POLYNOMIALS DECOMPOSITION AS SUMS OF POWERS

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Contents

Introduction

Let $F \in k[x_0, ..., x_n]_d$ be a homogeneous polynomial of degree d. Consider its decompositions as sum of linear forms

$$F = L_1^d + ... + L_h^d.$$

We know that in some cases the decomposition is unique. As instance the following.

d	n	h	*Reference*
$2h-1$	1	h	*Sylvester*
5	2	7	*Hilbert*
3	3	5	*Sylvester*

We will give some explicit methods to compute the decomposition in these cases, and compute some examples using symbolic and numerical calculus softwares such as *MacAulay*2 and *Bertini*.

1. Apolarity

We work over an algebraically closed field of characteristic zero. We mainly follow notations and definitions of [Do]. Let V be a vector space of dimension $n+1$ and let $\mathbb{P}(V) = \mathbb{P}^n$ the corresponding projective space. For any finite set of points $\{p_1, ..., p_h\} \subseteq \mathbb{P}^n$ we consider the linear space of homogeneous forms F of degree d on $\mathbb{P}^n$ such that $Z(F)$ contains the points $p_1, ..., p_h$, and we denote it by

$$L_d(p_1, ..., p_h) = \{F \in k[x_0, ..., x_n]_d \mid p_i \in Z(F)\ \forall\ 1 \leq i \leq h\}.$$

Definition 1.1. An unordered set of points $\{[L_1], ..., [L_h]\} \subseteq \mathbb{P}V^*$ is a polar h-polyhedron of $F \in k[x_0, ..., x_n]_d$ if

$$F = \lambda_1 L_1^d + ... + \lambda_h L_h^d,$$

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for some nonzero scalars $\lambda_1, ..., \lambda_h \in k$ and moreover the L_i^d are linearly independent in $k[x_0, ..., x_n]_d$.

We briefly introduce the concept of Apolar form to a given homogeneous form to state the connection between the set of h-polyhedra of F and the space of apolar forms of F. This correspondence will be very important to reconstruct the h-polyhedra of F.
We fix a system of coordinates $\{x_0, ..., x_n\}$ on V and the dual coordinates $\{\xi_0, ..., \xi_n\}$ on V^*.
Let $\phi = \phi(\xi_0, ..., \xi_n)$ be a homogeneous polynomial of degree t on V^*. We consider the differential operator

$$D_\phi = \phi(\partial_0, ..., \partial_n), \text{ with } \partial_i = \frac{\partial}{\partial x_i}.$$

This operator acts on ϕ substituting the variable ξ_i with the partial derivative $\partial_i = \frac{\partial}{\partial x_i}$. For any $F \in k[x_0, ..., x_n]_d$ we write

$$< \phi, F >= D_\phi(F).$$

We call this pairing the apolarity pairing.
In general ϕ is of the form $\phi(\xi_0, ..., \xi_n) = \sum_{i_0+...+i_n=t} \alpha_{i_0,...,i_n} \xi_0^{i_0}...\xi_n^{i_n}$ and F is of the form $F(x_0, ..., x_n) = \sum_{j_0+...+j_n=d} f_{i_0,...,i_n} x_0^{j_0}...x_n^{j_n}$. Then

$$D_\phi(F) = (\sum_{i_0+...+i_n=t} \alpha_{i_0,...,i_n} \partial_0^{i_0}...\partial_n^{i_n})(F).$$

We see that F is derived $i_0 + ... + i_n = t$ times. So we obtain a homogeneous polynomial of degree $d - t$ on V.
Fixed $F \in k[x_0, ..., x_n]_d$ we have the map

$$ap_F^t : k[\xi_0, ..., \xi_n]_t \to k[x_0, ..., x_n]_{d-t},\ \phi \mapsto D_\phi(F).$$

The map ap_F^t is linear and we can consider the subspace $Ker(ap_F^t)$ of $k[\xi_0, ..., \xi_n]_t$.

Definition 1.2. A homogeneous form $\phi \in k[\xi_0, ..., \xi_n]_t$ is called apolar to a homogeneous form $F \in k[x_0, ..., x_n]_d$ if $D_\phi(F) = 0$, in other words if $\phi \in Ker(ap_F^t)$. The vector subspace of $k[\xi_0, ..., \xi_n]_t$ of apolar forms of degree t to F is denoted by $AP_t(F)$.

Lemma 1.3. *The set $\mathcal{P} = \{[L_1], ..., [L_h]\}$ is a polar h-polyhedron of F if and only if*

$$L_d([L_1], ..., [L_h]) \subseteq AP_d(F),$$

and the inclusion is not true if we delete any $[L_i]$ from $\mathcal{P}$.

