_2(a_1E_1 + a_2E_2) < 0$. Furthermore $E_j^2 < 0$. We conclude that there exists an E_j such that $E_j \cdot (\sum_i a_iE_i) < 0$ and $E_j^2 < 0$. By adjunction on the curve E_j we get

$$2g(E_j) - 2 = K_Y \cdot E_j + E_j^2 < 0$$

Therefore, $g(E_j) = 0$ and $K_Y \cdot E_j + E_j^2 = -2$. This forces, $K_Y \cdot E_j = E_j^2 = -1$. By Castelnuovo contractibility criterion [Har77, Theorem 5.7] we can contract E_j on a smooth surface. Proceeding recursively we get that S is smooth. Therefore, a surface is terminal if and only if it is smooth.
Now, let S be a surface with canonical singularities, and let $f : Y \rightarrow S$ be a minimal resolution that is there are no (-1)-curves contracted by f. We may write $K_Y = f^*K_S + \sum_i a_iE_i$ with $a_i \geq 0$. If S is not smooth we have $a_i = 0$, and

$$K_Y = f^*K_S$$

If E is a curve contracted by f we get $K_Y \cdot E = 0$ and $E^2 < 0$. This imply $2g(E) - 2 = K_Y \cdot E + E^2 = E^2 < 0$, which in turn yields $g(E) = 0$ and $E^2 = -2$. Since the intersection matrix is negative definite $(E_i + E_j)^2 < 0$, and hence $E_i \cdot E_j \leq 1$. Therefore, any contracted fiber of f is a tree of rational curves corresponding to one of the Dynkin diagrams: A_n, D_n, E_6, E_7, and E_8. Canonical surface singularities are the so called Rational Double Points, also known as Du Val singularities or ADE singularities.

2.2. **Singularities of Pairs.** Let us consider a $\mathbb{Q}$-Weil divisor $D = \sum_i d_i D_i$ on a normal variety X. We assume that the D_i's are distinct. We want to give a reasonable notion of singularities of the pair (X, D). We require that $K_X + D$ is $\mathbb{Q}$-Cartier. Then for a resolution $f : Y \to X$ we have the formula

$$K_Y = f^*(K_X + D) + \sum_i a_i E_i - \widetilde{D}$$

where $\widetilde{D}$ is the strict transform. Even when X is smooth D could be very singular. A resolution of X is meaningless for the pair (X, D).

Definition 2.6. *A divisor $D = \sum_i D_i$ on a smooth variety X is simple normal crossing if D is reduced, any component D_i of D is smooth, and D is locally defined in a neighborhood of any point by an equation in local analytic coordinates of the type*

$$z_1 \cdot \ldots \cdot z_k = 0$$

with $k \leq \dim(X)$.

Roughly speaking the singularities of D should locally look no worse that those of a union of coordinate hyperplanes.

Example 2.7. Let $D = \sum_i D_i$ where the D_i's are hyperplanes in $\mathbb{P}^n$, and let $p_i \in \mathbb{P}^{n*}$ be the point corresponding to D_i. Then D is simple normal crossing if and only if the p_i's are in linear general position.

The following is a consequence of Hironaka's theorem on resolution [Hir64] of singularities.

Theorem 2.8. *Let X be an irreducible algebraic variety over $\mathbb{C}$, and let $D \subset X$ be an effective Cartier divisor on X.*

- *There exists a projective birational morphism $f : Y \to X$, where X is smooth and $f^{-1}D \cup \mathrm{Exc}(f)$ is simple normal crossing. The morphism f is called a log resolution of the pair (X, D).*
- *The smooth variety Y can be constructed as a sequence of blow-ups along smooth centers supported in the singular loci of D and X. In particular f is an isomorphism over $X \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(D))$.*

We will need many times the following result.

Proposition 2.9. *Let X be a smooth variety, $Z \subset X$ a smooth subvariety with $\mathrm{codim}_Z(Y) = c \geq 2$, and $\pi : Y \to X$ the blow-up of X along Z with exceptional divisor E. Then*

$$\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$$

Furthermore,

$$K_Y = \pi^* K_X + (c-1)E$$

Proof. Let us consider the map

$$\begin{array}{ccc} \psi : \mathbb{Z} & \to & \mathrm{Pic}(Y) \\ n & \longmapsto & nE \end{array}$$

By [Har77, Proposition 6.5] we have an exact sequence

$$\mathbb{Z} \to \mathrm{Pic}(Y) \to \mathrm{Pic}(Y \setminus E) \to 0$$

Let us assume that $nE \sim 0$ for some $n \neq 0$. Then there exists $f \in k(Y)$ with a zero of order n along E. Since π is surjective and birational, the function f induces a function $g \in k(X)$ having only a zero of order n a long Z. A contradiction because $c = \mathrm{codim}_Z(Y) \geq 2$. Therefore we have the exact sequence

$$0 \mapsto \mathbb{Z} \to \mathrm{Pic}(Y) \to \mathrm{Pic}(Y \setminus E) \mapsto 0. \tag{2.1}$$

Since π is an isomorphism outside E we have $\mathrm{Pic}(Y \setminus E) \cong \mathrm{Pic}(X \setminus Z)$, furthermore $c \geq 2$ yields $\mathrm{Pic}(X \setminus Z) \cong \mathrm{Pic}(X)$, and

$$\mathrm{Pic}(Y \setminus E) \cong \mathrm{Pic}(X \setminus Z) \cong \mathrm{Pic}(X)$$

Therefore, the pull-back map $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ gives a section of the second map in the exact sequence 2.1. This implies that the sequence 2.1 splits and $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$.
Now, we may write $K_Y = \pi^* D + qE$ for some $D \in \mathrm{Pic}(X)$. The isomorphism $X \setminus Z \cong Y \setminus E$ yields $K_{Y|Y\setminus E} \cong K_{X|X\setminus Z}$. Since $\mathrm{Pic}(X \setminus Z) \cong \mathrm{Pic}(X)$ we get $D = K_X$, and $K_Y = \pi^* K_X + qE$.
Now, our aim is to determine the integer q. By adjunction and using $\mathcal{O}_Y(E)_{|E} = \mathcal{O}_E(-1)$ we get

$$K_E \cong (K_Y + E)_{|E} \cong (\pi^* K_X + (q+1)E)_{|E} = \pi^* K_X - (q+1)E$$

Let $F = z \times_Z E$ be the fiber over a point $z \in Z$. Then

$$\omega_F = \pi_1^* \omega_z \otimes \pi_2^* \omega_E = \pi_1^* \omega_z \otimes \pi_2^* (\pi^* \omega_X \otimes \mathcal{O}_E(-q-1)) = \pi_2^* (\pi^* \omega_X \otimes \mathcal{O}_E(-q-1))$$

Now, a differential form on Y that is the pullback of a differential form on X must vanish on E. In particular $\pi_2^*(\pi^* \omega_X)$ is trivial, and

$$\omega_F \cong \pi_2^*(\mathcal{O}_E(-q-1)) \cong \mathcal{O}_F(-q-1)$$

On the other hand, $F \cong \mathbb{P}^{c-1}$. Therefore, $\omega_F \cong \mathcal{O}_F(-c)$ implies $q = c - 1$. □

