Some Examples of Quantum Algebras

Abstract

The term quantum groups stands for certain special Hopf algebras which are non-trivial deformations of the enveloping Hopf algebras of Lie algebras. Quantum groups have close connections with varied areas of mathematics and physics. In these notes we first introduce the concepts of Lie algebra, Hopf algebra and envoloping algebra. Then we will describe some important relations between two specific bialgebras or Hopf algebras. We will see some examples of quantum algebras that are deformations of well known algebras as M(*2*) and SL(*2*).

Contents

References

Christian Kassel, *Quantum Groups*, Springer-Verlag, Graduate Texts in Mathematics.

4 April 2009 - A.M.

1 Lie Algebras

Definition 1.1. *Let k be a field. A Lie algebra is a k-vector space L with an operation*

$$L \times L \to L, \text{ denoted by } (x,y) \mapsto [x,y]$$

and called the bracket or commutator of x and y, such that:

L1 *The bracket operation is bilinear.*

L2 $[x,x] = 0$ *for all* $x \in L$.

L3 $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ *for all* $x,y,z \in L$. *(Jocobi Identity)*

We note that $[x+y,x+y] = [x,x] + [x,y] + [y,x] + [y,y] = 0$ implies $[x,y] = -[y,x]$ and the bracket is anticommutative. Conversely if $char(k) \neq 2$ we have that $[x,y] = -[y,x]$ for all $x,y \in L$ implies $[x,x] = -[x,x]$, $2[x,x] = 0$ and $char(k) \neq 2$ implies $[x,x] = 0$.
A morphism of Lie algebras is a morphism of k-vector spaces that is compatible with the bracket operations.

Definition 1.2. *Let* $L,L^{'}$ *be two Lie algebras. A morphism of Lie algebras is a k-linear map* $\varphi: L \to L^{'}$ *such that* $\varphi([x,y]) = [\varphi(x),\varphi(y)]$ *for all* $x,y \in L$.
An isomorphism of Lie algebras is a morphism that is an isomorphism of k-vector spaces.

A *Lie subalgebra* of a Lie algebra L is a subvector space W of L such that $[x,y] \in W$ for all $x,y \in W$. In this way $(W,[,])$ becomes a Lie algebra. Note that any nonzero element $x \in L$ defines a one dimensional subalgebra $k \cdot x$ with a trivial multiplication since for any $u,v \in k \cdot x$ we have $[u,v] = [\alpha x,\beta x] = \alpha\beta[x,x] = 0$. Now we give some example of Lie algebras.

1.1 Linear Lie Algebras

LL1 If V is a finite dimension k-vector space with $dim(V) = n$ then

$$End(V) = \{f: V \to V \ / \ f \text{ is k-linear}\}$$

is a k-vector space of dimension n^2. We define the following operation on $End(V)$

$$End(V) \times End(V) \to End(V), \ (f,g) \mapsto fg - gf.$$

Now we verify the axioms of definition 1.1, (L1) and (L2) are immediate, for (L3) we compute $[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = [f,gh - hg] + [g,hf - fh] + [h,fg - gf] = fgh - fhg - ghf + hgf + ghf - gfh - hfg + fhg + hfg - hgf - fgh + gfh = 0$.
We note that $End(V)$ is a k-algebra with the usual associative decomposition of function. To distinguish the new algebra structure we write $\mathfrak{gl}(V)$ for $End(V)$ and call this Lie algebra the *general linear algebra*, since it is closely related to the general linear group $GL(V)$ consisting of all invertible endomorphisms of V. Any subalgebra of $\mathfrak{gl}(V)$ is called a linear Lie algebra.

LL2 We denote by $\mathfrak{sl}(V)$ the set of endomorphism of V having trace zero. Since $Tr(f + g) = Tr(f) + Tr(g)$ and $Tr(fg) = Tr(gf)$ implies $Tr(fg - gf) = Tr([f,g]) = 0$ we have that $\mathfrak{sl}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$ called the *special linear algebra*. It is closely related to the special linear group consisting of all endomorphism of V having determinant equal to one. The map

$$Tr{:}End(V) \to k,\ f \mapsto Tr(f)$$

is a surjective k-linear map and we note that $ker(Tr) = \mathfrak{sl}(V)$, by dimension theorem we have $dim(\mathfrak{sl}(V)) = dim(End(V)) - 1 = n^2 - 1$. We see that $sl(V)$ is a hyperplane in $End(V)$.

LL3 Let $dim(V) = 2n$ and let $(v_1,...,v_{2n})$ be a basis of V. We define a skew-symmetric form F on V by the matrix

$$\mathbb{M} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

The set $\mathfrak{sp}(V)$ of endomorphisms of V such that $\mathbb{M}f = f^t\mathbb{M}$. In terms of the skew-symmetric form F we have that $f \in \mathfrak{sp}(V)$ if and only if $F(f(v),w) = -F(v,f(w))$. Let $f,g \in \mathfrak{sp}(V)$ then $\mathbb{M}[f,g] = \mathbb{M}(fg - gf) = \mathbb{M}fg - \mathbb{M}gf = f^t\mathbb{M}g - g^t\mathbb{M}f = f^t g^t\mathbb{M} - g^t f^t\mathbb{M} = ((gf)^t - (fg)^t)\mathbb{M} = [f,g]^t\mathbb{M}$. So $\mathfrak{sp}(V)$ is a Lie subalgebra of $\mathfrak{gl}(V)$ called the *symplectic algebra*. We note that the condition $\mathbb{M}f = f^t\mathbb{M}$ forces $Tr(f) = 0$ so $\mathfrak{sp}(V) \subseteq \mathfrak{sl}(V)$.

LL4 Let $dim(V) = 2n + 1$ be odd and let F be a nondegenerate bilinear form on V whose matrix is

$$\mathbb{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}$$

The *orthogonal algebra* $\mathfrak{O}(V)$ consists of all endomorphisms f of V satisfying $F(f(v),w) = -F(v,f(w))$ in other words such that $\mathbb{N}f = f^t\mathbb{N}$.

LL5 In the case $dim(V) = 2n$ even and with the simpler matrix

$$\mathbb{N} = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$$

We consider the algebra consisting of all endomorphisms f such that $\mathbb{N}f = f^t\mathbb{N}$. This new algebra is called again *orthogonal algebra*.

LL6 We denote by $\mathfrak{t}(V)$ the set of upper triangular matrices. The product and the sum of upper triangular matrices are again upper triangular matrices, so $\mathfrak{t}(V)$ is closed under the bracket. The same is true for the set $\mathfrak{n}(V)$ of strictly upper triangular matrices and for the set $\mathfrak{d}(V)$ of diagonal matrices. Then $\mathfrak{t}(V)$, $\mathfrak{n}(V)$, $\mathfrak{d}(V)$ are Lie subalgebras of $\mathfrak{gl}(V)$.

1.2 Lie Algebras of Derivations

In what follows by a k-algebra (not necessarily associative) we mean a k-vector space $\mathcal{A}$ endowed with a bilinear map $\mathcal{A}\times\mathcal{A} \to \mathcal{A}$ (if $\mathcal{A}$ is a Lie algebra we use the bracket).

DEFINITION 1.3. *A derivation of $\mathcal{A}$ is a k-linear map $\delta{:}\mathcal{A} \to \mathcal{A}$ such that*

$$\delta(xy) = x\delta(y) + \delta(x)y \text{ for all } x,y \in \mathcal{A} \text{ (Leibniz rule).}$$

The set of all derivations of $\mathcal{A}$ is denoted by $Der(\mathcal{A})$.

Let $\delta,\delta':\mathcal{A} \to \mathcal{A}$ be two derivations of $\mathcal{A}$. We compute $(\delta + \delta')(xy) = \delta(xy) + \delta'(xy) = x\delta(y) + \delta(x)y + x\delta'(y) + \delta'(x)y = x(\delta + \delta')(y) + (\delta + \delta')(x)$. So $\delta + \delta' \in Der(\mathcal{A})$ and $Der(\mathcal{A})$ is a subvector space of $End(\mathcal{A})$.
Now we note that the usual pointwise product of two derivations is not necessarily a derivation. For example consider the $\mathbb{R}$-algebra

$$\mathcal{C}^\infty(\mathbb{R}^2) = \{f:\mathbb{R}^2 \to \mathbb{R} \;/\; f \text{ is indefinitely differentiable}\}.$$

We choose the functions $f(x,y) = xy$, $g(x,y) = e^x$ and as derivations the usual partial derivatives $\partial_x,\partial_y:\mathcal{C}^\infty(\mathbb{R}^2) \to \mathcal{C}^\infty(\mathbb{R}^2)$. Then

$$(\partial_x\cdot\partial_y)(ye^x) = \partial_x(ye_x)\cdot\partial_y(ye^x) = ye^{2x} \text{ and } y(\partial_x\cdot\partial_y)(e^x) + (\partial_x\cdot\partial_y)(y)e^x = 0.$$

We see that $(\partial_x\cdot\partial_y)(fg) \neq f(\partial_x\cdot\partial_y)(g) + (\partial_x\cdot\partial_y)(f)g$. On then contrary the following lemma is true

LEMMA 1.4. *Let $\delta,\delta' \in Der(\mathcal{A})$ be two derivations of $\mathcal{A}$. Then the bracket*

$$[\delta,\delta'] = \delta\delta' - \delta'\delta$$

is a derivation of $\mathcal{A}$.