Remark 1.4 (Partial Derivatives). Let $\{[L_1], ..., [L_h]\}$ be a h-polar polyhedron for the homogeneous polynomial $F \in k[x_0, ..., x_n]_d$. We write

$$F = \lambda_1 L_1^d + ... + \lambda_h L_h^d.$$

The partial derivatives of F are homogeneous polynomials of degree $d - 1$ decomposed in h linear factors

$$\frac{\partial F}{\partial x_i} = \lambda_1 \alpha_{i_1} d L_1^{d-1} + ... + \lambda_h \alpha_{i_h} d L_h^{d-1}, \text{ for any } i = 0, ..., n.$$

Then $VSP(F, h)^o \subseteq VSP(\frac{\partial F}{\partial x_i}, h)^o$, taking closures we have

$$VSP(F, h) \subseteq VSP(\frac{\partial F}{\partial x_i}, h).$$

The polynomial F has $\binom{n+l}{l}$ partial derivatives of order l. Clearly these derivatives are homogeneous polynomials of degree $d-l$ decomposed in h-linear factors. Then we have $VSP(F, h) \subseteq VSP(\frac{\partial^l F}{\partial x_0^{l_0}, ..., \partial x_n^{l_n}}, h)$, where $l_0 + ... + l_n = l$.

2. The easy Case

In this section we present a way to rebuild decomposition under some special hypothesis.

Construction 2.1. Let $F \in k[x_0, ..., x_n]_d$ be an homogeneous polynomial and let $F_1^l, ..., F_{D_l}^l \in k[x_0, ..., x_n]_{d-l}$ be the partial derivatives of order l, with $D_l = \binom{n+l}{l}$. We denote by $\mathbb{P}^{N_l}$ the projective space parametrizing the homogeneous polynomials of degree $d-l$ and consider the hyperplanes $AP^{d-l}(F_1^l), ..., AP^{d-l}(F_{D_l}^l) \subseteq \mathbb{P}^{N_l}$.
Let $h \in \mathbb{Z}$ be a positive integer such that $h-1 < N_l$ and let $\{[l_1], ..., [l_h]\}$ be an h-polar polyhedron of F. Then by remark 1.4 and lemma 1.3 we know that

$$L_{d-l}(l_1, ..., l_h) \subseteq \bigcap_{i=1}^{D_l} AP^{d-l}(F_i^l) = H^{d-l} \cong \mathbb{P}^{N_l - D_l}.$$

Since for a general h-polar polyhedron $\{[l_1], ..., [l_h]\}$ we have $dim(L_{d-l}(l_1, ..., l_h)) = N_l - h$, we get the rational map

$$\phi : VSP(F, h) \dashrightarrow \mathbb{G}(N_l - h, N_l - D_l),\ \{[l_1], ..., [l_h]\} \mapsto L_{d-l}(l_1, ..., l_h).$$

Suppose that the general $(h-1)$-plane containing $(AP^{d-l})^*$ intersects the corresponding Veronese variety in at least h points, so that the map ϕ is dominant.
In this case a general $(N_l - h)$-plane contained in H^{d-l} represents a linear system of the type $L_{d-l}(l_1, ..., l_h)$. If the intersection of n elements of this linear system consists of $(d-l)^n = t$ points $p_1, ..., p_t$ and if $h \le t$, then choosing h points from the p_i we get an h-polar polyhedron of F.
If $L_{d-l}(l_1, ..., l_h)$ has a base locus $\mathcal{B}$ of positive dimension we can construct an h-polar polyhedron of F simply by choosing h points on $\mathcal{B}$.
This construction gives a method to find the h-polihedra of F under the required hypothesis, in general to find the base locus of the linear system $L_{d-l}(l_1, ..., l_h)$ is not an easy task.

Example 2.2. *Consider the cubic polynomial*

$$F = x^3 + x^2y + x^2z + xy^2 + xyz + xz^2 + y^3 + y^2z + yz^2 + z^3.$$

The operator D_ϕ is given by

$$D_\phi = \alpha_0 \frac{\partial^2}{\partial x^2} + \alpha_1 \frac{\partial^2}{\partial y^2} + \alpha_2 \frac{\partial^2}{\partial z^2} + \alpha_3 \frac{\partial^2}{\partial x \partial y} + \alpha_4 \frac{\partial^2}{\partial x \partial z} + \alpha_5 \frac{\partial^2}{\partial y \partial z}.$$

We are in the situation of construction 2.1, an the spaces of apolar forms are the following

$$AP_2(\tfrac{\partial F}{\partial x}) = Z(6\alpha_0 + 2\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5);$$
$$AP_2(\tfrac{\partial F}{\partial y}) = Z(2\alpha_0 + 6\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5);$$
$$AP_2(\tfrac{\partial F}{\partial z}) = Z(2\alpha_0 + 2\alpha_1 + 6\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5).$$

Now we choose a line on the plane determined by these three equations, as instance intersecting with the hyperplane $H_0 = Z(\alpha_0)$. Choosing two conics in this pencil and computing the base locus we get the following decomposition for F.