Example 2.10. Let $Z \subset \mathbb{P}^n$ be a smooth variety of codimension c, $\pi : Y \to \mathbb{P}^n$ the blow-up of Z, H the pullback of the hyperplane class of $\mathbb{P}^n$ and E the exceptional divisor. Then

$$K_Y = (-n-1)H + (c-1)E$$

Now, let us assume that X and D are both smooth and consider $(1+\epsilon)D$. The $Id_X : X \to X$ is a log resolution and

$$K_X = Id_X^*(K_X + (1+\epsilon)D) - (1+\epsilon)D$$

Let $\pi_1 : X_1 \to X$ be the blow-up of a codimension two smooth subvariety $Z_1 \subset D$. Then

$$K_{X_1} = \pi_1^*(K_X + (1+\epsilon)D) - \epsilon E_1 - (1+\epsilon)D_1$$

where D_1 is the strict transform of D. Now, let $f : X_2 \to X_1$ be the blow-up of $D_1 \cap E_1$, and $\pi_2 = f \circ \pi_1$. Then

$$K_{X_2} = \pi_2^*(K_X + (1+\epsilon)D) - 2\epsilon E_2 - \epsilon E_1 - (1+\epsilon)D_2$$

Proceeding like this we see that starting with a discrepancy less than -1 we can produce arbitrarily negative discrepancies. This motivates the following definition.

Definition 2.11. *Let X be a normal variety and $D = \sum_j d_j D_j$ be a $\mathbb{Q}$-Weil divisor. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of the pair (X, D) and write*

$$K_Y = f^*(K_X + D) + \sum_i a_i E_i - \widetilde{D}$$

The pair (X, D) is

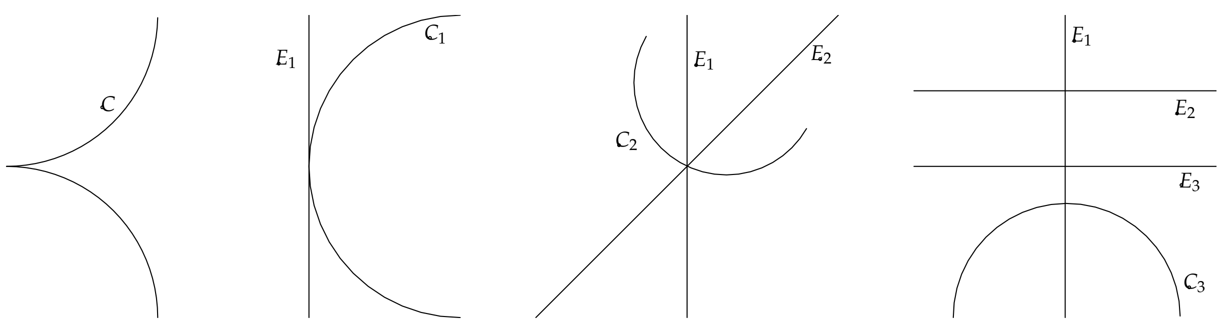
terminal	*if*	$a_i > 0$ *for any* i,
canonical	*if*	$a_i \geq 0$ *for any* i,
klt	*if*	$a_i > -1$ *and* $d_j < 1$ *for any* i, j,
plt	*if*	$a_i > -1$ *for any* i,
lc	*if*	$a_i \geq -1$ *for any* i.

Here klt, plt, lc stands for Kawamata log terminal, purely log terminal, and log canonical respectively.

Example 2.12. Assume that D is a simple normal crossing divisor, and that X is smooth. Then Id_X is a log resolution. If $0 < \epsilon < 1$ is a rational number then we have $K_X = Id_X^*(K_X + \epsilon D) - \epsilon D$. The pair $(X, \epsilon D)$ is Kawamata log terminal.
Let $D \subset \mathbb{P}^2$ an irreducible curve with one node, and let $f : Y \to \mathbb{P}^2$ be the blow-up of the node. Then $f^{-1}D \cup E$ is simple normal crossing. Furthermore $K_Y = f^*K_{\mathbb{P}^2} + E$ and $f^*D = \widetilde{D} + 2E$ where $\widetilde{D}$ is the strict transform of D, yield

$$K_Y = f^*(K_{\mathbb{P}^2} + D) - \widetilde{D} - E$$

Therefore the pair $(\mathbb{P}^2, D)$ is log canonical.
Now, let us consider a cusp $D \subset \mathbb{P}^2$ to have a log resolution we have to blow-up three times.



Let $\epsilon_1 : X_1 \to \mathbb{P}^2$ be the first blow-up. We have $K_{X_1} = \epsilon_1^* K_{\mathbb{P}^2} + E_1$ and $C_1 = \epsilon_1^* C - 2E_1$. If $\epsilon_2 : X_2 \to X_1$ is the second blow-up we have $K_{X_2} = \epsilon_2^*(\epsilon_1^* K_{\mathbb{P}^2} + E_1) + E_2 = \epsilon_2^*\epsilon_1^* K_{\mathbb{P}^2} + E_1 + 2E_2$ and $C_2 = \epsilon_2^* C_1 - E_2 = \epsilon_2^*\epsilon_1^* C - 2E_1 - 3E_2$. Finally, let $\epsilon_3 : X_3 \to X_2$ be the third blow-up. Then $K_{X_3} = \epsilon_3^*\epsilon_2^*\epsilon_1^* K_{\mathbb{P}^2} + E_1 + 2E_2 + 4E_3$ and $C_3 = \epsilon_3^* C_2 - E_3 = \epsilon_3^*\epsilon_2^*\epsilon_1^* C - 2E_1 - 3E_2 - 6E_3$. Let $\epsilon = \epsilon_1 \circ \epsilon_2 \circ \epsilon_3$. Summing up we have

$$K_{X_3} = \epsilon^* K_{\mathbb{P}^2} + E_1 + 2E_2 + 4E_3$$
$$C_3 = \epsilon^* C - 2E_1 - 3E_2 - 6E_3$$

Therefore, we get

$$K_{X_3} = \epsilon^*(K_{\mathbb{P}^2} + C) - C_3 - E_1 - E_2 - 2E_3$$

In particular, $a_i(E_3, \mathbb{P}^2, D) = -2$ and $(\mathbb{P}^2, D)$ is not log canonical.

Now, let us consider a slightly more complicated example.