Proof: We compute $[\delta,\delta'](xy) = \delta\delta'(xy) - \delta'\delta(xy) = \delta(xy)\delta'(xy) - \delta'(xy)\delta(xy) = x(\delta\delta')(y) + \delta(x)\delta'(y) + \delta'(x)\delta(y) + (\delta\delta')(x)y - x(\delta'\delta)(y) - \delta'(x)\delta(y) - \delta(x)\delta'(y) - (\delta'\delta)(x)y = x(\delta\delta' - \delta'\delta)(y) + (\delta\delta' - \delta'\delta)(x)y = x[\delta,\delta'](y) + [\delta,\delta'](x)y$. □

Then $Der(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{A})$.

2 Examples of Bialgebras and Hopf algebras

DEFINITION 2.1. *Let k be a field. A k-bialgebra is a 5th-uple $(B,m,u,\Delta,\varepsilon)$ such that (B,m,u) is a k-algebra, (B,Δ,ε) is a k-coalgebra and*

$$\Delta:B \to B\otimes B,\ \varepsilon:B \to k \text{ are } k\text{-algebras morphisms.}$$

DEFINITION 2.2. *A Hopf algebra is a 6th-uple $(H,m,u,\Delta,\varepsilon,S)$ where $(H,m,u,\Delta,\varepsilon)$ is a bialgebra and $S:H \to H$ is a linear map that is an inverse for Id_H in the convolution algebra $Hom(H^C,H^A)$ with $H^C = (H,\Delta,\varepsilon)$ and $H^A = (H,m,u)$ i.e.*

$$S \star Id_H = u \circ \varepsilon = Id_H \star S$$

The map S is called an antipode for H.

OBSERVATION 2.3. *Consider the polynomial ring $k\{X_1,\ldots,X_n\}$ and let R be a k-algebra. Then any algebra morphism of $k\{X_1,\ldots,X_n\}$ in R is uniquely determined by its values r_i on X_i. We denote the evaluation morphism by*

$$E_{r_1,\ldots,r_n}:\ k\{X_1,\ldots,X_n\} \to R,\ F \mapsto F(r_1,\ldots,r_n) = E_{r_1,\ldots,r_n}(F).$$

Let $F(r_1,\dots,r_n) = E_{r_1,\dots,r_n}(F)$ *for any* $F \in k\{X_1,\dots,X_n\}$.
Let $F_1,\dots,F_n \in k\{X_1,\dots,X_n\}$ *and let* I *be the ideal of* $k\{X_1,\dots,X_n\}$ *generated by* $F_1,\dots,F_n$. *For any* $i=1,\dots,n$ *we denote by* $x_i = X_i+I$ *the class of* X_i *and let* $A = k\{X_1,\dots,X_n\}/I$.
Let R *be a* k*-algebra then to give a morphism of algebras* $\Phi:A\to R$ *is equivalent to give a* n*-uple* $(r_1,\dots,r_n)$ *of elements of* R *such that* $F_i(r_1,\dots,r_n) = E_{r_1,\dots,r_n}(F_i) = 0$ *for any* $i=1,\dots,n$.

2.1 The Tensor Algebra

Let V be a k-vector space. We define

$$T^0(V) = k,\ T^1(V) = V \text{ and } T^n(V) = V^{\otimes n} \text{ if } n > 1.$$

The isomorphism $T^n(V) \otimes T^m(V) \cong T^{n+m}(V)$ induces an associative product on the vector space $T(V) = \bigoplus_{n\geq 0} T^n(V)$ explicitly given by

$$(x_1\otimes\dots\otimes x_n)(x_{n+1}\otimes\dots\otimes x_{n+m}) = x_1\otimes\dots\otimes x_n\otimes x_{n+1}\otimes\dots\otimes x_{n+m}$$

The k-vector space $T(V) = \bigoplus_{n\geq 0} T^n(V)$ equipped with this structure is a k-algebra called the *Tensor Algebra of V*.
One can prove the following universal property of $T(V)$.
Let $i_0:k \to T(V)$ *and* $i_1:V \to T(V)$ *the canonical embeddings. Then for any* k*-algebra* A *if* $f_1:V \to A$ k*-linear map, there exists a unique* k*-algebras morphisms* $F:T(V) \to A$ *such that* $F\circ i_1 = f_1$.
We note that the k-algebra $T(V)$ is graded and $T^n(V)$ is the subspace of degree n elements.
In particular if V is finite dimensional and $\{e_1,\dots,e_n\}$ is a basis of V, then $T(V)$ is isomorphic to the algebra $k\{X_1,\dots,X_n\}$ of the polynomials in the noncommutative variables $X_1,\dots,X_n$, where $X_i = i_1(e_i)$.
We consider the two-sided ideal I of $T(V)$ generated by the elements of type xy-yx where x,y run in V. Then $S(V) = T(V)/I$ is a k-algebra called the *symmetric algebra of V*. If $dim(V) = n$ then $S(V)$ is isomorphic to $k[X_0,\dots,X_n]$.

2.2 The Quantum Plane

We consider the k-algebra $k\{X,Y\}$. Using the universal property of the tensor algebra we define the following two algebras morphisms

$$\Delta:k\{X,Y\} \to k\{X,Y\}\otimes k\{X,Y\},\ \Delta(X) = X\otimes X,\ \Delta(Y) = Y\otimes 1 + X\otimes Y$$
$$\varepsilon:k\{X,Y\} \to k,\ \varepsilon(X) = 1,\ \varepsilon(Y) = 0.$$

By the fundamental theorem of the tensor algebra we get that $(k\{X,Y\},\Delta,\varepsilon)$ is a bialgebra. Let q be an element in k such that $q \neq 0$ and let I be the two-sided ideal of $k\{X,Y\}$ generated by XY - qYX. We compute
$\Delta(XY - qYX) = \Delta(X)\Delta(Y) - q\Delta(Y)\Delta(X) = (X\otimes X)(Y\otimes 1 + X\otimes Y) - q(Y\otimes 1 + X\otimes Y)(X\otimes X) = XY\otimes X + XX\otimes XY - q(YX\otimes X + YX\otimes X) = XY\otimes X + XX\otimes XY - YX\otimes qX - XX\otimes qYX = (XY - qYX)\otimes X + XX\otimes(XY - qYX) \in I\otimes k\{X,Y\} + k\{X,Y\}\otimes I$.

$\varepsilon(XY - qYX) = \varepsilon(X)\varepsilon(Y) - q\varepsilon(Y)\varepsilon(X) = 0.$
Then I is a biideal of $k\{X,Y\}$ and $k\{X,Y\}/I$ is a bialgebra. This bialgebra is denoted by $\mathcal{O}_q(k^2)$ or by $k_q[x,y]$ and is called the *Quantum Plane*. Let $x = X + I$ and $y = Y + I$ then the comultiplication and the counit of $\mathcal{O}_q(k^2)$ are defined by

$$\Delta_{\mathcal{O}_q(k^2)}(x) = x \otimes x,\ \Delta_{\mathcal{O}_q(k^2)}(y) = y \otimes 1 + x \otimes y,$$
$$\varepsilon_{\mathcal{O}_q(k^2)}(x) = 1,\ \varepsilon_{\mathcal{O}_q(k^2)}(y) = 0.$$

3 The bialgebra $M_q(2)$

We construct a deformation of the algebra $M(2)$. Let $q \in k$, $q \neq 0$ and $q^2 \neq -1$. By using the universal property of the tensor algebra we define on $R = k\{A,B,C,D\}$ two algebras morphisms

$$\Delta{:}R \to R\otimes R \text{ and } \varepsilon{:}R \to k$$

uniquely determined by

$$\Delta(A) = A\otimes A + B\otimes C,\ \Delta(B) = A\otimes B + B\otimes D,$$
$$\Delta(C) = C\otimes A + D\otimes C,\ \Delta(D) = C\otimes B + D\otimes D,$$
$$\varepsilon(A) = \varepsilon(D) = 1,\ \varepsilon(B) = \varepsilon(C) = 0.$$

In this way the algebra R becomes a bialgebra. Let us consider the two-sided ideal I of R generated by the following elements

$$BA - qAB,\ DB - qBD,$$
$$CA - qAC,\ DC - qCD,\ BC - CB,\ AD - DA - (q^{-1} - q)BC,$$

(1)

where $q \in k$, $q \neq 0$ and $q^2 \neq -1$. We want to prove that I is a biideal of R.

Let $p{:}R \to R/I$ be the projection, we denote by

$$a = p(A),\ b = p(B),\ c = p(C),\ d = p(D)$$

the classes of A,B,C,D.