$L_1 = (-0.005006 - i0.278616)x + (-0.008344 - i0.464361)y + (-0.012516 - i0.696541)z;$

$L_2 = (0.438881 - i0.986000)x;$

$L_3 = (-0.579402 - i0.878415)y;$

$L_4 = (-0.027303 - i0.199112)x + (-0.081910 - i0.597338)y + (-0.081910 - i0.597338)z.$

3. Polynomials on $\mathbb{P}^1$

In this section we consider the decomposition of a polynomial $F \in k[x,y]_{2h-1}$ as sum of h linear forms.

Theorem 3.1. *(Sylvester) Let F be a generic homogeneous polynomial of degree $2h-1$ in two variables. There exists a unique decomposition of F as sum of h linear forms.*

Proof. : Let X be the rational normal curve of degree $2h-1$ in $\mathbb{P}^{2h-1}$. Since $dim(Sec_{h-1}(X)) = h + (h-1) = 2h-1$ there exists a decomposition of F.
Suppose that $\{l_1, ..., l_h\}$ and $\{L_1, ..., L_h\}$ are two distinct decomposition of F. Let Λ_l and Λ_L the two $(h-1)$-planes generated by the decompositions. The point F_{2h-1} belongs to $\Lambda_l \cap \Lambda_L$ so the linear space$\Gamma = < \Lambda_l, \Lambda_L >$ has dimension

$$dim(\Gamma) \leq (h-1) + (h-1) = 2h-2.$$

If $\Lambda_l \cap \Lambda_L = \{F\}$, then $dim(\Gamma) = (h-1)+(h-1) = 2h-2$. So Γ is a hyperplane in $\mathbb{P}^{2h-1}$ and $\Gamma \cdot X \geq 2h$. A contradiction because $deg(X) = 2h-1$.
If Λ_l and Λ_L have k common points, then Λ_l and Λ_L intersect in $k+1$ points $Q_1, ..., Q_k, F$. In this case $\Lambda_l \cap \Lambda_L$ is a $\mathbb{P}^k$ and $dim(\Gamma) = 2h-2-k$. We choose k points $P_1, ..., P_k$ on X in general position so $H = < \Gamma, P_1, ..., P_k >$ is a hyperplane such that $H \cdot X \geq 2h-k+k = 2h$, a contradiction. We conclude that the decomposition of F in h linear factors is unique. □

In order to reconstruct the decomposition we consider the following construction

Construction 3.2. The partial derivatives of order $h-2$ of F are $\binom{h-2+1}{1} = h-1$ homogeneous polynomials of degree $h+1$. Let $\nu_{h+1} : \mathbb{P}^1 \to \mathbb{P}^{h+1}$ be the $(h+1)$-Veronese embedding and let $X = \nu_{h+1}(\mathbb{P}^1)$ be the corresponding rational normal curve. Consider the projection

$$\pi : \mathbb{P}^{h+1} \setminus H_\partial \to \mathbb{P}^2$$

from the $(h-2)$-plane H_∂ spanned by the partial derivatives. Since the decomposition $\{L_1, ..., L_h\}$ of F is unique, the projection $\overline{X} = \pi(X)$ will have an unique singular point $p_L = \pi(< L_1^{h+1}, ..., L_h^{h+1} >)$ of multiplicity h. Now to find the decomposition, we have to compute the intersection $H \cdot X = \{L_1^{h+1}, ..., L_h^{h+1}\}$, where $H = < H_\partial, p_L >$.

Example 3.3. *We consider the polynomial*

$$F = x^3 + x^2y - xy^2 + y^3 \in k[x,y]_3.$$

i.e. the point $[F] = [1:1:1:1] \in \mathbb{P}^3$. The projection from $[F]$ to the plane $(X=0) \cong \mathbb{P}^2$ is given by

$$\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2,\ [X:Y:Z:W] \mapsto [Y-X : X+Z : W-X].$$

Using the following sequence of MacAulay2 we compute the projection $C = \pi(X)$ of the twisted cubic curve X.

```
Macaulay2, version1.3.1
i1 :  P3 = QQ[X,Y,Z,W]
o1 = P3
o1 :  PolynomialRing
```

```
i2 :  P1 = QQ[s,t]
o2 = P1
o2 :  PolynomialRing
i3 :  TC = map(P1, P3, s^3, 3s^2t, 3st^2, t^3)
o3 = map(P1, P3, s^3, 3s^2t, 3st^2, t^3)
o3 :  RingMap P1 <   P3
i4 :  ITC = kernelTC
o4 = ideal(Z^2-3YW, YZ-9XW, Y^2-3XZ)
o4 :  Idealof P3
i5 :  RTC = P3/ITC
o5 = RTC
o5 :  QuotientRing
i6 :  P2 = QQ[A,B,C]
o6 = P2
o6 :  PolynomialRing
i7 :  projmap = map(RTC, P2, Y-X, X+Z, W-X)
o7 = map(RTC, P2, -X+Y, X+Z, -X+W)
o7 :  RingMap RTC <   P2
i8 :  I = kernelprojmap
o8 = ideal(14A^3+15A^2B+15AB^2-13B^3-18A^2C+45ABC-18B^2C+54AC^2)
o8 :  Ideal of P2
```

