Projective Geometry

Perspective and real-world applications

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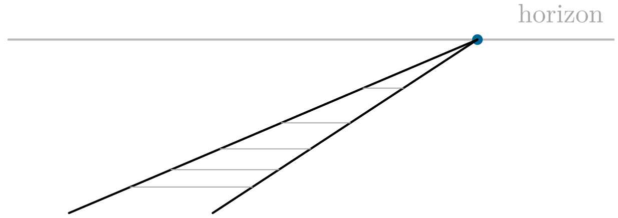
Abstract

Projective geometry is the geometry of perspective: it studies properties invariant under projecting a 3D scene onto a 2D image. Its fundamental idea is to add "points at infinity" so that any two distinct lines meet exactly once—hence there are no parallel lines. We introduce projective space $\mathbb{P}^n$, focus on the projective plane, explain homogeneous coordinates, describe lines and intersections, and illustrate key phenomena with drawings: vanishing points, the line at infinity, and the projective closure of curves. We end with applications in art, photography, computer vision, and engineering.

1 Historical motivation: perspective and vanishing points

Renaissance artists discovered that parallel edges in the real world (railroad tracks, building edges) appear to meet in a painting. That "meeting point" is a *vanishing point*. Projective geometry formalizes this.



parallel lines in 3D
meet at a point in the image

We extend the Euclidean plane by adding points "at infinity" so that *every pair of distinct lines meets.*

2 Projective space: definition and homogeneous coordinates

2.1 Projective space as lines through the origin

Definition 1 (Real projective space)**.** The *real projective space* $\mathbb{RP}^n$ is the set of all lines through the origin in $\mathbb{R}^{n+1}$. Equivalently,

$$\mathbb{RP}^n \;=\; (\mathbb{R}^{n+1} \setminus \{0\})/\sim, \qquad (x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n) \text{ for } \lambda \neq 0.$$

An equivalence class is denoted by $[x_0 : \cdots : x_n]$ and is called *homogeneous coordinates*. Multiplying all coordinates by the same nonzero scalar does not change the point.

Example 1 (A point in $\mathbb{RP}^2$)**.**

$$[2 : 4 : 6] = [1 : 2 : 3] = [-1 : -2 : -3].$$

All represent the same projective point.

2.2 Affine charts: recovering usual Euclidean space

Inside $\mathbb{RP}^n$, the subset where $x_0 \neq 0$ can be normalized to $x_0 = 1$:

$$[1 : x_1 : \cdots : x_n] \longleftrightarrow (x_1, \ldots, x_n) \in \mathbb{R}^n.$$

Thus Euclidean space $\mathbb{R}^n$ sits inside $\mathbb{RP}^n$ as a dense "affine patch".

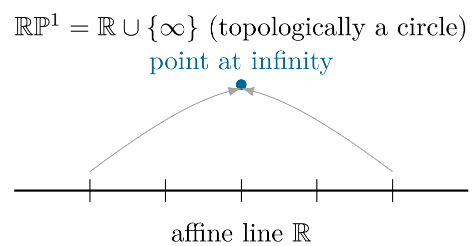
Remark 1. Points with $x_0 = 0$ form the *hyperplane at infinity*. In $\mathbb{RP}^2$ this is a line (the *line at infinity*).

3 The projective line and plane

3.1 The projective line $\mathbb{RP}^1$

$$\mathbb{RP}^1 = \{[x : y]\}.$$

If $x \neq 0$, write $[1 : t]$ with $t = y/x \in \mathbb{R}$. If $x = 0$, we get the special point $[0 : 1]$, which is the *point at infinity* for slopes.



3.2 The projective plane $\mathbb{RP}^2$ and "no parallel lines"

A point of $\mathbb{RP}^2$ is $[x : y : z]$ not all zero, modulo scaling. The affine patch $z \neq 0$ corresponds to the usual plane:

$$[x : y : 1] \longleftrightarrow (x, y) \in \mathbb{R}^2.$$

Points with $z = 0$ form the *line at infinity*:

$$\ell_\infty = \{[x : y : 0]\} \cong \mathbb{RP}^1.$$

Lines in $\mathbb{RP}^2$. A projective line is the set of solutions to a linear equation

$$ax + by + cz = 0$$

in homogeneous coordinates (not all a, b, c zero).