THEOREM 3.1. *The ideal I generated by relations 1 is a biideal of R and R/I is a bialgebra. The comultiplication* $\Delta{:}R/I \to R/I\times R/I$ *and the counit* $\varepsilon{:}R/I \to k$ *are defined by*

$$\Delta(a) = a\otimes a + b\otimes c,\ \Delta(b) = a\otimes b + b\otimes d,$$
$$\Delta(c) = c\otimes a + d\otimes c,\ \Delta(d) = c\otimes b + d\otimes d,$$
$$\varepsilon(a) = \varepsilon(d) = 1,\ \varepsilon(b) = \varepsilon(c) = 0.$$

In matrix form we have

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \varepsilon\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proof: We compute
$\Delta(BA - qAB) = AA\otimes BA + AB\otimes BC + BA\otimes DA + BB\otimes DC - qAA\otimes AB - qAB\otimes AD - qBA\otimes CB - qBB\otimes CD$. Then
$((p\otimes p)\circ\Delta)(BA - qAB) = qaa\otimes ab + ab\otimes bc + qab\otimes da + qbb\otimes cd - qaa\otimes ab - qab\otimes ad - qba\otimes cb - qbb\otimes cd = ab\otimes(qda - qad) + (ab - qba)\otimes bc$.
Now $qad - qda = (1 - q^2)bc$, so $((p\otimes p)\circ\Delta)(BA - qAB) = ab\otimes(q^2 - 1)bc + (ab - qba)\otimes bc = ab\otimes q^2bc - ab\otimes bc + ab\otimes bc - qba\otimes bc = q(qab - ba)\otimes bd = 0\otimes bc = 0$.
We compute
$\Delta(BC\text{-}CB) = (A\otimes B+B\otimes D)(C\otimes A+D\otimes C)\text{-}(C\otimes A+D\otimes C)(A\otimes B+B\otimes D)$. Then
$((p\otimes p)\circ\Delta)(BC\text{-}CB) = ac\otimes ba+(ad\text{-}da)\otimes bc+bc\otimes(da\text{-}ad)+bc\otimes dc\text{-}ca\otimes ab\text{-}db\otimes cd = ac\otimes ba+(q^{-1}\text{-}q)bc\otimes bc\text{-}bc\otimes(q^{-1}\text{-}q)bc+bd\otimes dc\text{-}ca\otimes ab\text{-}db\otimes cd = (qac\text{-}ca)\otimes ab+(qbd\text{-}db)\otimes cd = 0$.
Finally we compute
$\Delta(AD - DA - (q^{-1} - q)BC) = (A\otimes A + B\otimes C)(C\otimes B + D\otimes D) - (C\otimes B + D\otimes D)(A\otimes A + B\otimes C) - (q^{-1} - q)(A\otimes B + B\otimes D)(C\otimes A + D\otimes C) = AC\otimes AB+AD\otimes AD+BC\otimes CB+BD\otimes CD\text{-}CA\otimes BA\text{-}CB\otimes BC\text{-}DA\otimes DA\text{-}DB\otimes DC\text{-}(q^{-1}\text{-}q)(AC\otimes BA+AD\otimes BC+BC\otimes DA+BD\otimes DC)$. Then $((p\otimes p)\circ\Delta)(AD - DA - (q^{-1} - q)BC) = q^{-1}ac\otimes ba+ad\otimes ad+bc\otimes bc+q^{-1}db\otimes cd\text{-}qac\otimes ba\text{-}cb\otimes bc\text{-}da\otimes da\text{-}qbd\otimes dc\text{-}q^{-1}ac\otimes ba\text{-}q^{-1}ad\otimes bc\text{-}q^{-1}bc\otimes daq^{-1}bd\otimes dc+qac\otimes ba+qad\otimes bc+qbc\otimes da+qbd\otimes dc = ad\otimes ad\text{-}da\otimes da+(ad\otimes bc)(q\text{-}q^{-1})\text{-}(bc\otimes da)(q\text{-}q^{-1}) = ad\otimes ad\text{-}da\otimes da+ad\otimes(ad\text{-}da)\text{-}(ad\text{-}da)\otimes da = 0$.
The other relations are similar. Furthermore we have $\varepsilon(BA - qAB) = \varepsilon(BC\text{-}CB) = 0$ and $\varepsilon(AD - DA - (q^{-1} - q)BC) = 1\text{-}1 = 0$.
We have seen that $\Delta(I) \subseteq R\otimes I+I\otimes R$ and $\varepsilon(I) = 0$. Then I is a biideal of R and R/I is a bialgebra.

□

The k-algebra R/I is denoted by $M_q(2)$. When $q = 1$ the algebra $M_q(2)$ is isomorphic to $M(2)$.

Definition 3.2. *Let R be a k-algebra. An R-point of $M_q(2)$ is a quadruple $(A,B,C,D) \in R^4$ such that*

$$AB = qBA,\ BD = qDB,$$
$$AC = qCA,\ CD = qDC,$$
$$BC = CB,\ AD - DA = (q^{-1} - q)BC.$$

We consider the element $det_q = DA - qBC$ of R. Then $\det_q = da - qbc$ is well defined on $M_q(2)$. If $a = p(A) = p(A'),b = p(B) = p(B'),c = p(C) = p(C'),d = p(D) = p(D')$ we have

$$p(D)p(A) - qp(B)p(C) = p(D')p(A') - qp(B')p(C') \text{ and}$$
$$p(DA - qBC) = p(D'A' - qB'C').$$

In other words if $[x]$ denote the class of the element $x \in R$ we have

$$[det_q] = [D][A] - q[B][C].$$

LEMMA 3.3. *We have* $\Delta(det_q) = det_q \otimes det_q$ *and* $\varepsilon(det_q) = 1$.

Proof: We compute $\Delta(det_q) = \Delta(da\text{-}qbc) = ca\otimes ba+cb\otimes bc+da\otimes da+db\otimes dc\text{-}qac\otimes ba\text{-}qad\otimes bc\text{-}qbc\otimes da\text{-}qbd\otimes dc = qac\otimes ba+bc\otimes bc+da\otimes da+qbd\otimes dc\text{-}qac\otimes ba\text{-}qad\otimes bc\text{-}qbc\otimes da\text{-}qbd\otimes dc = (bc\text{-}qad)\otimes bc+(da\text{-}qbc)\otimes da$. In view of the relation $ad = da+(q^{-1}\text{-}q)bc$ we have $qad = qda+(1\text{-}q^2)bc$ so
$\Delta(det_q) = (bc\text{-}qda\text{-}bc+q^2bc)\otimes bc+(da\text{-}qbc)\otimes da = (da\text{-}qbc)\otimes(da\text{-}qbc) = det_q\otimes det_q$.
Finally $\varepsilon(det_q) = \varepsilon(da\text{-}qbc) = \varepsilon(d)\varepsilon(a) = 1$. □

LEMMA 3.4. *There exists a bijective correspondence between the algebra morphisms of* $M_q(2)$ *in* R *and the* R*-points of* $M_q(2)$.

3.1 The Hopf Algebra $SL_q(2)$

The special linear group $SL(2)$ consists of all matrices in $M(2)$ whose determinant is equal to one. We consider the ideal I of $M_q(2)$ generated by det_q - $1 = da$ - qbc - 1. We will prove that $M_q(2)/I$ has a structure of Hopf algebra and we will denote it by $SL_q(2)$.

THEOREM 3.5. *The ideal* $I = (det_q$ - $1)$ *is a biideal in* $M_q(2)$. *The quotient algebra* $SL_q(2)$ *is a Hopf algebra with antipode defined by*

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = det_q^{-1}\begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix}$$

Proof: We have $\Delta(det_q\text{-}1) = det_q\otimes det_q\text{-}1\otimes 1 = det_q\otimes det_q\text{-}det_q\otimes 1+det_q\otimes 1\text{-}1\otimes 1 = det_q\otimes(det_q\text{-}1)+det_q\text{-}1+ \in M_q(2)\otimes I+I\otimes M_q(2)$.
$\varepsilon(det_q\text{-}1) = 1\text{-}1 = 0$.
Then Δ and ε are well defined on $SL_q(2)$ and $SL_q(2)$ is a bialgebra.
We check that S is an antipode
$(S\star Id)(a) = S(a)a+S(b)c = det_q^{-1}(da\text{-}qbc) = 1 = \varepsilon(a)$,
$(S\star Id)(b) = S(a)b+S(b)d = det_q^{-1}(db\text{-}qbd) = 0 = \varepsilon(b)$.
In a similar way one prooves that $(S\star Id)(c) = \varepsilon(c)$ and $(S\star Id)(d) = \varepsilon(d)$. □

3.1.1 Coaction of $M_q(2)$ and $SL_q(2)$ on the quantum plane

We begin this section giving the definition of H-comodule-algebra.

DEFINITION 3.6. *Let* $(H,m_H,u_H,\Delta_H,\varepsilon_H)$ *be a bialgebra and let* (A,m_A,u_A) *be an algebra. We say that* A *is a left* H*-comodule-algebra if*

1. *The vector space* A *has a left* H*-comodule structure given by a map*

$$\rho_A:A \to H\otimes A.$$

2. *The maps*

$$m_A:A\otimes A \to A \text{ and } u_A:k \to A$$

are morphisms of H-comodules.