The latter is the equation of $C = \pi(X)$. *Using the following function of Bertini*

```
CONFIG
END;
INPUT
homvariablegroup A,B,C;
function f1, f2, f3, f4;
f1 = 14A^3+15A^2B+15AB^2-13B^3-18A^2C+45ABC-18B^2C+54AC^2);
f2 = (42(A^2))+(30AB)+(45CB)-(36CA)+(15(B^2))+(54(C^2));
f3 = (15(A^2))+(30AB)+(45AC)-(39(B^2))-(36*B*C);
f4 = (45AB)+(108AC)-(18(A^2))-(18(B^2));
END;
```

we compute the singular point of C,

$$P = Sing(C) = [4 : 10 : 9].$$

The line generated by P *and* $[F]$ *is given by the following equations*

$$L = (6X - 10Y - 4Z = 5X - 9Y + 4W = 0).$$

We compute the intersection $X \cdot L$, *where* X *is the twisted cubic curve, with Bertini and we find* $L_1^3 = [0.0515957 : 0.4157801 : 1.1168439 : 1]$ *and* $L_2^3 = [155.0515957 : 86.5842198 : 16.1168439 : 1]$. *These points correspond to the linear forms*

$$L_1 = -0.3722812x + y \text{ and } L_2 = 5.3722813x + y.$$

Indeed we have

$$F = 0.99322 \cdot (-0.3722812x + y)^3 + 0.00678 \cdot (5.3722813x + y)^3.$$

4. Hilbert Theorem

Consider the case $d=5,\ n=2,\ h=7$.

Theorem 4.1. *(Hilbert) Let $P\in k[x,y,z]_5$ be a generic homogeneous polynomial of degree five in three variables. Then P can be decomposed as sum of seven linear forms*

$$P = L_1^5 + ... + L_7^5.$$

Furthermore the decomposition is unique.

The following construction provides a method to reconstruct the decomposition starting from the polynomial.

Construction 4.2. If $\{[L_1],...,[L_7]\}$ is a decomposition of P, then it is also a decomposition for its partial derivatives of any order. In particular P has six partial derivatives of order 2 that are homogeneous polynomials of degree three in x,y,z. We consider these derivatives as points in the projective space $\mathbb{P}^9=\mathbb{P}(k[x,y,z]_3)$, parametrizing the homogeneous polynomials of degree three in three variables. We denote by $H_\partial\subseteq\mathbb{P}^9$ the 5-plane spanned by the derivatives, and with V the Veronese variety $V=\nu(\mathbb{P}^2)$, where $\nu:\mathbb{P}^2\rightarrow\mathbb{P}^9$ is the Veronese embedding of degree 3.
Since all the derivatives can be decomposed as sum of $L_1^3,...,L_7^3$ the 5-plane H_∂ is contained in the 6-plane 7-secant to the the Veronese variety $V\subseteq\mathbb{P}^9$, given by $H_L=<L_1^3,...,L_7^3>$.
Consider now the projection

$$\pi:\mathbb{P}^9 \dashrightarrow \mathbb{P}^3$$

form the linear space H_∂. The image of the Veronese variety $\pi(V)=\overline{V}$ is a surface of degree 9 in $\mathbb{P}^3$, furthermore it has a point p_L of multiplicity 7, which comes from the contraction of H_L. This is the unique point of multiplicity 7 on $\overline{V}$ by the uniqueness of the decomposition.
From this discussion we derive an algorithm to find the decomposition divided into the following steps.

(1) Compute the partial derivative of order 2 of P.
(2) Compute the equation of the 5-plane H_∂ spanned by the derivatives.
(3) Project the Veronese variety V in $\mathbb{P}^3$ from H_∂.
(4) Compute the point p_L of multiplicity 7 on $\overline{V}$.
(5) Compute the 6-plane $H=<H_\partial,p_L>$ spanned by H_∂ and the point p_L.
(6) Compute the intersection $V\cdot H=\{L_1^3,...,L_7^3\}$.