Example 2.13. Let us consider the cubic surface

$$S = \{x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0\} \subset \mathbb{P}^3$$

known as the Cayley nodal cubic surface. By taking partial derivatives it is easy to see that the singular locus of S consists of the four coordinates points of $\mathbb{P}^3$, and that each of them is a point of multiplicity two for S. Let us consider the point $p = [1 : 0 : 0 : 0]$. In the chart $\mathcal{U}_0 := \{x_0 \neq 0\}$ the equation of S is given by $\{x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3 = 0\}$. Therefore, the projective tangent cone of S in p is the conic $\{x_1x_2 + x_1x_3 + x_2x_3 = 0\} \subset \mathbb{P}^3$. Since this conic is smooth p is an ordinary double point. We conclude that the fundamental points of $\mathbb{P}^3$ are ordinary singularities for S, and hence S can be resolved simply by blowing-up these four points. Now, let $\pi : Y \to \mathbb{P}^3$ be the blow-up with exceptional divisors $E_1, ..., E_4$. Then we may write

$$K_Y = \pi^* K_{\mathbb{P}^3} + 2(E_1 + E_2 + E_3 + E_4)$$

and

$$\epsilon\widetilde{D} = \pi^*(\epsilon D) - 2\epsilon(E_1 + E_2 + E_3 + E_4)$$

Therefore

$$K_Y = \pi^*(K_{\mathbb{P}^3} + \epsilon D) + (2 - 2\epsilon)(E_1 + E_2 + E_3 + E_4)$$

and since $2 - 2\epsilon > -1$ if and only if $\epsilon < \frac{3}{2}$ we get that $(\mathbb{P}^3, \epsilon S)$ is klt if and only if $\epsilon < 1$.

3. Mori Dream Spaces

In this section we address the notion of Mori dream space.

Definition 3.1. *A normal projective variety X is a Mori Dream Space if*

(a) *X is $\mathbb{Q}$-factorial and* $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathbb{Q}}$*;*
(b) $\mathrm{Nef}(X)$ *is generated by finitely many semi-ample line bundles;*
(c) *there exist finitely many small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$ such that each X_i satisfies* (a), (b)*, and* $\mathrm{Mov}(X)$ *us the union of* $f_i^* \mathrm{Nef}(X_i)$*.*

Remark 3.2. Condition (a) is equivalent to the finite generation of $\mathrm{Pic}(X)$ which is equivalent to $h^1(X, \mathcal{O}_X) = 0$. Note that if X is a Mori Dream Space then the X_i are Mori Dream Spaces as well.

- A normal $\mathbb{Q}$-factorial projective variety of Picard number is one is a Mori Dream Space if and only if $\mathrm{Pic}(X)$ is finitely generated.
- Let X be a normal $\mathbb{Q}$-factorial projective surface satisfying (a), (b), then $\mathrm{Nef}(X) = \mathrm{Mov}(X)$ and, by taking Id_X, we see that (c) is satisfied as well.
- Any projective $\mathbb{Q}$-factorial toric variety and any smooth Fano variety is a Mori Dream Space.
- If X is a smooth rational surface and $-K_X$ is big the X is a Mori Dream Space.
- A smooth $K3$ surface is a Mori Dream Space if and only if its automorphism group is finite.

We recall two important facts about Mori Dream Space.

Proposition 3.3. *Let X a be a Mori Dream Space.*

- *Any normal projective variety Y which is a small $\mathbb{Q}$-factorial modification of X is a Mori Dream Space. Furthermore the f_i of Definition 3.1 are the only small $\mathbb{Q}$-factorial transformations of X,* [HK00, Proposition 1.11].
- *If there is a surjective morphism $X \to Y$ on a normal $\mathbb{Q}$-factorial projective variety Y, then Y is a Mori Dream Space,* [Oka16, Theorem 1.1].

Definition 3.4. *Let Γ be a semigroup of Weil divisors on X. We can consider the Γ-graded ring:*

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D))$$

If the divisor class group $\mathrm{Cl}(X)$ *is finitely generated and Γ is a group of Weil divisors such that* $\Gamma_{\mathbb{Q}} \cong \mathrm{Cl}(X)_{\mathbb{Q}}$ *then the ring $R_X(\Gamma)$ is denoted by* $\mathrm{Cox}(X)$*, and called the Cox ring of X.*

Remark 3.5. Let X be a normal and $\mathbb{Q}$-factorial projective variety with finitely generated and free Picard group and Picard number ρ_X. Let $D_1, ..., D_{\rho_X}$ be a basis of Cartier divisors of $\mathrm{Pic}(X)$. Then

$$\mathrm{Cox}(X) = \bigoplus_{m_1,...,m_{\rho_X} \in \mathbb{Z}} H^0(X, \sum_{i=1}^{\rho_X} m_i D_i)$$

Different choices of divisors $D_1, ..., D_{\rho_X}$ yield isomorphic algebras.

For the details of the proof of the following Theorem we refer to [HK00, Proposition 2.9].

Theorem 3.6. *A $\mathbb{Q}$-factorial projective variety X with* $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathbb{Q}}$ *is a Mori Dream Space if and only if* $\mathrm{Cox}(X)$ *is finitely generated. In this case X is a GIT quotient of the affine variety* $Y = \mathrm{Spec}(\mathrm{Cox}(X))$ *by a torus of dimension ρ_X.*

Proof. Let X be a Mori Dream Space. Then the effective cone is rational and polyhedral and we have a decomposition:

$$\mathrm{Eff}(X) = \bigcup_{i=1}^{k} P_i$$

where the P_i's are rational polyhedra. Furthermore there are finitely many rational maps $f_i : X \dashrightarrow X_i$ such that if $D \in \mathrm{Eff}(X)$ then $f_D = f_i$ for some $i = 1, ..., k$. Let us take $D_1, ..., D_h$ divisors generating the cone P_i. The cone $R_X(D_1, ..., D_h)$ does not change by replacing X with X_i and $D_1, ..., D_h$ by the corresponding divisors $D_{1,i}, ..., D_{h,i}$ on X_i. On X_i the divisors $D_{1,i}, ..., D_{h,i}$ are semi-ample. Then $R_{X_i}(D_{1,i}, ..., D_{h,i})$, and hence $R_X(D_1, ..., D_h)$ are finitely generated.
Now, let us assume that $\mathrm{Cox}(X)$ is finitely generated. Then we have an equivariant embedding, with respect a torus G, of $Y = \mathrm{Spec}(\mathrm{Cox}(X))$ is $\mathbb{A}^n$. Taking the GIT quotient we have an embedding $Y \subseteq Q = \mathbb{A}^n // G$. Since G is a torus Q is a toric variety and hence a Mori Dream Space. Furthermore if $r : X \dashrightarrow Y$ is a rational map then there is a rational map of toric varieties $t : M \dashrightarrow N$ inducing r by restriction. Therefore X is a Mori Dream Space. □

3.1. **Fano-type varieties.** In this section we address log Fano and weak Fano varieties.

Definition 3.7. *Let X be a smooth projective variety. We say that X is:*

- *weak Fano if $-K_X$ is nef and big,*
- *log Fano if there exists an effective divisor D such that $-(K_X + D)$ is ample and the pair (X, D) is Kawamata log terminal. In particular if $D = 0$ we have terminal Fano varieties,*
- *weak log Fano if there exists an effective divisor D such that $-(K_X + D)$ is neg and big, and the pair (X, D) is Kawamata log terminal.*

For instance, any toric variety is log Fano, a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d is log Fano if and only if $d \leq n$.
If X is a normal $\mathbb{Q}$ projective variety with $\rho(X) = 1$ then X is a Mori Dream Space if and only if $\mathrm{Pic}(X)$ is finitely generated. For instance, the only Mori Dream Space of dimension one is $\mathbb{P}^1$.

The bridge between Mori Dream Spaces and log Fano varieties is the content of the following proposition.