PROPOSITION 3.7. *Let H be a bialgebra and let A be an algebra. Then A is a left H-comodule-algebra if and only if*

1. *The vector space A has a left H-comodule structure given by a map*

$$\rho_A : A \to H \otimes A.$$

2. *The map $\rho_A : A \to H \otimes A$ is a morphism of algebras.*

<u>*Proof*</u>: The commutativity of the two following diagrams means that $m_A : A \otimes A \to A$ *and* $u_A : k \to A$ are morphisms of H-comodules.

$$\begin{tikzcd}
H \otimes A \otimes H \otimes A \arrow[d, "H \otimes \tau_{A,H} \otimes A"] & A \otimes A \arrow[l, "\rho_A \otimes \rho_A"'] \arrow[r, "m_A"] \arrow[d, "\rho_{A \otimes A}"'] & A \arrow[d, "\rho_A"] \\
H \otimes H \otimes A \otimes A \arrow[r, "m_H \otimes A \otimes A"'] & H \otimes A \otimes A \arrow[r, "H \otimes m_A"'] & H \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
 & k \arrow[r, "u_A"] \arrow[dl, "l_k^{-1}"'] \arrow[dd, "\rho_k"] & A \arrow[dd, "\rho_A"] \\
k \otimes k \arrow[dr, "u_H \otimes k"'] & & \\
 & H \otimes k \arrow[r, "H \otimes u_A"'] & H \otimes A
\end{tikzcd}$$

The fact that ρ_A is a morphism of algebras is equivalent to the commutativity of the following squares

$$\begin{tikzcd}
 & H \otimes A \otimes H \otimes A \arrow[dl, "H \otimes \tau_{A,H} \otimes A"'] \arrow[dd, "m_{H \otimes A}"] & A \otimes A \arrow[l, "\rho_A \otimes \rho_A"] \arrow[dd, "m_A"] \\
H \otimes H \otimes A \otimes A \arrow[dr, "m_H \otimes m_A"'] & & \\
 & H \otimes A & A \arrow[l, "\rho_A"']
\end{tikzcd}$$

$$\begin{tikzcd}
k \arrow[r, "l_k^{-1}"] \arrow[d, "u_A"'] \arrow[dr, "u_{H \otimes A}"] & k \otimes k \arrow[d, "u_H \otimes u_A"] \\
A \arrow[r, "\rho_A"'] & H \otimes A
\end{tikzcd}$$

Cleary the first two diagrams and the last two diagrams are equivalent.

□

THEOREM 3.8. *There exists a unique $M_q(2)$-comodule-algebra structure and a unique $SL_q(2)$-comodule-algebra structure on the quantum plane $A = k_q[x,y]$ such that*

$$\rho_A(x) = a\otimes x + b\otimes y \text{ and } \rho_A(y) = c\otimes x + d\otimes y$$

We rewrite these formulas in the matrix form

$$\rho_A(x,y) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}$$

Proof: We first check that ρ_A defines an algebra morphism from A to $M_q(2)\otimes A$. In view of 2.3 we have to verify that

$$\rho_A(y)\rho_A(x) = q\rho_A(x)\rho_A(y).$$

We have
$\rho_A(y)\rho_A(x) = (c\otimes x+d\otimes y)(a\otimes x+b\otimes y) = ca\otimes x^2+cb\otimes xy+da\otimes yx+db\otimes y^2 = qac\otimes x^2+bc\otimes xy+qda\otimes xy+qbd\otimes y^2 = qac\otimes x^2+(bc+qda)\otimes xy+qbd\otimes y^2$.
On the other hand we have
$q\rho_A(x)\rho_A(y) = q(a\otimes x+b\otimes y)+(c\otimes x+d\otimes y) = q(ac\otimes x^2+ad\otimes xy+bc\otimes yx+bd\otimes y^2) = qac\otimes x^2+qad\otimes xy+qbc\otimes yx+qbd\otimes y^2$.
We note that $qad\otimes xy+qbc\otimes yx = qad\otimes xy+q^2bc\otimes xy = (qad+q^2bc)\otimes xy$ and $ad = da+(q^{-1}\text{-}q)bc$ implies $qad = qda+(1\text{-}q^2)bc$ so $(qad+q^2bc)\otimes xy = (qda+bc\text{-}q^2bc+q^2bc)\otimes xy = qda\otimes xy+bc\otimes xy$.
Since the projection map of $M_q(2)$ onto $SL_q(2)$ is a morphism of algebras the resulting map $A\rightarrow SL_q(2)\otimes A$ is an algebra morphism. It remains to check that ρ_A defines a comodule structure on the quantum plane. We compute
$(Id\otimes\rho_A)\circ\rho_A(x) = (Id\otimes\rho_A)(a\otimes x+b\otimes y) = a\otimes\rho_A(x)+b\otimes\rho_A(y) = a\otimes(a\otimes x+b\otimes y)+b\otimes(c\otimes x+d\otimes y) = a\otimes a\otimes x+a\otimes b\otimes y+b\otimes c\otimes x+b\otimes d\otimes y$.
On the other hand
$(\Delta\otimes Id)\circ\rho_A(x) = (\Delta\otimes Id)(a\otimes x+b\otimes y) = \Delta(a)\otimes x+\Delta(b)\otimes y = (a\otimes a+b\otimes c)\otimes x+(a\otimes b+b\otimes d)\otimes y = a\otimes a\otimes x+b\otimes c\otimes x+a\otimes b\otimes y+b\otimes d\otimes y$.
Finally we have
$(\varepsilon\otimes Id)\circ\rho_A(x) = (\varepsilon\otimes Id)(a\otimes x+b\otimes y) = \varepsilon(a)\otimes x+\varepsilon(b)\otimes y = 1\otimes x$. □

LEMMA 3.9. *For any $i,j \geq 0$ we have*

$$\rho_A(x^iy^j) = \sum_{r=0}^{i}\sum_{s=0}^{j} q^{(i-r)s}\binom{i}{r}_{q^2}\binom{j}{s}_{q^2} a^r b^{i-r} c^s d^{j-s}\otimes x^{r+s}y^{i+j-r-s}.$$

Proof: Since ρ_A is a morphism of algebras we have $\rho_A(x^iy^j) = \rho_A(x^i)\rho_A(y^j)$. Next we have $(b\otimes y)(a\otimes x) = ba\otimes yx$ and $q^2(a\otimes x)(b\otimes y) = q(ab)\otimes q(xy) = ba\otimes yx$ so

$$(b\otimes y)(a\otimes x) = q^2(a\otimes x)(b\otimes y).$$

Similarly $(d\otimes y)(c\otimes a) = dc\otimes ya$ and $q^2(c\otimes a)(d\otimes y) = q(cd)\otimes q(ay) = dc\otimes ya$ so

$$(d\otimes y)(c\otimes a) = q^2(c\otimes a)(d\otimes y).$$

Then we have

$$(b\otimes y)(a\otimes x) = q^2 (a\otimes x)(b\otimes y) \text{ and } (d\otimes y)(c\otimes a) = q^2 (c\otimes a)(d\otimes y).$$

in the algebra $M_q(2)\otimes A$.
We can apply the formula

$$(x+y)^n = \sum_{0\leq k\leq n} \binom{n}{k}_q x^k y^{n-k}$$

to both the expressions

$$\rho_A(x)^i = (a\otimes x+b\otimes y)^i \text{ and } \rho_A(y)^j = (c\otimes a+d\otimes y)^j.$$

In this way we complete the proof. □

We note that for the term $x^{r+s}y^{i+j-r-s}$ we have $r+s+i+j\text{-}r\text{-}s = i+j$. We see that the set $k_q[x,y]_n$ of homogeneous degree n elements of the quantum plane is a $SL_q(2)$-subcomodule of the quantum plane $k_q[x,y]$.