Remark 4.3. To apply the algorithm is necessary to ensure that a point $p\in X$ of multiplicity r, where $X\subseteq\mathbb{P}^n$ is a hypersurface, is mapped by an automorphism $\omega:\mathbb{P}^n\rightarrow\mathbb{P}^n$, in a point $\omega(p)\in\omega(X)$ of multiplicity r.

Lemma 4.4. *Let $X\subseteq\mathbb{P}^n$ be a hypersurface, $p\in X$ a point, and $\omega:\mathbb{P}^n\rightarrow\mathbb{P}^n$ an automorphism of $\mathbb{P}^n$. Then $p\in X$ is a point of multiplicity r if and only if $\omega(p)\in\omega(X)$ is a point of multiplicity r.*

Proof. Let $F\in k[x_0,...,x_n]_d$ be the polynomial of X i.e. $X=Z(F)$ and let

$$\omega(x_0,...,x_n)=(\omega_0(x_0,...,x_n),...,\omega_n(x_0,...,x_n)),$$

be the automorphism of $\mathbb{P}^n$. Then $\omega(X) = Z(\overline{F})$, where $\overline{F} \in k[\omega_0, ..., \omega_n]_d$ is such that $F = \overline{F} \circ \omega$. The partial derivatives of F are given by

$$\frac{\partial^k F(x_0, ..., x_n)}{\partial x_0^{k_0} ... \partial x_n^{k_n}} = \sum_{j_0+...+j_n=k} \frac{\partial^k \overline{F}(\omega_0, ..., \omega_n)}{\partial \omega_0^{j_0} ... \partial \omega_n^{j_n}} H_{k_0,...,k_n}.$$

The matrix $H = (H_{k_0,...,k_n})$ is a $\binom{n+k}{n} \times \binom{n+k}{n}$ square matrix. Since it is formed by blocks that are products of the matrix of ω, it is non singular. Then the linear system

$$\frac{\partial^k F(x_0, ..., x_n)}{\partial x_0^{k_0} ... \partial x_n^{k_n}} = \sum_{j_0+...+j_n=k} \frac{\partial^k \overline{F}(\omega_0, ..., \omega_n)}{\partial \omega_0^{j_0} ... \partial \omega_n^{j_n}} H_{k_0,...,k_n} = 0, \; k_0 + ... + k_n = k,$$

has a unique trivial solution. In other words $\frac{\partial^k F(\overline{x_0}, ..., \overline{x_n})}{\partial x_0^{k_0} ... \partial x_n^{k_n}} = 0$ for any $k_0 + ... + k_n = k$ if and only if $\frac{\partial^k \overline{F}(\overline{\omega_0}, ..., \overline{\omega_n})}{\partial \omega_0^{j_0} ... \partial \omega_n^{j_n}} = 0$ for any $j_0 + ... + j_n = k$, where $\overline{\omega_i} = \omega_i(\overline{x_0}, ..., \overline{x_n})$. □

Example 4.5. *Consider the polynomial* $P \in k[x, y, z]_5$ *given by*
$P = x^5 + x^4y^2 - x^2y^3 - y^5 + z^5 + x^3z^2 + x^2z^3 - x^4y + x^4z - 4x^3yz + 6x^2y^2z$
$- 6x^2yz^2 + xy^4 - 4xy^3z + 6xy^2z^2 - 4xyz^3 + xz^4 + y^4z - 2y^3z^2 + 2y^2z^3 - yz^4$.
On $\mathbb{P}^9 = \mathbb{P}(k[x, y, z]_3)$ *we fix homogeneous coordinates* $[X_0 : ... : X_9]$ *corresponding respectively to the monomials* $\{x^3, x^2y, x^2z, xyz, xy^2, xz^2, y^3, y^2z, yz^2, z^2\}$. *In these coordinates the linear space* H_∂ *spanned by the second partial derivatives is given by the following equations.*

$H_{\partial,1}$: $-1701X_0 - 4455X_1 + 567X_2 - 4455X_3 - 567X_5 - 1458X_6 + 81X_7 = 0;$
$H_{\partial,2}$: $-4536X_0 - 13392X_1 - 13392X_3 - 4455X_6 + 216X_7 - 567X_9;$
$H_{\partial,3}$: $216X_1 + 216X_2 + 216X_3 - 216X_5 + 81X_6 + 81X_9 = 0;$
$H_{\partial,4}$: $13392X_4 - 26784X_8 = 0.$