Proposition 3.8. [BCHM10, Corollary 1.3.2] *Let X be a smooth projective variety. If X is log Fano then X is a Mori Dream Space .*

Remark 3.9. On the other hand a Mori Dream Space is not necessarily log Fano. Indeed, by Grothendieck-Lefschetz theorem if $X \subset \mathbb{P}^n$ is a general hypersurface and $n \geq 4$ then $\text{Pic}(X) \cong \mathbb{Z}$ is generated by $X \cap H$ where H is a general hyperplane in $\mathbb{P}^n$. Therefore, X is a Mori Dream Space. On the other hand, if $d = \deg(X)$ then X is not rationally connected as soon as $d \geq n+1$. In particular if $d \geq n+1$ the hypersurface X is not log Fano.
By Noether-Lefschetz theorem we have $\text{Pic}(S_d) \cong \mathbb{Z}$ and generated by the restriction of the hyperplane section of $\mathbb{P}^3$ for a general surface of degree $d \geq 4$ in $\mathbb{P}^3$. These give other examples of Mori Dream Spaces that are not log Fano.
Even when X is a Mori Dream Space with big and movable anti-canonical divisor it is not necessarily log Fano. Indeed we have the following:

Proposition 3.10. [CG13, Proposition 2.6] *Let X be a projective $\mathbb{Q}$-factorial variety which is a Mori Dream Space, and let $L_1, ..., L_m$ be ample line bundles on X. Then*

$$Y = \mathbb{P}(\bigoplus_{i=1}^{m} L_i)$$

is a Mori Dream Space.

Now, following [CG13, Example 5.1] we consider a smooth projective variety X of general type such that $H^1(X, \mathcal{O}_X) = 0$ and $\rho(X) = 1$. Let $\mathcal{E} = L_1 \oplus L_2 \oplus (\omega_X^\vee \oplus L_1^\vee \oplus L_2^\vee)$, and $Y = \mathbb{P}(\mathcal{E})$. Then $-K_Y$ is big and movable. On the other hand if Y would be rationally connected then X would be rationally connected as well. A contradiction because X is of general type. Therefore Y is not rationally connected and in particular it is not log Fano.

The following is an important result in order to achieve, among other things, an useful characterization of big divisors.

Lemma 3.11. *(Kodaira's Lemma) Let D and E be respectively a big and an effective Cartier divisor on a projective variety X. Then*

$$H^0(X, mD - E) \neq 0$$

for $m \gg 0$.

Proof. Since D is big there exists a constant $c > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq c \cdot m^{\dim(X)}$ for $m \gg 0$. On the other hand $\dim(E) = \dim(X) - 1$ implies that $h^0(X, \mathcal{O}_E(mD))$ grows at most like $m^{\dim(X)-1}$, and $h^0(X, \mathcal{O}_X(mD)) > h^0(X, \mathcal{O}_E(mD))$ for $m \gg 0$.
Now, let us consider the following exact sequence:

$$0 \to \mathcal{O}_X(mD - E) \to \mathcal{O}_X(mD) \to \mathcal{O}_E(mD) \to 0$$

By taking cohomology we get

$$h^0(X, \mathcal{O}_X(mD - E)) \geq h^0(X, \mathcal{O}_X(mD)) - h^0(X, \mathcal{O}_E(mD)) > 0$$

for $m \gg 0$. □

Lemma 3.12. *Let D be a divisor on an irreducible projective variety X then D is big if and only if for any integer ample divisor A on X there exist an integer m and an effective divisor E such that $mD \sim_{lin} A + E$.*

Proof. Assume that D is big and consider $mD - rA$ with $r \gg 0$. Then rA and $(r-1)A$ are both effective and by Lemma 3.11 we get $H^0(X, mD - rA) \neq 0$. Therefore, there exists an effective divisor E such that $mD - rA \sim_{lin} E$. That is

$$mD \sim_{lin} A + (r-1)A + E = A + E^{'}$$

where $E^{'} = (r-1)A + E$ is effective.
Now, let $mD \sim_{lin} A + E$ with A ample and E effective. Therefore, possibly passing to an higher multiple, we have $r \cdot mD \sim_{lin} rA + rE$ with $H = rA$ very ample, and rE effective. Then

$$\mathrm{kod}(X, D) \geq \mathrm{kod}(X, H) = \dim(X)$$

and D is big. $\square$

Remark 3.13. Note that in the proof of Lemma we have to consider a multiple of A in order to have an effective divisor. To see this for instance consider three general points $p_1, p_2, p_3 \in C$ where C is a smooth curve of genus $g = 2$. The divisor $D = p_1 + p_2 - p_3$ is ample, indeed $\deg(5D) = 5 = 2g + 1$ and by [Har77, Corollary 3.2] $5D$ is very ample. Then D is ample. Now, let us consider $D^{'} = p_1 + p_2$. Then $\deg(K_C - D^{'}) = 0$. If $h^0(K_C - D^{'}) \neq 0$ then $\deg(K_C - D^{'}) = 0$ yields $K_C - D^{'} \sim 0$ and $h^0(K_C - D^{'}) = 1$. On the other hand $h^0(K_C) = 2$, and since p_1, p_2 are general they impose independent conditions to the differential forms on C, that is $h^0(K_C - D^{'}) = 0$. By Riemann-Roch this gives $h^0(p_1 + p_2) = 1$. Now, assume that $h^0(p_1 + p_2 - p_3) \neq 0$. The inclusion $H^0(C, p_1 + p_2 - p_3) \subseteq H^0(C, p_1 + p_2)$ forces $H^0(C, p_1 + p_2 - p_3) = H^0(C, p_1 + p_2)$, that is any global section $s \in H^0(C, p_1 + p_2) \cong k$ vanishes at p_3. Therefore s is zero because it is constant. This implies $h^0(p_1 + p_2) = 0$, a contradiction. We conclude that $H^0(C, p_1 + p_2 - p_3) = 0$, that is there is no effective divisor on C linearly equivalent to $p_1 + p_2 - p_3$.

Lemma 3.14. *Let D be a nef and big divisor on an irreducible projective variety X. Then there exist an effective divisor E such that $D - \epsilon E$ is ample for $0 < \epsilon \ll 1$.*

Proof. Let D be a nef and big divisor. Since D is big, by Lemma 3.13, there exist an ample divisor A, an effective divisor E, and a positive integer k such that $kD \equiv A + E$. If $h > k$ we can write $hD \equiv (h-k)D + A + E$. The divisor $D^{'} = (h-k)D + A$ is a sum of a nef and an ample divisor. Therefore $D^{'}$ is ample. If $\epsilon = \frac{1}{h}$ we get that

$$D - \epsilon E \equiv \epsilon D^{'}$$

is ample. $\square$

Proposition 3.15. *Let X be normal, irreducible, projective variety with at most klt singularities. If X is weak Fano then X is log Fano.*

Proof. Since X is weak Fano $-K_X$ is nef and big. By Lemma 3.14 there exists an effective divisor D and a rational number $0 < \epsilon \ll 1$ such that $-K_X - \epsilon D = -(K_X + \epsilon D)$ is ample. The pair $(X, \epsilon D)$ is klt for $\epsilon \ll 1$ because X has at most klt singularities. $\square$

Remark 3.16. The converse of Proposition 3.15 is false. For instance the Hirzebruch surface $X_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ is a toric surface and hence log Fano. The anti-canonical divisor is $-K_{X_e} = 2C_0 + (2+e)F$, where C_0 is the section and F is the fiber. Therefore $-K_{X_e} \cdot C_0 = 2C_0^2 + 2 + e =$

$-e+2$, and $-K_{X_e}$ is not nef for $e > 2$. We conclude that for any $e > 2$ the Hirzebruch surface X_e is log Fano but not weak Fano.