Proposition 1 (No parallel lines in $\mathbb{RP}^2$)**.** *Any two distinct projective lines in $\mathbb{RP}^2$ meet at exactly one point.*

Idea of proof. Two distinct lines correspond to two distinct linear equations in (x, y, z). Solving the 2×3 linear system gives a one-dimensional solution space in $\mathbb{R}^3$, i.e. a unique line through the origin, hence a unique point in $\mathbb{RP}^2$. □

Remark 2. In the affine patch $z = 1$, two Euclidean parallel lines have no intersection. In $\mathbb{RP}^2$ they intersect at a point with $z = 0$ (a point on ℓ_∞), encoding their common direction.

4 Drawings: parallel lines meet at infinity

Example: two parallel affine lines

In the affine plane (x, y), consider

$$L_1: \ y = 1, \qquad L_2: \ y = 2.$$

In homogeneous coordinates (using $z = 1$), these become

$$L_1: \ y - z = 0, \qquad L_2: \ y - 2z = 0.$$

Their intersection in $\mathbb{RP}^2$ solves $y = z$ and $y = 2z$, so $z = 0$ and $y = 0$, leaving x free:

$$[1 : 0 : 0] \in \ell_\infty.$$

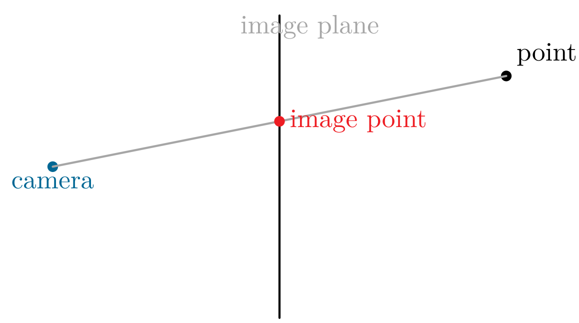
So parallel lines meet at the point $[1 : 0 : 0]$ at infinity.

5 Projective space $\mathbb{P}^3$ and perspective projection

Projective 3-space $\mathbb{RP}^3$ is defined similarly:

$$\mathbb{RP}^3 = (\mathbb{R}^4 \setminus \{0\})/\sim, \quad [x : y : z : w].$$

A *pinhole camera* model can be described projectively: a 3D point projects onto an image plane along a line through the camera center.



projection = intersection of a line with the image plane

6 Worked examples

6.1 Example 1: intersection of two lines via cross product

In $\mathbb{RP}^2$, a line is given by coefficients (a, b, c) in $ax + by + cz = 0$. Two lines $L = (a, b, c)$ and $M = (a', b', c')$ intersect at a point

$$P = L \times M \quad \text{(cross product in } \mathbb{R}^3\text{)},$$

interpreted as homogeneous coordinates $[x : y : z]$.

Example 2. Let $L : x + y - z = 0$ and $M : x - y = 0$ (so $(a, b, c) = (1, 1, -1)$ and $(a', b', c') = (1, -1, 0)$). Then

$$L \times M = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{vmatrix} = (-1)\mathbf{i} + (-1)\mathbf{j} + (-2)\mathbf{k}.$$

So the intersection point is $[-1 : -1 : -2] = [1 : 1 : 2]$. In the affine chart $z = 1$, this is $(1/2, 1/2)$.

6.2 Example 2: projective closure of a parabola

In the affine plane, the parabola is $y = x^2$. To homogenize, replace $x \mapsto x/z$ and $y \mapsto y/z$ and clear denominators:

$$\frac{y}{z} = \left(\frac{x}{z}\right)^2 \quad \Longrightarrow \quad yz = x^2.$$

So the projective curve is

$$C : \; x^2 - yz = 0 \subset \mathbb{RP}^2.$$

Points at infinity satisfy $z = 0$, so $x^2 = 0$, hence $x = 0$ and we get the single point at infinity

$$[0 : 1 : 0] \in \ell_\infty.$$

Thus the parabola has exactly one point at infinity in $\mathbb{RP}^2$.

7 Why projective geometry is useful

7.1 1. Art and architecture

Perspective drawing is projective geometry in action: parallel lines in space intersect at vanishing points, and sets of parallel planes generate vanishing lines (horizons).

7.2 2. Cameras, computer vision, and robotics

Modern computer vision uses projective geometry to:

- model camera images (homographies, projection matrices);
- stitch panoramas (mapping one image plane to another);
- reconstruct 3D scenes from multiple views (triangulation);
- understand constraints such as epipolar geometry.

7.3 3. Engineering and metrology

Measuring real objects from images (photogrammetry) relies on projective invariants: straight lines map to straight lines, and cross-ratios are preserved under projective transformations.

7.4 4. "Points at infinity" simplify formulas

Many statements become cleaner:

- "Any two lines intersect" (no special case for parallel lines).
- Many theorems about conics and intersections work uniformly after adding points at infinity.