We project on the linear space $(X_0 = X_1 = X_2 = X_3 = X_4 = X_5 = 0) \cong \mathbb{P}^3$. *The projection* $\pi : \mathbb{P}^9 \setminus H_\partial \to \mathbb{P}^3$ *has equations*
$\pi(X_0, ..., X_9) = [-(42X_0 + 110X_1 - 14X_2 + 110X_3 + X_4 + 14X_5 + 36X_6) : -18(X_4 + 2X_7) : 18(X_4 - 2X_8) : (42X_0 + 14X_1 - 110X_2 + 14X_3 + X_4 + 110X_5 - 36X_9)]$.
We compute the projection of the Veronese variety V *by the following function in MacAulay2*

```
Macaulay2, version 1.3.1
i1 :  P2 = QQ[x,y,z]
o1 = P2
o1 :  PolynomialRing
i2 :  P9 = QQ[X0,X1,X2,X3,X4,X5,X6,X7,X8,X9]
o2 = P9
o2 :  PolynomialRing
i3 :  VerMap = map(P2,P9,x^3,3x^2y,3x^2z,6xyz,3xy^2,3xz^2,y^3,3y^2z,3yz^2,z^3)
o3 = map(P2,P9,x^3,3x^2y,3x^2z,6xyz,3xy^2,3xz^2,y^3,3y^2z,3yz^2,z^3)
o3 :  RingMap P2 <--- P9
i4 :  IVer = kernel VerMap
o4 :  Ideal of P9
i5 :  RVer = P9/IVer
o5 = RVer
o5 :  QuotientRing
```

```
i6 :  P3 = QQ[X,Y,Z,W]
o6 = P3
o6 :  PolynomialRing
i7 :  Projection = map(RVer,P3,"Equations of the Projection")
o7 = map(RVer,P3,"Equations of the Projection")
o7 :  RingMap RVer <--- P3
i8 :  IProjVer = kernel Projection
o8 :  Ideal of P3
```

In this way we obtain the equation of $\overline{V} = Z(F)$ *where* $F = F(X, Y, Z, W)$ *is a homogeneous polynomial of degree* $9 = deg(V)$*. Now we use Bertini to compute the point of multiplicity* 7 *on* $\overline{V}$*.*

```
CONFIG
TRACKTOLBEFOREEG: 1e-8;
TRACKTOLDURINGEG: 1e-11;
FINALTOL: 1e-14;
MPTYPE: 1;
PRECISION: 128;
END;
INPUT
homvariablegroup X,Y,Z,W;
function f1, f2, f3, f4, f5;
f1 = F;
```
$f2 = \frac{\partial^6 F}{\partial X^6};$
$f3 = \frac{\partial^6 F}{\partial Y^6};$
$f4 = \frac{\partial^6 F}{\partial Z^6};$
$f5 = \frac{\partial^6 F}{\partial W^6};$
```
END;
```

The singular point is $p_L = [-5.0632364198314 : 0 : 0 : 35.442654938835]$*. Again using Bertini we compute the intersection* $V \cdot H = \{L_1^3, ..., L_7^3\}$ *and we obtain the linear forms*

$L_1 = 0.98274177184x - 0.12482457140y;$
$L_2 = -0.65071281231x + 0.65071281231y;$
$L_3 = 0.12482457140x - 0.98274177184y;$
$L_4 = (0.18975376061 - i0.33683479696)x + (0.83442021400 - i0.082003524422)z;$
$L_5 = (0.04447250903 - i0.38403953709)x - (0.62685967129 + i0.556802140865)z;$
$L_6 = (-0.12154672768 + i0.37408236279)x + (0.18089826609 - i0.55674761546)z;$
$L_7 = 0.72477966367x - 0.72477966495y + 0.72477965837z.$

These forms give the unique decomposition of our polynomial.

5. Sylvester Theorem

Consider the case $d = 3,\ n = 3,\ h = 5$.

Theorem 5.1. *(Sylvester) Let* $F \in k[x, y, z, w]_3$ *be a generic homogeneous polynomial of degree three in four variables. Then* F *can be decomposed as sum of seven* *linear forms*

$$F = L_1^3 + ... + L_5^3.$$

Furthermore the decomposition is unique.

Proof. Let $F = F_3 \in \mathbb{P}^9$ be a homogeneous form of degree three. We know that a 5-polar polyhedron of F exists. The polar form of F in a point $\xi = [\xi_0 : \xi_1 : \xi_2 : \xi_3] \in \mathbb{P}^3$ is the quadric

$$P_\xi F = \xi_0 \frac{\partial F}{\partial x_0} + \xi_1 \frac{\partial F}{\partial x_1} + \xi_2 \frac{\partial F}{\partial x_2} + \xi_3 \frac{\partial F}{\partial x_3}.$$