It is quite easy to see that projective toric varieties are log Fano.

Lemma 3.17. *Let $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on a normal projective variety X such that $d_i < 1$ and the pair $(X, \lceil D \rceil)$ is lc. Then (X, D) is klt.*

Proof. Let $f : Y \rightarrow X$ be a log resolution of the pair $(X, \lceil D \rceil)$. We have

$$K_Y = f^* K_X + \sum_i a_i E_i$$

and

$$\lceil \widetilde{D} \rceil = f^* \lceil D \rceil - \sum_i b_i E_i$$

where $\lceil \widetilde{D} \rceil$ is the strict transform of $\lceil D \rceil$. Therefore,

$$K_Y = f^*(K_X + \lceil D \rceil) + \sum_i (a_i - b_i) - \lceil \widetilde{D} \rceil$$

and since $(X, \lceil D \rceil)$ is lc we have $a_i - b_1 \geq -1$. On the other hand

$$\widetilde{D} = f^* D - \sum_i t_i E_i$$

with $t_i < b_i$ because $d_i < 1$ for any i. This yields $a_i - t_i > a_i - b_i \geq -1$, and the pair (X, D) is klt. □

Proposition 3.18. *Let X be a projective toric variety. Then X is log Fano.*

Proof. Let $D_1^X, ..., D_r^X$ be the irreducible toric invariant divisors on X. Then we have $K_X = -\sum_i D_i^X$. Now, let $A = \sum_i a_i D_i^X$ be an ample toric invariant divisor, and ϵ a rational number $0 < \epsilon \ll 1$. Therefore

$$-K_X - \epsilon A = \sum_i (1 - \epsilon a_i) D_i^X$$

with $1 - \epsilon a_i < 1$. The divisor $D = \sum_i (1 - \epsilon a_i) D_i^X$ is such that $\epsilon A = -K_X - D$ is ample. Note that $\lceil D \rceil \sim -K_X$. Let $f : Y \rightarrow X$ be a toric log resolution of $(X, \lceil D \rceil)$, and let $D_1^Y, ..., D_h^Y$ be the invariant toric divisors on Y. We have

$$K_Y = f^*(K_X + \lceil D \rceil) + \sum a_i E_i - \lceil \widetilde{D} \rceil = \sum a_i E_i - \lceil \widetilde{D} \rceil$$

because $\lceil D \rceil \sim -K_X$. On the other hand $K_Y = -\sum_i D_i^Y$ yields

$$K_Y = \sum a_i E_i - \lceil \widetilde{D} \rceil = -\sum_i D_i^Y$$

This forces $a_i = -1$ for any i. Therefore, the pair $(X, \lceil D \rceil)$ is lc. To conclude it is enough to apply Lemma 3.17. □

It turns out that weak log Fano is equivalent to log Fano.

Proposition 3.19. *Let X be a projective variety with at most klt singularities. Then X is log Fano if and only if X is weak log Fano.*

Proof. Clearly X log Fano implies X weak log Fano. Now, let X be weak log Fano. Then there exists an effective divisor D such that $-K_X - D$ is big and nef and (X, D) is klt. By Lemma 3.14 there exists an effective divisor E such that $-K_X - D - \epsilon E = -K_X - (D + \epsilon E)$ is ample for $0 < \epsilon \ll 1$. Let $D' = D + \epsilon E$. Therefore, D' is effective and $-K_X - D'$ is ample. Furthermore, since X has at most klt singularities and (X, D) is klt we get that (X, D') is klt for $0 < \epsilon \ll 1$. □

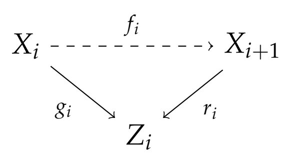
Finally, we have two important facts about log Fano varieties. We will prove just the latter, for the first one we refer to [GOST15].

Lemma 3.20. [GOST15, Corollary 1.3] *Let $f : X \to Y$ be a projective surjective morphism between normal projective varieties over an algebraically closed field of characteristic zero. If X is log Fano then Y is log Fano.*

The second result says that being log Fano is preserved under small transformations.

Lemma 3.21. *Let X and Y be normal varieties over a field of characteristic zero that are isomorphic in codimension one. Then X is log Fano if and only if Y is so.*

Proof. There exists a small transformation $f : X \dashrightarrow Y$. Such a small transformation can be factored as $f = f_k \circ ... \circ f_1$ where any $f_i : X_i \dashrightarrow X_{i+1}$ is small, and fits in a diagram of the following form



where f_i is a small projective birational contraction. To conclude, we have to prove that if X and Y are normal varieties over a field of characteristic zero and $f : X \to Y$ is a small birational morphism the X is log Fano if and only if Y is log Fano.
Assume that X is log Fano. Then there exists D effective such that $-K_X - D$ is ample and (X, D) is klt. Let us take an ample divisor H on Y such that $-K_X - D - \epsilon f^*H$ is ample and $(X, D + \epsilon f^*H)$ is klt. Note that since f is small $f_*(D + \epsilon f^*H)$ may not be $\mathbb{Q}$-Cartier. To deal with this we need the following trick. We take an ample divisor A on X such that $(X, D + \epsilon f^*H + A)$ is klt and

$$K_X + D + \epsilon f^*H + A \sim_{\mathbb{Q}} 0$$

Therefore,

$$K_Y + f_*D + \epsilon H + f_*A = f_*(K_X + D + \epsilon f^*H + A) \sim_{\mathbb{Q}} 0$$

Now, since f is small we have

$$f^*(K_Y + f^*D + \epsilon H + f_*A) = K_X + D + \epsilon f^*H + A$$

We conclude that $(Y, f_*D + f_*A)$ is klt and $-(K_Y + f_*D + f_*A) \sim_{\mathbb{Q}} \epsilon H$ is ample.
Now, let us assume that Y is log Fano, and let D an effective divisor on Y such that $-K_Y - D$ is ample and (Y, D) is klt. Let $\widetilde{D}$ be the strict transform of D in X. Since f is small we have

$$K_X + \widetilde{D} = f^*(K_Y + D)$$

Therefore, $(X, \widetilde{D})$ is klt and $-K_X - \widetilde{D}$ is nef and big. This means that X is weak log Fano, and by Proposition 3.19 it is log Fano. □

4. Decompositions of the effective cone

Let X be a Mori dream space. We describe a fan structure on the effective cone $\mathrm{Eff}(X)$, called the *Mori chamber decomposition*. We refer to [HK00, Proposition 1.11] and [Oka16, Section 2.2] for details. There are finitely many birational contractions from X to Mori dream spaces, denoted by $g_i : X \dashrightarrow Y_i$. The set $\mathrm{Exc}(g_i)$ of exceptional prime divisors of g_i has cardinality $\rho(X/Y_i) = \rho(X) - \rho(Y_i)$. The maximal cones $\mathcal{C}$ of the Mori chamber decomposition of $\mathrm{Eff}(X)$ are of the form: $\mathcal{C}_i = \langle g_i^*(\mathrm{Nef}(Y_i)), \mathrm{Exc}(g_i)\rangle$. We call $\mathcal{C}_i$ or its interior $\mathcal{C}_i^{\circ}$ a *maximal chamber* of $\mathrm{Eff}(X)$.