3.2 The algebra $\mathcal{U}_q(\mathfrak{sl}(2))$

First we define the enveloping algebra of a Lie algebra

Definition 3.10. *Let $(L,[,])$ be a Lie algebra. The enveloping algebra $\mathcal{U}(L)$ of L is the quotient of the tensor algebra $T(L)$ modulo the ideal I generated by the elements of the form*

$$i_1([x,y]) - i_2(x\otimes y - y\otimes x), \text{ where } x,y \in L.$$

One can prove that the tensor algebra $T(L)$ induces a Hopf algebra structure on $\mathcal{U}(L)$.
We have seen that $sl(2)$ is a Lie algebra with the bracket defined by $[x,y] = xy - yx$. The matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are linearly independent and since $dim(\mathfrak{sl}(2)) = 3$ form a basis of $\mathfrak{sl}(2)$. We note that

$$[e,f] = h,\ [h,e] = 2e,\ [h,f] = -2f.$$

So the enveloping algebra $\mathcal{U}(\mathfrak{sl}(2))$ is the quotient of the algebra $k\{E,F,H\}$ in non commutative variables modulo the ideal generated by

$$EF - FE - H,\ HE - EH - 2E,\ HF - FH + 2F.$$

We consider the algebra $k\{A,B,C,C'\}$ and define a comultiplication and a counit setting

$$\Delta(A) = 1\otimes A + A\otimes C,\ \Delta(B) = C'\otimes B + B\otimes 1,$$
$$\Delta(C) = C\otimes C,\ \Delta(C') = C'\otimes C',$$
$$\varepsilon(A) = \varepsilon(B) = 0,\ \varepsilon(C) = \varepsilon(C') = 1.$$

In this way $k\{A,B,C,C'\}$ becomes a bialgebra. Let now $q \in k$, $q \neq 0$, $q^2 \neq 1$ and let I be the two-sided ideal of $k\{A,B,C,C'\}$ generated by

$$CC' - 1,\ C'C - 1,\ AB - BA - \tfrac{C-C'}{q-q^{-1}},\ CA - q^2AC,\ CB - q^2BC.$$

One can prove that I is a biideal of $k\{A,B,C,C'\}$ and that the map

$$S{:}k\{A,B,C,C'\} \to k\{A,B,C,C'\},\ S(A) = -AC',\ S(B) = -CB,\ S(C) = C',\ S(C') = C,$$

is such that $S(I) \subseteq I$ and passing to the quotient defines an antipode on $k\{A,B,C,C'\}/I$. In this way $k\{A,B,C,C'\}/I$ becomes an Hopf algebra denoted by $\mathcal{U}_q(\mathfrak{sl}(2))$. We denote by E, F, K, K' the classes of A, B, C, C' in the quotient algebra $\mathcal{U}_q(\mathfrak{sl}(2))$.

3.2.1 Action of $\mathcal{U}_q(\mathfrak{sl}(2))$ on the Quantum Plane

We start with some generalities on skew-derivation of an algebra A. If $a \in A$ is a element we denote by

$$a_{\boldsymbol{l}}{:}A \to A,\ x \mapsto ax \text{ and } a_{\boldsymbol{r}}{:}A \to A,\ x \mapsto xa,$$

the left and right multiplications.
If $\sigma{:}A \to A$ is an automorphism we have

$$\sigma a_{\boldsymbol{l}} = \sigma(a)_l\sigma \text{ and } \sigma a_{\boldsymbol{r}} = \sigma(a)_r\sigma.$$

In fact

$$\begin{aligned}\sigma a_{\boldsymbol{l}}(x) = \sigma(ax) = \sigma(a)\sigma(x) = (\sigma(a)_l\sigma)(x)\\ \sigma a_{\boldsymbol{r}}(x) = \sigma(xa) = \sigma(x)\sigma(a) = (\sigma(a)_r\sigma)(x)\end{aligned}$$

Definition 3.11. *Let $\sigma,\tau{:}A \to A$ be two automorphisms of the algebra A. A linear endomorphism $\delta{:}A \to A$ is called a (σ,τ)-derivation if*

$$\delta(xx') = \sigma(x)\delta(x') + \delta(x)\tau(x') \text{ for all } x,x' \in A.$$

Lemma 3.12. *Let δ be a (σ,τ)-derivation of A and a be an element of A. If there exist two algebra automorphisms σ', τ' such that*

$$a_{\boldsymbol{r}}\sigma' = a_{\boldsymbol{l}}\sigma \text{ and } a_{\boldsymbol{l}}\tau' = a_{\boldsymbol{r}}\tau$$

then the linear endomorphism $a_{\boldsymbol{l}}\delta$ is a (σ',τ)-derivation and $a_{\boldsymbol{r}}\delta$ is a (σ,τ')-derivation.

Proof: We compute $\sigma'(x)(a_{\boldsymbol{l}}\delta)(x') + (a_{\boldsymbol{l}}\delta)(x)\tau(x') = \sigma'(x)a\delta(x') + a\delta(x)\tau(x') = a_l\sigma(x)\delta(x') + a_l\delta(x)\tau(x') = a_l(\sigma(x)\delta(x') + \delta(x)\tau(x')) = a_l\delta(xx')$.
$\sigma(x)(a_r\delta')(x') + (a_r\delta)(x')\tau(x') = \sigma(x)\delta(x')a + a\delta(x)\tau(x') = a_l\sigma(x)\delta(x') + a_l\delta(x)\tau(x') = (\sigma(x)\delta(x') + \delta(x')(a_r\tau)(x)) = a_r(\sigma(x)\delta(x') + \delta(x)\tau(x')) = (a_r\delta)(xx')$. □

We consider the algebra morphisms σ_x, σ_y of the quantum plane $R = k_q[x,y]$ defined by

$$\sigma_x(x) = qx,\ \sigma_x(y) = y,\ \sigma_y(x) = x,\ \sigma_y(y) = qy.$$

When $q = 1$ we have $\sigma_x = \sigma_y = Id$. These morphisms are well defined, consider the morphism $\Phi_X : k\{X,Y\} \to \{X,Y\}$ defined by $X \mapsto qX$, $Y \mapsto Y$ then $\Phi_X(YX - qXY) = YqX - q^2XY = q(YX - qXY) \in I$. Then the morphism Φ_X passing to the quotient defines the morphism σ_x.
For any $n > 1$ we define

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

We define the q-analogues of the partial derivatives

$$\frac{\partial_q(x^m y^n)}{\partial x} = [m]x^{m-1}y^n \text{ and } \frac{\partial_q(x^m y^n)}{\partial y} = [n]x^m y^{n-1}.$$

for all $m,n \geq 0$. Now we describe the commutation relation between the endomorphisms $x_l, y_l, x_r, y_r, \sigma_x, \sigma_y, \frac{\partial_q}{\partial x}, \frac{\partial_q}{\partial y}$.

Within the algebra of linear endomorphisms of $k_q[x,y]$, all commutation relations between the above six endomorphisms are trivial, except the following cases

$y_l x_l = q x_l y_l$,	$x_r y_r = q y_r x_r$,
$\sigma_x x_{l,r} = q x_{l,r} \sigma_x$	$\sigma_y y_{l,r} = q y_{l,r} \sigma_y$,
$\frac{\partial_q}{\partial x} \sigma_x = q \sigma_x \frac{\partial_q}{\partial x}$,	$\frac{\partial_q}{\partial y} \sigma_y = q \sigma_y \frac{\partial_q}{\partial y}$,
$\frac{\partial_q}{\partial x} y_l = q y_l \frac{\partial_q}{\partial x}$,	$\frac{\partial_q}{\partial y} x_r = q x_r \frac{\partial_q}{\partial y}$,
$\frac{\partial_q}{\partial x} x_l = q^{-1} x_l \frac{\partial_q}{\partial x} + \sigma_x = q x_l \frac{\partial_q}{\partial x} + \sigma_x^{-1}$,	$\frac{\partial_q}{\partial y} y_r = q^{-1} y_r \frac{\partial_q}{\partial y} + \sigma_y = q y_r \frac{\partial_q}{\partial y} + \sigma_y^{-1}$.

We also have

$$x_l \frac{\partial_q}{\partial x} = \frac{\sigma_x - \sigma_x^{-1}}{q - q^{-1}} \text{ and } y_r \frac{\partial_q}{\partial y} = \frac{\sigma_y - \sigma_y^{-1}}{q - q^{-1}}.$$

Furthermore the endomorphism

$$\frac{\partial_q}{\partial x} \text{ is a } (\sigma_x^{-1}\sigma_y, \sigma_x)\text{-derivation and } \frac{\partial_q}{\partial y} \text{ is a } (\sigma_y, \sigma_x \sigma_y^{-1})\text{-derivation.}$$

Definition 3.13. *Let $(H, m_H, u_H, \Delta_H, \varepsilon_H)$ be a bialgebra and let (A, m_A, u_A) be an algebra. We say that A is a H-module-algebra if*

1. *The vector space A has a H-module structure.*
2. *The maps*

$$m_A : A \otimes A \to A \text{ and } u_A : k \to A$$

are morphisms of H-modules.

We recall that $A \otimes A$ becomes a H-module defining

$$h(a \otimes b) = \Delta_H(h)(a \otimes b) = h_1 a \otimes h_2 b.$$

Now $m_A(h(a \otimes b)) = m_A(h_1 a \otimes h_2 b) = (h_1 a)(h_2 b)$ and $h m_A(a \otimes b) = h(ab)$. Then the fact that m_A is a morphism of H-modules is equivalent to the relation

$$\sum (h_1 a)(h_2 b) = h(ab). \tag{2}$$

The ground field k becomes a H-module defining

$$ht = \varepsilon_H(h)t \text{ for any } h \in H,\ t \in k.$$

We have $hu_A(t1_k) = ht1_A$ and $u_A(h(t1_k)) = u_A(\varepsilon_H(h)t1_k) = \varepsilon_H(h)t1_A$.
Then u_A is a morphism of H-modules is equivalent to the relation

$$\varepsilon_H(h)t1_A = h(t1_A) \text{ for any } h \in H, t \in k$$

that is equivalent to the relation

$$h1_A = \varepsilon_H(h)1_A, \text{for any } h \in H. \tag{3}$$

THEOREM 3.14. *For any $P \in k_q[x,y]$, set*

$$EP = x\frac{\partial_q P}{\partial y},\ FP = \frac{\partial_q P}{\partial x}y,$$
$$KP = (\sigma_x\sigma_y^{-1})(P),\ K'P = (\sigma_y\sigma_x^{-1})(P).$$

Formulas above defines the structure of $\mathcal{U}_q(\mathfrak{sl}(2))$-module-algebra on $k_q[x,y]$.