Let $\{L_1, ..., L_5\}$ be a 5-polar polyhedron of F, then $F = L_1^3 + ... + L_5^3$. The polar form is of the type

$$P_\xi F = \sum_{i=1}^{5} \xi_i \lambda_i L_i^2$$

and it has rank 2 on the points $\xi \in \mathbb{P}^3$ on which three of the linear form L^i vanish simultaneously. These points are $\binom{5}{3} = 10$.
Now we consider the subvariety X_2 of $\mathbb{P}^9$ parametrizing the quadrics of rank 2. A quadric Q of rank 2 is the union of two plane, then $dim(X_2) = 6$. To find the degree of X_2 we have to intersect with a 3-plane, that is intersection of 6 hyperplanes. So the degree of X_2 is equal to the number of quadrics of rank 2 passing through 6 general points of $\mathbb{P}^3$. If we choose three points then the plane through these points is determined, and also the quadric is determined. Then these quadric are $\frac{1}{2}\binom{6}{3} = 10$. We have seen that $dim(X_2) = 6$ and $deg(X_2) = 10$.
Now the linear space

$$\Gamma = \{P_\xi F \mid \xi \in \mathbb{P}^3\} \subseteq \mathbb{P}^9$$

is clearly a 3-plane in $\mathbb{P}^9$.
Then $\Gamma \cap X_2 = \{P_\xi F | rank(P_\xi F) = 2\}$ is a set of 10 points. These points have to be the 10 points we have found in the first part of the proof. Then the decomposition of F in five linear factor is unique. □

This proof suggests us an algorithm to reconstruct the decomposition.

Construction 5.2. Consider F and its first partial derivatives.

(1) Compute the 3-plane Γ spanned by the partial derivatives of F.
(2) Compute the intersection $\Gamma \cdot X_2$, where X_2 is the variety parametrizing the rank 2 quadrics in $\mathbb{P}^3$.
(3) Consider the 10 points in the intersection. By construction on each plane we are looking for there are 6 of these points, furthermore on each plane there are 4 triples of collinear points. Then with these 10 points we can construct exactly $\frac{\binom{10}{3}}{\binom{6}{3}+4} = 5$ planes. These planes gives the decomposition of F. Note that a priori we have $\binom{10}{6} = 210$ choices, but we are interested in combinations of six points $\{P_{j_1}, ..., P_{j_6}\}$ which lie on the same plane. We know that there are exactly five of these. To find the five combinations we use the following script in *Matlab*.

```
P1 = input('Point 1:');
⋮
P10 = input('Point 10:');
```

```
q = input('Precision:');
A = [P1;P2;P3;P4;P5;P6;P7;P8;P9;P10];
t = 1;
B = [];
for a=1:5,
for b=a+1:6,
for c=b+1:7,
for d=c+1:8,
for f=d+1:9,
for g=f+1:10,
M = [A(a,:);A(b,:);A(c,:);A(d,:);A(f,:);A(g,:)];
disp(t);
t = t+1;
v = [];
for a1 = 1:3,
for a2 = a1+1:4,
for a3 = a2+1:5,
for a4 = a3+1:6,
v = [v,det([M(a1,:);M(a2,:);M(a3,:);M(a4,:)])];
end; end; end; end;
if abs(v(1))<q,abs(v(2))<q,abs(v(3))<q,abs(v(4))<q,abs(v(5))<q,
abs(v(6))<q,abs(v(7))<q,abs(v(8))<q,abs(v(9))<q,abs(v(10))<q,
abs(v(11))<q,abs(v(12))<q,abs(v(13))<q,abs(v(14))<q,abs(v(15))<q,
B = [B M];
end; end; end; end; end; end; end;
[n,m] = size(B);
s = 1;
for r=1:4:m-3,
disp('Matrix'), disp(s),
s = s+1;
B(:,r:r+3),
end;
```

This script constructs a matrix A whose lines are the then points and then computes the 6×4 submatrices of rank 3 of A.

Example 5.3. *Consider the polynomial*
$F = x^3 + x^2y + x^2z + x^2w + xy^2 + xyz + xyw + xz^2 + xzw + xw^2 + y^3 + y^2z + y^2w + yz^2 + yzw + yw^2 + z^3 + z^2w + zw^2 + w^3$.
We compute the equations of the linear space Γ*, the equations of the variety* X_2*, and verify that their intersection is a subscheme of dimension zero and length* 10*. In the* $\mathbb{P}^9$ *parametrizing the quadrics on* $\mathbb{P}^3$ *we fix homogeneous coordinates* $[X_0 : ... : X_9]$*, corresponding to the monomials* $\{x^2, xy, xz, xw, y^2, yz, yw, z^2, zw, w^2\}$.

```
Macaulay2, version 1.3.1
i1 :  P9 = QQ[X_0,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9]
o1 = P9
o1 :  PolynomialRing
i2 :  MDer = matrix {{X_0,X_1,X_2,X_3,X_4,X_5,X_6,X_7,X_8,X_9},{3,2,2,2,1,1,1,1,1,1},
{1,2,1,1,3,2,2,1,1,1},{1,1,2,1,1,2,1,3,2,1},{1,1,1,2,1,1,2,1,2,3}}
```

```
o2 :  Matrix P9 <--- P9
i3 :  IDer = minors(5,MDer)
o3 :  Ideal of P9
i4 :  MQuad = matrix {{X0,X1/2,X2/2,X3/2},{X1/2,X4,X5/2,X6/2},{X2/2,X5/2,X7,X8/2},
{X3/2,X6/2,X8/2,X9}}
o4 :  Matrix P9 <--- P9
i5 :  IRTQuad = minors(3,MQuad)
o5 :  Ideal of P9
i6 :  X2 = variety IRTQuad
o6 = X2
o6 :  ProjectiveVariety
i7 :  DerSpace = variety IDer
o7 = DerSpace
o7 :  ProjectiveVariety
i8 :  IdInt = IDer+IRTQuad
o8 :  Ideal of P9
i9 :  Int = variety IdInt
o9 = Int
o9 :  ProjectiveVariety
i10 :  dim Int
o10 = 0
i11 :  degree Int
o11 = 10
```