Let X be a normal $\mathbb{Q}$-factorial projective variety, and let D be an effective $\mathbb{Q}$-divisor on X. The stable base locus $\mathbf{B}(D)$ of D is the set-theoretic intersection of the base loci of the complete linear systems $|sD|$ for all positive integers s such that sD is integral

$$\mathbf{B}(D) = \bigcap_{s>0} B(sD)$$

Since stable base loci do not behave well with respect to numerical equivalence, we will assume that $h^1(X, \mathcal{O}_X) = 0$ so that linear and numerical equivalence of $\mathbb{Q}$-divisors coincide.

Then numerically equivalent $\mathbb{Q}$-divisors on X have the same stable base locus, and the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ of X can be decomposed into chambers depending on the stable base locus of the corresponding linear series called *stable base locus decomposition*. The *movable cone* of X is the convex cone $\mathrm{Mov}(X) \subset N^1(X)$ generated by classes of *movable divisors* that is divisors whose stable base locus has codimension at least two in X.

If X is a Mori dream space, satisfying then the condition $h^1(X, \mathcal{O}_X) = 0$, determining the stable base locus decomposition of $\mathrm{Eff}(X)$ is a first step in order to compute its Mori chamber decomposition.

Remark 4.1. Recall that two divisors D_1, D_2 are said to be *Mori equivalent* if $\mathbf{B}(D_1) = \mathbf{B}(D_2)$ and the following diagram of rational maps is commutative

$$\begin{array}{ccc} & X & \\ {\scriptstyle\phi_{D_1}}\swarrow & & \searrow{\scriptstyle\phi_{D_2}} \\ X(D_1) & \xrightarrow{\ \sim\ } & X(D_2) \end{array}$$

where the horizontal arrow is an isomorphism. Therefore, the Mori chamber decomposition is a refinement of the stable base locus decomposition.

4.1. The Cox ring of a toric variety and the GKZ decomposition.

Recall that given a toric variety X_Σ we have the total coordinate ring

$$S = \mathbb{C}[x_\rho \mid \rho \in \Sigma(1)]$$

and the Cox ring

$$\mathrm{Cox}(X_\Sigma) = \bigoplus_{(m_1,\ldots,m_r)\in\mathbb{Z}^r} H^0(X_\Sigma, \mathcal{O}_{X_\Sigma}(m_1D_1 + \cdots + m_rD_r))$$

where $\{D_1, \ldots, D_r\}$ is a basis of $\mathrm{Cl}(X_\Sigma)$.

Theorem 4.2. *Let X_Σ be a simplicial projective toric variety. The degree D part S_D of S coincides with $H^0(X_\Sigma, D)$. In particular $\mathrm{Cox}(X_\Sigma) \cong S$ is finitely generated and hence X_Σ is a Mori dream space.*

Proof. Let $D = \sum_{\rho\in\Sigma(1)} \alpha_\rho D_\rho$. Then $H^0(X_\Sigma, D)$ is generated by monomials x^m such that $\langle m, u_\rho\rangle \geq -\alpha_\rho$ for all $\rho \in \Sigma(1)$. Consider the map

$$\begin{array}{cccc} \psi: & H^0(X_\Sigma, D) & \longrightarrow & S_D \\ & x^m & \longmapsto & (x_1^{\langle m,u_1+\alpha_1\rangle}, \dots, x_k^{\langle m,u_k+\alpha_k\rangle})) \end{array}$$

where $k = |\Sigma(1)|$. Note that $\deg(x_1^{\langle m,u_1+\alpha_1\rangle} \dots x_k^{\langle m,u_k+\alpha_k\rangle})) = \sum_\rho(\langle m, u_\rho + \alpha_\rho\rangle)D_\rho = \sum_\rho \langle m, u_\rho\rangle D_\rho + \sum_\rho \alpha_\rho D_\rho = \operatorname{div}(\chi^m) + D \sim D$.

Since the u_ρ span $N_\mathbb{R}$ the morphism ψ is injective. Now, let $x_1^{\beta_1} \dots x_k^{\beta_k} \in S_D$. Then $\sum_\rho(\beta_\rho - \alpha_\rho)D_\rho \sim D - D = 0$. So the exact sequence

$$M \to \mathbb{Z}^{|\Sigma(1)|} \to \operatorname{Cl}(X_\Sigma) \to 0$$

yields that there is an $m \in M$ such that $\beta_\rho - \alpha_\rho = \langle m, u_\rho\rangle$ for all $\rho \in \Sigma(1)$. Furthermore, $\beta_\rho - \alpha_\rho \geq -\alpha_\rho$ implies that $x^m \in H^0(X_\Sigma, D)$, and by construction $\psi(x^m) = x_1^{\beta_1} \dots x_k^{\beta_k}$. Hence ψ is an isomorphism. □

Our aim in to describe the so called secondary fan of a toric variety that is the Mori chamber decomposition of its effective cone. The first step consists in determining the effective cone itself.

Proposition 4.3. *Let X_Σ be a simplicial toric variety. The effective cone* $\operatorname{Eff}(X_\Sigma)$ *is generated by classes of torus invariant divisors, that is*

$$\operatorname{Eff}(X_\Sigma) = \overline{\operatorname{Eff}}(X_\Sigma) = \langle [D_\rho] \mid \rho \in \Sigma(1)\rangle$$

Proof. Since X_Σ is simplicial any D_ρ is $\mathbb{Q}$-Cartier and effective, so

$$\langle [D_\rho] \mid \rho \in \Sigma(1)\rangle \subseteq \overline{\operatorname{Eff}}(X_\Sigma)$$

Now, let D be a torus invariant divisor. Since $\operatorname{Cl}(X_\Sigma)$ is generated by torus invariant divisors we may assume that D is torus invariant. Since D is effective we have that $H^0(X_\Sigma, \mathcal{O}_\Sigma(D)) \neq 0$. Moreover, since

$$H^0(X_\Sigma, \mathcal{O}_\Sigma(D)) = \bigoplus_{\operatorname{div}(\chi^m)+D\geq 0} \mathbb{C}\chi^m$$

there exists $m \in M$ such that $\operatorname{div}(\chi^m) + D \geq 0$ and hence we conclude that $[D] = [\operatorname{div}(\chi^m) + D] \in \langle [D_\rho] \mid \rho \in \Sigma(1)\rangle$. □

Furthermore, we have the following description of the movable cone of a Mori dream space.