Proof: We consider the algebra morphism

$$\Phi\colon k\{A,B,C,C'\} \longrightarrow \mathrm{End}_k k_q[x,y], \text{ defined by}$$
$$A \mapsto x\frac{\partial_q}{\partial y}, B \mapsto \frac{\partial_q}{\partial x}y, C \mapsto \sigma_x\sigma_y^{-1}, C' \mapsto \sigma_y\sigma_x^{-1}.$$

To say that $k_q[x,y]$ is a $\mathcal{U}_q(\mathfrak{sl}(2))$-module is equivalent to give a ring morphism $\mathcal{U}_q(\mathfrak{sl}(2)) \longrightarrow End_k k_q[x,y]$. Then to conclude that $k_q[x,y]$ is a $\mathcal{U}_q(\mathfrak{sl}(2))$-module we have only to check that $\Phi(I) = 0$.
We compute
$KEK' = \sigma_x\sigma_y^{-1}x_l\frac{\partial_q}{\partial y}\sigma_y\sigma_x^{-1} = \sigma_x\sigma_y^{-1}x_l q\sigma_y\frac{\partial_q}{\partial y}\sigma_x^{-1} = \sigma_x\sigma_y^{-1}x_l q\sigma_y\sigma_x^{-1}\frac{\partial_q}{\partial y} = \sigma_y^{-1}q^2x_l\sigma_y\frac{\partial_q}{\partial y} = q^2x_l\frac{\partial_q}{\partial y} = q^2E$.
Then we have $KK' = (\sigma_x\sigma_y^{-1})(\sigma_y\sigma_x^{-1}) = 1$ *and* $K'K = (\sigma_y\sigma_x^{-1})(\sigma_x\sigma_y^{-1}) = 1$.
Finally we compute
$EF - FE = x_l\frac{\partial_q}{\partial y}y_r\frac{\partial_q}{\partial x} - y_r\frac{\partial_q}{\partial x}x_l\frac{\partial_q}{\partial y} = x_l(q^{-1}y_r\frac{\partial_q}{\partial y}+\sigma_y)\frac{\partial_q}{\partial x} - y_r(q^{-1}x_l\frac{\partial_q}{\partial x}+\sigma_x)\frac{\partial_q}{\partial y} = q^{-1}x_ly_r\frac{\partial_q}{\partial y}\frac{\partial_q}{\partial x}+x_l\sigma_y\frac{\partial_q}{\partial x} - q^{-1}y_rx_l\frac{\partial_q}{\partial x}\frac{\partial_q}{\partial y} - y_r\sigma_x\frac{\partial_q}{\partial y} = \sigma_yx_l\frac{\partial_q}{\partial x} - \sigma_xy_r\frac{\partial_q}{\partial y} = \sigma_y\frac{\sigma_x-\sigma_x^{-1}}{q-q^{-1}} - \sigma_x\frac{\sigma_y-\sigma_y^{-1}}{q-q^{-1}} = \frac{\sigma_x\sigma_y^{-1}-\sigma_y\sigma_x^{-1}}{q-q^{-1}} = \frac{K-K'}{q-q^{-1}}$.
Now we check relations 2 and 3. For any $P, Q \in k_q[x,y]$ we have to check that

$$E(PQ) = \sum(E_1P)(E_2Q) = PE(Q)+E(P)K(Q)$$
$$F(PQ) = \sum(F_1P)(F_2Q) = K'(P)F(Q)+F(P)Q$$
$$K(PQ) = \sum(K_1P)(K_2Q) = K(P)k(Q)$$
$$K'(PQ) = \sum(K'_1P)(K'_2Q) = K'(P)K'(Q)$$
$$u1 = \varepsilon(u)1 \text{ for any } u \in \mathcal{U}_q(\mathfrak{sl}(2)).$$

The endomorphism $x_l\frac{\partial_q}{\partial y}$ is a $(Id,\sigma_x\sigma_y^{-1})$-derivation then we have
$E(PQ) = x_l\frac{\partial_q(PQ)}{\partial y} = Id(P)E(Q)+E(P)(\sigma_x\sigma_y^{-1})(Q) = PE(Q)+E(P)K(Q)$.
The endomorphism $y_r\frac{\partial_q}{\partial x}$ is a $(\sigma_x^{-1}\sigma_y,Id)$-derivation then we have
$F(PQ) = y_r\frac{\partial_q(PQ)}{\partial x} = (\sigma_x^{-1}\sigma_y)(P)F(Q)+F(P)Id(Q) = K'(P)F(Q)+F(P)Q$.
We have $K(PQ) = (\sigma_x\sigma_y^{-1})(PQ) = \sigma_x(\sigma_y^{-1}(P)\sigma_y^{-1}(Q)) = (\sigma_x\circ\sigma_y^{-1})(P)(\sigma_x\circ\sigma_y^{-1})(Q) = K(P)K(Q)$.

A similar computation show that $K'(PQ) = K'(P)K'(Q)$.
We have $E1 = 0 = \varepsilon(E)1$, $F1 = 0 = \varepsilon(F)1$, $K1 = 1 = \varepsilon(K)1$, $K'1 = 1 = \varepsilon(K')1$. $\square$

4 Duality between the Hopf Algebras $\mathcal{U}_q(\mathfrak{sl}(2))$ and $SL_q(2)$

In this section we speak of duality in the sense of the following definition.

DEFINITION 4.1. *Let $(U, m_U, u_U, \Delta_U, \varepsilon_U)$ and $(H, m_H, u_H, \Delta_H, \varepsilon_H)$ be bialgebras and let $<,>$ be a bilinear form on $U \times H$. We say that the bilinear form realizes a duality between U and H if we have*

$$\begin{aligned} <uv,x> &= \sum <u,x'><v,x''>; \\ <u,xy> &= \sum <u',x><u'',y>; \\ <1,x> &= \varepsilon_H(x); \\ <u,1> &= \varepsilon_U(u). \end{aligned}$$

for all $u,v \in V$ and $x,y \in H$, where $\Delta_H(x) = \sum x' \otimes x''$ and $\Delta_U(u) = \sum u' \otimes u''$.
If U and H are Hopf algebras with antipode S then they are said to be in duality if the underlying bialgebras are in duality and if we have

$$<S_U(u),x> = <u,S_H(x)>$$

for all $u \in U$ and $x \in H$.

We assume that k is an algebraically closed field and that q is not a root of unit. We want to determine all simple $\mathcal{U}_q$-module of finite dimension.
For any $\mathcal{U}_q$-module V and any scalar $\lambda \neq 0$ we denote by V^λ the subspace of all vectors in V such that $Kv = \lambda v$.

$$V^\lambda = \{v \in V \,/\, Kv = \lambda v\} \subseteq V.$$

The scalar λ is called a *weight* of V if $V^\lambda \neq \{0\}$.

LEMMA 4.2. *We have $EV^\lambda \subset V^{q^2\lambda}$ and $FV^\lambda \subset V^{q^{-2}\lambda}$.*

Proof: Let $v \in V^\lambda$ we have

$$K(Ev) = q^2 E(Kv) = q^2 \lambda Ev \text{ and } K(Fv) = q^{-2} F(Kv) = q^{-2} \lambda Fv.$$

$\square$

DEFINITION 4.3. *Let V be an $\mathcal{U}_q$-module and let λ be a scalar. An element $v \neq 0$ of V is a highest weight vector of weight λ if $Ev = 0$ and if $Kv = \lambda v$. An $\mathcal{U}_q$-module is a highest weight module of highest weight λ if it is generated by a highest weight vector of weight λ.*

PROPOSITION 4.4. *Any $\mathcal{U}_q$-module $V \neq \{0\}$ of finite dimension contains a highest weight vector.*

Proof: The field k is algebraically closed and V is finite-dimensional. The characteristic polynomial P_K of K has its roots in k that are the eigenvalues of K. Then there exists a non-zero vector w and a scalar $\alpha \neq 0$ such that $Kw = \alpha w$. If $Ew = 0$ then w is a highest weight vector. If not, we consider the sequence of vectors $E^n w$, with $n \in \mathbb{Z}$ $n \geq 0$. By lemma 4.2 it is a sequence of eigenvectors with distinct eigenvalues then there exists an integer n such that $E^n w \neq 0$ and $E^{n+1} w = 0$. So $E^n w$ is a highest weight vector. □

Now we state the following lemma omitting the proof

LEMMA 4.5. *Let v be a highest vector of weight λ. Set $v_0 = v$ and $v_p = \frac{1}{[p]!} F^p v$ for $p > 0$. Then*

$$Kv_p = \lambda q^{-2p} v_p,\ Ev_p = \frac{q^{-(p-1)}\lambda - q^{p-1}\lambda^{-1}}{q-q^{-1}} v_{p-1},\ Fv_{p-1} = [p]v_p$$

THEOREM 4.6. *Let V be a finite dimensional $\mathcal{U}_q$-module generated by a highest weight vector v of weight λ. Then*

i *The scalar λ is of the form $\lambda = \varepsilon q^n$ with $\varepsilon = \pm 1$ and n is such that $dim(V) = n+1$.*

ii *Setting $v_p = \frac{F^p v}{[p]!}$, we have $v_p = 0$ for $p > n$ and the set $\{v = v_0, v_1, \ldots, v_n\}$ is a basis of V.*

iii *The operator K acting on V is diagonalizable with the $n+1$ distinct eigenvalues*

$$\{\varepsilon q^n, \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n+2}, \varepsilon q^{-n}\}.$$

iv *Any other highest weight vector in V is a scalar multiple of v and of weight λ.*

v *The module V is simple.*

Furthermore any simple finite-dimensional $\mathcal{U}_q$-module is generated by a highest weight vector and two simple finite $\mathcal{U}_q$-module generated by highest vectors of the same weight are isomorphic.