In these coordinates the 3*-plane spanned by the partial derivatives has equations*

$H_{\partial,1}$: $X_7 - 2X_8 + X_9 = 0;$
$H_{\partial,2}$: $X_5 - X_6 - X_8 + X_9 = 0;$
$H_{\partial,3}$: $X_4 - 2X_6 + X_9 = 0;$
$H_{\partial,4}$: $X_2 - X_3 - X_8 + X_9 = 0;$
$H_{\partial,5}$: $X_1 - X_3 - X_6 + X_9 = 0;$
$H_{\partial,6}$: $X_0 - 2X_3 + X_9 = 0.$

The following function in Bertini allows us to calculate the intersection of H_∂ *with the variety* X_2 *parametrizing the quadrics of rank* 2*.*

```
CONFIG
END;
INPUT
homvariablegroup X0,X1,X2,X3,X4,X5,X6,X7,X8,X9;
function f1,f2,f3,f4,f5,f6,f7,...,f22;
f1 = X7-2X8+X9;
f2 = X5-X6-X8+X9;
f3 = X4-2X6+X9;
f4 = X2-X3-X8+X9;
f5 = X1-X3-X6+X9;
f6 = X0-2X3+X9;
f7 = ....;
⋮
```

```
f22 = ...;
END;
```

Where $f7,...,f22$, are the equations cutting X_2 in $\mathbb{P}^9$. We find $10 = deg(X_2)$ points on H_∂ that corresponds to the following points in $\mathbb{P}^3$.

$P_1 = [-0.0538 - 0.0089i : -0.0538 - 0.0089i : -0.0538 - 0.0089i : 0.2692 + 0.0447i]$;
$P_2 = [0.9291 + 0.1127i : 0 - 0.9291 - 0.1127i : 0]$;
$P_3 = [0 : 0 : -0.3198 - 0.0488i : 0.3198 + 0.0488i]$;
$P_4 = [0 : 0.4297 + 0.7502i : -0.4297 - 0.7502i : 0]$;
$P_5 = [0 : -0.3850 + 0.0834i : 0 : 0.3850 - 0.0834i]$;
$P_6 = [0.4850 - 0.8736i : -0.4850 + 0.8736i : 0 : 0]$;
$P_7 = [-0.4873 - 0.0825i : 0 : 0 : 0.4873 + 0.0825i]$;
$P_8 = [0.7990 + 0.1275i : -0.1598 - 0.0255i : -0.1598 - 0.0255i : -0.1598 - 0.0255i]$;
$P_9 = [2.3960 - 1.8505i : 2.3960 - 1.8505i : -11.9800 + 9.2523i : 2.3960 - 1.8505i]$;
$P_{10} = [-0.0652 - 0.1273i : 0.3260 + 0.6364i : -0.0652 - 0.1273i : -0.0652 - 0.1273i]$.

Thanks to the previous Matlab script we can compute the five combinations of six coplanar points, and then the linear forms.

$L_1 = (0.0149652 + 0.0069738i)x + (0.0449377 + 0.020996i)y + (0.0149652 + 0.0069738i)z + (0.0149652 + 0.0069738i)w$;
$L_2 = (0.00927286 + 0.0448705i)x + (0.00310162 + 0.0149327i)y + (0.00310162 + 0.0149327i)z + (0.00310162 + .0149327i)w$;
$L_3 = (0.0278039 - 0.0573066i)x + (0.0278039 - 0.0573066i)y + (0.0834118 - 0.17192i)z + (0.02780390.0573066i)w$;
$L_4 = (-0.0642594 - 0.253748i)x + (-0.0642594 - 0.253748i)y + (-0.0642594 - 0.253748i)z + (-0.06425940.253748i)w$;
$L_5 = (-0.0312783 - 0.127146i)x + (-0.0312783 - 0.127146i)y + (-0.0312783 - 0.127146i)z + (-0.0938348 - 0.381437i)w$.

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