Proposition 4.4. [ADHL15, Proposition 3.3.2.3] *Let X be a Mori dream space and $\{f_i,\ i \in I\}$ be any system of pairwise distinct generators of* $\operatorname{Cox}(X)$*. Then*

$$\operatorname{Mov}(X) = \bigcap_{i\in I} \langle \deg(f_j),\ j \in I \setminus \{i\}\rangle$$

Now, consider on $\operatorname{Eff}(X_\Sigma$ the subdivision in cones obtained by considering all the subspaces of $\operatorname{Cl}(X_\Sigma)_\mathbb{R}$ generated by all the subsets of a fixed set of generators of $\operatorname{Cox}(X_\Sigma)$. This subdivision induces a wall-and-chamber decomposition on $\operatorname{Eff}(X_\Sigma$ called Gelfand-Kapranov-Zelevinsky decomposition, GKZ for short, of $\operatorname{Eff}(X_\Sigma$.

A divisor $D \in \operatorname{Eff}(X_\Sigma$ yields a map $\phi_D : X_\sigma \dashrightarrow X_D$, and

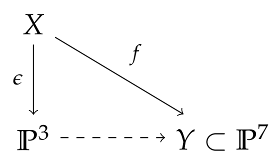
- if $[D]$ lies in the interior of $\operatorname{Nef}(X_\Sigma)$, that is D is ample, then ϕ_D is an isomorphism;

- if $[D]$ lies in the interior of a chamber of $\mathrm{Mov}(X_\Sigma)$ but not in the interior of $\mathrm{Nef}(X_\Sigma)$ then ϕ_D is a small $\mathbb{Q}$-factorial transformation;
- if $[D]$ lies in the interior of a chamber of $\mathrm{Eff}(X_\Sigma)$ but not of $\mathrm{Mov}(X_\Sigma)$ then ϕ_D is a divisorial contraction;
- if $\mathrm{Eff}(X_\Sigma)$ and $\mathrm{Mov}(X_\Sigma)$ share a wall and $[D]$ lies in this wall then ϕ_D is a rational fibration.

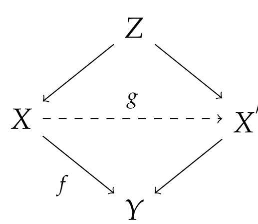
In particular, all the birational geometry of X_Σ is encoded in its secondary fan, and for any effective divisor D on X_Σ the MMP with respect to D terminates.

Example 4.5. Let X be the blow-up of $\mathbb{P}^3$ at two distinct points x_1, x_2. Let H be the pullback of the hyperplane section and E_1, E_2 the two exceptional divisors. The anti-canonical divisor of X is $-K_X = 4H - 2E_1 - 2E_2$. If L is the strict transform of the line $\langle x_1, x_2\rangle$ we have $-K_X \cdot L = 0$. Therefore X is not Fano. The Picard group of X is generated by H, E_1, E_1 and $\rho_X = 3$. Clearly X is a toric variety. Therefore it is a Mori Dream Space.

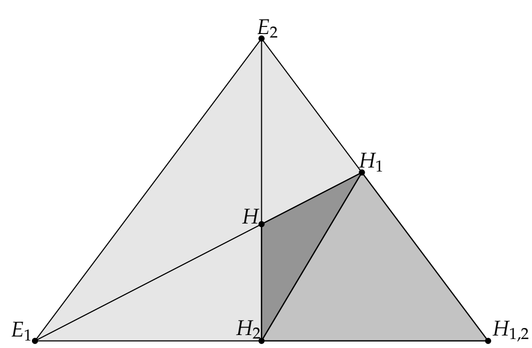
Let $|\mathcal{I}_{x_1,x_2}(2)|$ be the linear system of quadrics in $\mathbb{P}^3$ through x_1, x_2. The corresponding linear system on X induces an morphism



contracting L. Since the normal bundle of L is $\mathcal{O}_L(-1)^{\oplus 2}$ the singular point $f(L) \in f(X) = Y$ is a node. Furthermore f is a small contraction and $f(X)$ is not $\mathbb{Q}$-factorial. Let us blow-up the curve L and let Z be the blow-up. The exceptional divisor is isomorphic two $\mathbb{P}^1 \times \mathbb{P}^1$. By contracting one ruling we get X. On the other hand by contracting the other ruling we find another smooth variety $X^{'}$. The birational map $g : X \dashrightarrow X^{'}$ is the flip of f. The situation is summarized in the following diagram.



The following is a section of $\mathrm{Eff}(X)$:



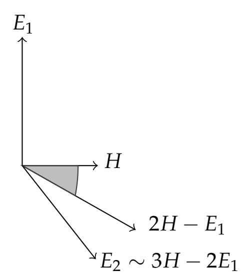
where $H_{1,2} = H - E_1 - E_2$, $H_1 = H - E_1$, $H_2 = H - E_2$. Let L be the strict transform of a general line and R_1, R_2 the classes of a line in the exceptional divisors E_1, E_2. Then the strict transform of the line through x_1, x_2 is given by $C = L - E_1 - E_2$. Now, let H_1, H_2, H_{12} be strict transforms of planes through x_1, x_2 and containing the line $\langle x_1, x_2 \rangle$ respectively. Consider $D = aH_{12} + bH_1 + cH_2$. We have $D \cdot C = -a$. Therefore $D \cdot C$ is always less or equal that zero and its zero if and only if $a = 0$. On the other hand after the contraction of C any divisor of this form becomes nef.
The variety X has exactly two small $\mathbb{Q}$-factorial transformations: the identity and the flip g. Furthermore we have $\operatorname{Mov}(X) = \operatorname{Nef}(X) \cup g^* \operatorname{Nef}(X^{'})$. In the picture $\operatorname{Nef}(X)$ is the cone generated by H, H_1, H_2, and $\operatorname{Nef}(X^{'})$ is the cone generated by $H_{1,2}, H_1, H_2$.

4.2. **A non toric example: complete collineations.** Let V be a K-vector spaces of dimensions $n+1$ over an algebraically closed field K of characteristic zero. We will denote by $\mathcal{X}(n)$ and $\mathcal{Q}(n)$ the spaces of complete collineations and complete quadrics of V, respectively. These spaces are very particular compactifications of the spaces of full rank linear maps and full-rank symmetric linear maps of V, respectively.

In [Vai82], [Vai84], I. Vainsencher showed that these spaces can be understood as sequences of blow-ups of the projective spaces parametrizing $(n+1) \times (n+1)$ matrices and symmetric matrices modulo scalars along the subvariety parametrizing rank one matrices and the strict transforms of their secant varieties in order of increasing dimension.