Proof: By lemma 4.5 the sequence $\{v_p\}$ is a sequence of eigenvectors for K with distinct eigenvalues. Now V is finite dimensional and then there exists an integer n such that $v_n \neq 0$ and $v_{n+1} = 0$.

i,ii By formulas of lemma 4.5 we have $v_h = 0$ for all $h > n$ and $v_h \neq 0$ for all $h \leq n$. By lemma 4.5 we also have

$$Ev_{n+1} = \frac{q^{-n}\lambda - q^n\lambda^{-1}}{q-q^{-1}} v_n \text{ but } Ev_{n+1} = 0.$$

Then we have $q^{-n}\lambda = q^n\lambda^{-1}$ which is equivalent to $\lambda = \pm q^n$.
We have $v_p = \frac{F^p v}{[p]!} = \frac{F^{p-n} v_n}{[p-n]!} = 0$ for any $p > n$. Any element of V, which is generated by v as a module is a linear combination of the set $\{v_i\}$ so $dim(V) = n+1$ and $\{v_0, \ldots, v_n\}$ is a basis of V.

iii We note that

$$Kv_0 = \varepsilon q^n v_0,\ Kv_1 = \varepsilon q^n q^{-2} v_1 = \varepsilon q^{n-2} v_1, \ldots,\ Kv_{n-1} = \varepsilon q^n q^{-2n+2} v_{n-1} = \varepsilon q^{-n+2} v_{n-1},\ Kv_n = \varepsilon q^n q^{-2n} v_n = \varepsilon q^{-n} v_n.$$

Then the matrix of K in the basis $\{v = v_0, v_1, \ldots, v_n\}$ is diagonal with $\{\varepsilon q^n, \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n+2}, \varepsilon q^{-n}\}$ in the diagonal.

iv Let v' be another highest weight vector. Then v' is an eigenvector for the action of K and hence it is a scalar multiple of some vector v_i. But $E(v_i) = 0$ if and only if $i = 0$. So $v' = \alpha v_0$.

v Let V' be a non-zero $\mathcal{U}_q$-submodule of V and let v' be a highest weight vector of V'. Then v' is a highest weight vector of V and by [**iv**] it has to be a scalar multiple of v_0. Therefore $v' = \alpha v_0$ and $v_0 = \frac{1}{\alpha} v'$ is in V′. Then $V \subseteq V'$, we conclude that $V = V'$ and V is simple.

Let v be a highest weight vector of V. Now V is simple and then the submodule generated by v has to be equal to V and V is generated by a highest weight vector. If V and V' are generated by highest weight vectors of the same weight then the linear map

$$V \to V', \; v_i \mapsto v_i'$$

is an isomorphism of $\mathcal{U}_q$-modules. □

By theorem 4.6 we have that, up to isomorphism, there exists a unique simple $\mathcal{U}_q$-module of dimension $n+1$ and generated by a highest weight vector of weight εq^n. We denote this module by $V_{\varepsilon,n}$ and with $\rho_{\varepsilon,n}:\mathcal{U}_q \to End(V_{\varepsilon,n})$ the corresponding morphism of algebras. On $V_{\varepsilon,n}$ we have

$$Kv_p = \varepsilon q^{n-2p} v_p, \; Ev_p = \frac{\varepsilon q^{n-p+1} - \varepsilon^{-1} q^{-n+p-1}}{q - q^{-1}} v_{p-1}, \; Fv_{p-1} = [p] v_p.$$

We want to construct an algebra morphism

$$\psi : M_q(2) \to \mathcal{U}_q{}^*$$

and deduce a bilinear form on $\mathcal{U}_q \times M_q(2)$ defined by $<u,x> = \psi(x)u$ realizing a duality. To give the morphism ψ is equivalent to give four elements A,B,C,D of $\mathcal{U}_q^*$ satisfying the six relations defining $M_q(2)$, in other words it is equivalent to give an $\mathcal{U}_q^*$-point of $M_q(2)$. We consider the simple $\mathcal{U}_q$-module $V_{1,1}$ of highest weight q and basis $\{v_0, v_1\}$. Setting $\rho_{1,1} = \rho$ we have

$$Kv_0 = qv_0, \; Kv_1 = q^{-1} v_1$$

and in matrix form

$$\rho(K) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

$$Ev_0 = 0, \; Ev_1 = v_0$$

and in matrix form

$$\rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$Fv_0 = [1] v_1, \; Fv_1 = [1] v_2 = 0$$

and in matrix form

$$\rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

We extend ρ on $\mathcal{U}_q$ by linearity, in this way we obtain

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}$$

where A,B,C,D are four linear forms on $\mathcal{U}_q$, hence four elements A,B,C,D of $\mathcal{U}_q^*$.

Lemma 4.7. *The quadruple (A,B,C,D) is an $\mathcal{U}_q^*$-point of $M_q(2)$.*

We deduce that there exists a unique morphism of algebras ψ:$M_q(2) \rightarrow \mathcal{U}_q{}^*$ such that

$$\psi(a) = A,\ \psi(b) = B,\ \psi(c) = C,\ \psi(d) = D.$$

Then for any $x \in M_q(2)$ we have that $\psi(x) \in \mathcal{U}_q{}^*$ is a linear form on $\mathcal{U}_q$ and we can consider $\psi(x)(u)$ for any $u \in \mathcal{U}_q$. In this way we get a bilinear map

$$\mathcal{U}_q \times M_q(2) \rightarrow k,\ (u,x) \mapsto <u,x> = \psi(x)(u).$$

Proposition 4.8. *The bilinear map*

$$\mathcal{U}_q \times M_q(2) \rightarrow k,\ (u,x) \mapsto <u,x> = \psi(x)(u)$$

realizes a duality between the bialgebras $\mathcal{U}_q$ and $M_q(2)$.

Proof: We have $\rho(1) = 1$ then

$$\begin{pmatrix} <1,a> & <1,b> \\ <1,c> & <1,d> \end{pmatrix} = \begin{pmatrix} A(1) & B(1) \\ C(1) & D(1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \varepsilon(a) & \varepsilon(b) \\ \varepsilon(c) & \varepsilon(d) \end{pmatrix}$$

Since $<1,xy> = <1,x><1,y>$ the map $<1,x>$ and ε are both algebra morphisms and they coincide on the generators a,b,c,d. Then they have to be equal and $<1,x> = \varepsilon(x)$. We denote by $P(x)$ the following conditions on an element $x \in M_q(2)$. For any pair (u,v) of elements in $\mathcal{U}_q$ we have

$$<uv,x> = \sum <u,x'><v,x''>.$$

We note that $<uv,1> = \varepsilon(uv) = \varepsilon(u)\varepsilon(v) = <u,1><v,1>$. Then $P(1)$ is satisfied. By definition we have

$$\rho(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix} = \begin{pmatrix} <u,a> & <u,b> \\ <u,c> & <u,d> \end{pmatrix}$$

Then by $\rho(uv) = \rho(u)\rho(v)$ we have

$$\begin{pmatrix} <uv,a> & <uv,b> \\ <uv,c> & <uv,d> \end{pmatrix} = \begin{pmatrix} <u,a> & <u,b> \\ <u,c> & <u,d> \end{pmatrix} \begin{pmatrix} <v,a> & <v,b> \\ <v,c> & <v,d> \end{pmatrix}.$$

We get $<uv,a> = <u,a><v,a>+<u,b><v,c>$ and we recall that $\Delta(a) = a\otimes a+b\otimes c$. Then $P(a),P(b),P(c),P(d)$ are satisfied.
If $P(x)$ and $P(y)$ are satisfied then so is $P(\lambda x+y)$ for any scalar λ. We have $<uv,\lambda x+y> = <uv,\lambda x> + <uv,y> = \sum<u,(\lambda x)'><v,(\lambda x)''> + \sum<u,y'><v,y''> = \sum<u,(\lambda x+y)'><v,(\lambda x+y)''>$.
Finally we prove that if $P(x)$ and $P(y)$ are verified then so is $P(xy)$. We have
$<uv,xy> = \sum<(uv)',x><(uv)'',y> = \sum<u'v',x><u''v'',y> = \sum<u',x'><v',x''><u'',y'><v'',y''>$.
On the other hand we have
$\sum<u,(xy)'><v,(xy)''> = \sum<u,x'y'><v,x''y''> = \sum<u',x'><u'',y'><v',x''><v'',y''> = <uv,xy>$.
We conclude that $<uv,xy> = \sum<u,(xy)'><v,(xy)''>$. □

Lemma 4.9. *For the quantum determinant $det_q = da - qbc$ of $M_q(2)$ we have*

$$\psi(det_q) = 1.$$

Equivalently $<u,det_q> = \varepsilon(u)$ for any $u \in \mathcal{U}_q$.