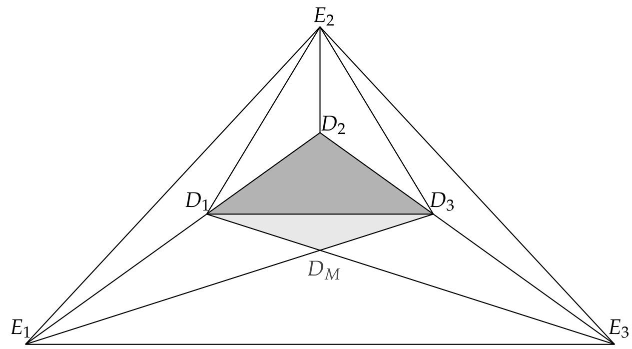
Recall that given an irreducible and reduced non-degenerate variety $X \subset \mathbb{P}^N$, and a positive integer $h \leq N$, the *h-secant variety* $\mathbb{S}ec_h(X)$ of X is the subvariety of $\mathbb{P}^N$ obtained as the closure of the union of all $(h-1)$-planes spanned by h general points of X. Spaces of matrices and symmetric matrices admit a natural stratification dictated by the rank and observe that a general point of the h-secant variety of a Segre, or a Veronese, corresponds to a matrix of rank h. More precisely, let $\mathbb{P}^N$ be the projective space parametrizing $(n+1) \times (n+1)$ matrices modulo scalars, $\mathbb{P}^{N_+}$ the subspace of symmetric matrices, $\mathcal{S} \subset \mathbb{P}^N$ the Segre variety, and $\mathcal{V} \subset \mathbb{P}^{N_+}$ the Veronese variety. Since $\mathbb{S}ec_h(\mathcal{V}) = \mathbb{S}ec_h(\mathcal{S}) \cap \mathbb{P}^{N_+}$, the natural inclusion $\mathbb{P}^{N_+} \hookrightarrow \mathbb{P}^N$ lifts to an embedding $\mathcal{Q}(n) \hookrightarrow \mathcal{X}(n)$.

Example 4.6. The space $\mathcal{Q}(2)$ is well-known: the space of complete conics. It is the blow-up of the projective space $\mathbb{P}^5$ of 3×3 symmetric matrices along the Veronese surface $\mathcal{V} \subset \mathbb{P}^5$. Similarly $\mathcal{X}(2)$ is the blow-up of the projective space $\mathbb{P}^8$ along the Segre variety $\mathcal{S} \cong \mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$. These spaces have both Picard rank two. We will denote by H the strict transform of a general hyperplane of $\mathbb{P}^8$, by E_1 the exceptional divisor over $\mathcal{S}$, by E_2 the strict transform of $\mathbb{S}ec_2(\mathcal{S})$, and by H^+, E_1^+, E_2^+ the analogous divisors in $\mathcal{Q}(2)$. Then $\operatorname{Eff}(\mathcal{X}(2))$ is generated by E_1, E_2, $\operatorname{Nef}(\mathcal{X}(2))$ and $\operatorname{Mov}(\mathcal{X}(2))$ are generated by $H, 2H - E_1$, the Mori chamber decomposition of $\operatorname{Eff}(\mathcal{X}(2))$ is the following:



and the Mori chamber decomposition of $\operatorname{Eff}(\mathcal{Q}(2))$ is obtained from the previous one substituting H, E_1, E_2 with H^+, E_1^+, E_2^+.

Example 4.7. The space $\mathcal{X}(3)$ is the blow-up of the projective space $\mathbb{P}^{15}$ along the Segre variety $\mathcal{S} \cong \mathbb{P}^3 \times \mathbb{P}^3 \subset \mathbb{P}^{15}$, and then along the strict transform of $\mathbb{S}ec_2(\mathcal{S})$. We will denote by H the strict transform of a general hyperplane of $\mathbb{P}^{15}$ and by E_1, E_2 the exceptional divisors over $\mathcal{S}$ and $\mathbb{S}ec_2(\mathcal{S})$ respectively. Similarly, $\mathcal{Q}(3)$ is the blow-up of the projective space $\mathbb{P}^9$ along the Veronese variety $\mathcal{V} \subset \mathbb{P}^9$, and then along the strict transform of $\mathbb{S}ec_2(\mathcal{V})$. As before we will denote by H^+, E_1^2, E_2^+ the divisors on $\mathcal{Q}(3)$ corresponding to H, E_1, E_2. The Mori chamber decomposition of $\mathrm{Eff}(\mathcal{X}(3))$ is displayed in the following two dimensional section of $\mathrm{Eff}(\mathcal{X}(3))$:



where $D_M \sim 6D_1 - 3E_1 - 2E_2$, $D_1 \sim H$, $D_2 \sim 2H - E_1$, $D_3 \sim 3H - 2E_1 - E_2$, $E_3 \sim 4H - 3E_1 - 2E_2$ is the strict transform of $\mathbb{S}ec_3(\mathcal{S})$, and $\mathrm{Mov}(\mathcal{X}(3))$ is generated by D_1, D_2, D_3, D_M. Furthermore, the same statements hold, by replacing H, E_1, E_2 with H^+, E_1^+, E_2^+ for the space of complete quadrics $\mathcal{Q}(3)$.

For details on the constructions in Examples 4.6, 4.7 we refer to [Mas18], [Mas20], [HM18].

References

[ADHL15] I. Arzhantsev, U. Derenthal, J. Hausen, and A. Laface, *Cox rings*, Cambridge Studies in Advanced Mathematics, vol. 144, Cambridge University Press, Cambridge, 2015. MR 3307753

[BCHM10] C. Birkar, P. Cascini, C. D. Hacon, and J. McKernan, *Existence of minimal models for varieties of log general type*, J. Amer. Math. Soc. **23** (2010), no. 2, 405–468. MR 2601039

[CG13] P. Cascini and Y. Gongyo, *On the anti-canonical ring and varieties of Fano type*, Saitama Math. J. **30** (2013), 27–38 (2013). MR 3203718

[CLS11] D. A. Cox, J. B. Little, and H. K. Schenck, *Toric varieties*, Graduate Studies in Mathematics, vol. 124, American Mathematical Society, Providence, RI, 2011. MR 2810322

[GOST15] Y. Gongyo, S. Okawa, A. Sannai, and S. Takagi, *Characterization of varieties of Fano type via singularities of Cox rings*, J. Algebraic Geom. **24** (2015), no. 1, 159–182. MR 3275656

[Har77] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, New York-Heidelberg, 1977, Graduate Texts in Mathematics, No. 52. MR 0463157

[Hir64] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) **79** (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR 0199184

[HK00] Y. Hu and S. Keel, *Mori dream spaces and GIT*, vol. 48, 2000, Dedicated to William Fulton on the occasion of his 60th birthday, pp. 331–348. MR 1786494

[HM18] C. Lozano Huerta and A. Massarenti, *The Lefschetz principle in birational geometry: birational twin varieties*, `https://arxiv.org/abs/1811.07714`, 2018.

[Mas18] A. Massarenti, *On the birational geometry of spaces of complete forms I: collineations and quadrics*, `https://arxiv.org/abs/1803.09161`, 2018.

[Mas20] ———, *On the birational geometry of spaces of complete forms II: Skew-forms*, J. Algebra **546** (2020), 178–200. MR 4032731

[Oka16] S. Okawa, *On images of Mori dream spaces*, Math. Ann. **364** (2016), no. 3-4, 1315–1342. MR 3466868

[Vai82] I. Vainsencher, *Schubert calculus for complete quadrics*, Enumerative geometry and classical algebraic geometry (Nice, 1981), Progr. Math., vol. 24, Birkhäuser, Boston, Mass., 1982, pp. 199–235. MR 685770

[Vai84] ———, *Complete collineations and blowing up determinantal ideals*, Math. Ann. **267** (1984), no. 3, 417–432. MR 738261

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