<u>*Proof*</u>: We know that $\Delta(det_q) = det_q\otimes det_q$, so the map $u\mapsto<u,det_q>$ is a morphism of algebras from $\mathcal{U}_q$ in k. We show that this morphism coincide with the counit ε. We have
$<E,det_q> = <E,da> - q<E,bc> = \varepsilon(d)<E,a> + <E,d><K,a> - q\varepsilon(b)<E,c> - q<E,b><K,c> = 0 = \varepsilon(E)$.
$<F,det_q> = <F,da> - q<F,bc> = K'(d)F(a) + F(d)\varepsilon(a) - qK'(b)F(c) - qF(b)\varepsilon(c) = 0 =\varepsilon(F)$
$<K,det_q> = <K,da> - q<K,bc> = <K,d><K,a> - q<K,b><K,c> = q^{-1}q = 1 = \varepsilon(K)$.
$<K',det_q> = <K',da> - q<K',bc> = <K',d><K',a> - q<K',b><K',c> = qq^{-1} = 1 = \varepsilon(K')$. □

By lemma 4.9 we have that the morphism ψ form $M_q(2)$ to $\mathcal{U}_q$ factors through $SL_q(2)$. We will denote by φ the induced morphism of algebras between $SL_q(2)$ and $\mathcal{U}_q{}^*$ and by $<,>$ the corresponding bilinear form.

Lemma 4.10. *Let $u,v \in \mathcal{U}_q$. If*

$$<S_U(u),x> = <u,S_H(x)> \text{ and } <S_V(v),x> = <v,S_H(x)>$$

for all $x \in SL_q(2)$, then $<S_U(uv),x> = <uv,S_H(x)>$.
Similarly let x,y be elements of $SL_q(2)$. If

$$<S_U(u),x> = <u,S_H(x)> \text{ and } <S_U(u),y> = <u,S_H(y)>$$

for all $U \in \mathcal{U}_q$ then $<S_U(u),xy> = <u,S_H(xy)>$.

Theorem 4.11. *The bilinear map $<u,x> = \psi(x)(u)$ realizes a duality between the Hopf algebras $\mathcal{U}_q$ and $SL_q(2)$.*

Proof: We compute

$$< S_{\mathcal{U}_q}(E), \begin{pmatrix} a & b \\ c & d \end{pmatrix} >= \rho(S_{\mathcal{U}_q}(E)) = -\rho(E)\rho(K^{'}) = -det_q^{-1} \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$$

We note that

$$<E, \det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} >= det_q^{-1}$$

$$\begin{pmatrix} 1 \otimes E(d) + E(d) \otimes K(d) & 1 \otimes E(-qb) + E(-qb) \otimes K(-qb) \\ 1 \otimes E(-q^{-1}c) + E(-q^{-1}c) \otimes K(-q^{-1}c) & 1 \otimes E(a) + E(a) \otimes K(a) \end{pmatrix} =$$

$$det_q^{-1} \begin{pmatrix} 0 & -q \\ 0 & 0 \end{pmatrix}$$

Then we have

$$< S_{\mathcal{U}_q}(E), \begin{pmatrix} a & b \\ c & d \end{pmatrix} >=< E, det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} >=< E, \begin{pmatrix} S_{SL_q(2)}(a) & S_{SL_q(2)}(b) \\ S_{SL_q(2)}(c) & S_{SL_q(2)}(d) \end{pmatrix} > .$$

For F we have

$$< S_{\mathcal{U}_q}(F), \begin{pmatrix} a & b \\ c & d \end{pmatrix} >=< F, det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} >=< F, \begin{pmatrix} S_{SL_q(2)}(a) & S_{SL_q(2)}(b) \\ S_{SL_q(2)}(c) & S_{SL_q(2)}(d) \end{pmatrix} > .$$

For K we have

$$< S_{\mathcal{U}_q}(K), \begin{pmatrix} a & b \\ c & d \end{pmatrix} >=< K, det_q^{-1} \begin{pmatrix} d & -qb \\ -q^{-1}c & a \end{pmatrix} >=< K, \begin{pmatrix} S_{SL_q(2)}(a) & S_{SL_q(2)}(b) \\ S_{SL_q(2)}(c) & S_{SL_q(2)}(d) \end{pmatrix} > .$$

One proceeds with $K^{'}$ similarly. By lemma 4.10 the proof is complete. □

4.1 Duality between $\mathcal{U}_q$-Modules and $SL_q(2)$-Comodules

The vector space $k_q[x,y]_n$ of homogeneous degree n elements of the quantum plane has a structure of $SL_q(2)$-comodule. By duality the dual vector space $k_q[x,y]_n^*$ has a structure of $SL_q(2)^*$-module and we have a morphism of rings $\xi{:}SL_q(2)^* \to \mathrm{End}_k k_q[x,y]_n^*$. Dualizing the morphism

$$\varphi{:}SL_q(2) \to \mathcal{U}_q^* \text{ we have a morphism } \psi{:}\mathcal{U}_q \to SL_q(2)^*.$$

The composition

$$\xi \circ \psi\colon \mathcal{U}_q \to \mathrm{End}_k k_q[x,y]_n^*$$

endows $k_q[x,y]_n^*$ of a structure of $\mathcal{U}_q$-module.

THEOREM 4.12. *The $\mathcal{U}_q$-module $k_q[x,y]_n^*$ is isomorphic to the simple module $V_{1,n}$ of highest weight q^n.*

Proof: We consider the linear form on $k_q[x,y]_n$ defined by

$$f(x^i y^{n-i}) = \delta_{n,i}.$$

If we show that f is a highest weight vector of weight q^n of the $\mathcal{U}_q$-module $k_q[x,y]_n{}^*$ then $k_q[x,y]_n{}^*$ contains a submodule isomorphic to the simple module $V_{1,n}$. Since

$$dim(V_{1,n}) = n+1 = dim(k_q[x,y]_n{}^*),$$

we have $k_q[x,y]_n{}^* \cong V_{1,n}$.
We denote $C_{r,s} = q^{(i-r)s}\binom{i}{r}_{q^2}\binom{n-i}{s}_{q^2}$ and compute
$(uf)(x^i y^{n-i}) = (u\otimes f)(\Delta(x^i y^{n-i})) = (u\otimes f)(\sum_{r=0}^{i}\sum_{s=0}^{n-i} C_{r,s}\, a^r b^{i-r} c^s d^{n-i-s} \otimes x^{r+s} y^{n-r-s}) = \sum_{r=0}^{i}\sum_{s=0}^{n-i} C_{r,s} <u, a^r b^{i-r} c^s d^{n-i-s}> f(x^{r+s} y^{n-r-s}) = \sum_{r=0}^{i}\sum_{s=0}^{n-i} C_{r,s} <u, a^r b^{i-r} c^s d^{n-i-s}> \delta_{n,r+s}$
$= \sum_{r=0}^{i}\sum_{s=0}^{n-i} C_{r,s} <u, a^r b^{i-r} c^s d^{n-i-s}> \delta_{i,r}\delta_{n-i,s} = <u, a^i c^{n-i}>$.
We compute

$$<K, a^i c^j> = K(a^i)K(c^j) = \delta_{j,0}\, q^i.$$

Then we have

$$(Kf)(x^i y^{n-i}) = <K, a^i c^{n-i}> = \delta_{n-i,0}\, q^i = \delta_{n,i}\, q^n = q^n f(x^i y^{n-i})$$

which implies $Kf = q^n f$. It remains to prove that $Ef = 0$.
We have

$$<E,a^i> = <E,aa^{i-1}> = \varepsilon(a)<E,a^{i-1}> + <E,a><K,a^{i-1}> = <E,a^{i-1}> =..= <E,a> = 0$$
$$<E,c^j> = <E,cc^{j-1}> = \varepsilon(c)<E,c^{j-1}> + <E,c><K,c^{j-1}> = 0.$$
$$\text{Then } <E,a^i c^j> = \varepsilon(a^i)<E,c^j> + <E,a^i><K,c^j> = 0.$$

We conclude that $(Ef)(x^i y^{n-i}) = <E, a^i c^{n-i}> = 0$ and $Ef = 0$. □