THE RIEMANN–HURWITZ FORMULA

ALEX MASSARENTI

Let X, Y be smooth projective curves over the complex numbers, and $f : X \to Y$ a surjective morphism. Fix a meromorphic 1-form ω on Y give by the data $\{(V_i, g_i(v_i)dv_i))\}$, where $\{V_i\}$ is a finite collection of open subsets, in the Euclidean topology, of Y, and v_i is a local coordinate on V_i. Let $\{(U_i, u_i)\}$ be local coordinates on X such that $f(U_i) \subseteq V_i$. In these local coordinates f can be written as $v_i = f_i(u_i)$ where $f_i = f_{|U_i}$ and the collection

$$\left\{ \left(U_i, g_i(f_i(u_i)) \frac{\partial f_i(u_i)}{\partial u_i} du_i \right) \right\}$$

yields a meromorphic 1-form on X that well be denoted by $f^*\omega$ and called the pull-back via f of the 1-form ω.

Now, fix a point $x \in X$, set $y = f(x)$, and chose local coordinates u on X and v on Y such that $u(x) = 0$ and $v(y) = 0$. Locally, in these coordinates, f may be written as

$$v = u^{\nu_f(x)} \text{ with } u \in B_X,\ v \in B_Y, \nu_f(x) \geq 1,$$

where $B_X, B_Y \subset \mathbb{C}$ are neighborhoods of the origin.

Since Y is connected the sum $\sum_{x \mid f(x)=y} \nu_f(x)$ does not depend on the choice of $y \in Y$. We define

$$f^{-1}(y) = \sum_{x \mid f(x)=y} \nu_f(x)x \in \mathrm{Div}(X) \text{ and } \deg(f) = \sum_{x \mid f(x)=y} \nu_f(x).$$

Remark 0.1. Note that $Z_f = \{x \mid \nu_f(x) > 1\} \subset X$ is a proper Zariski closed subset and hence a finite set. For all $x \notin Z_f$ the fiber $f^{-1}(f(x))$ consists of $\deg(f)$ distinct points. If $x \in Z_f$ then $\nu_f(x) > 1$ is the number of leaves of the branched covering $f : X \to Y$ coming together at x.

The divisor

$$R_f = \sum_{x \in X} (\nu_f(x) - 1)x \in \mathrm{Div}(X) \qquad (0.2)$$

is the ramification divisor of the morphism $f : X \to Y$ and

$$B_f = \sum_{y \mid f^{-1}(y) \in R_f} y \in \mathrm{Div}(Y)$$

is its branch divisor.

Definition 0.3. Let X be a smooth projective curve and $g \in \mathbb{C}(X)$ a meromorphic function on X. We define $\nu_x(g)$ as the order of vanishing of g at x if g has a zero in x, and as the order of vanishing of $1/g$ at x if g has a pole in x.

Now, let ω be a meromorphic 1-form on C. In a neighborhood U of $x \in X$ we may write $\omega_{|U} = g(u)du$ where u is a local coordinate on U. We define $\nu_x(\omega) = \nu_x(g)$. Note that the number $\nu_x(\omega)$ does not depend on the choice of the local coordinate.

Finally, we define the divisor of the 1-form ω as $\mathrm{div}(\omega) = \sum_{x \in X} \nu_x(\omega)x$.

Lemma 0.4. *Let X be a smooth projective curve. There exists a non trivial meromorphic 1-form on X.*

Proof. Let g_X be the genus of X, fix a point $x \in X$ and consider the divisor $D = kx$. The Riemann-Roch theorem yields that

$$h^0(X, D) - h^1(X, D) = deg(D) - g + 1 = k - g + 1.$$

Hence, $h^0(X, D) > 0$ for $k \gg 0$. Note that a non trivial element of $h^0(X, D)$ is a non constant function on X having a pole of order at least k at x and holomorphic elsewhere. In particular, there exists a non constant meromorphic function η on X. Therefore, $d\eta$ is non trivial meromorphic 1-form on X. □

Theorem 0.5. *(Riemann-Hurwitz) Let X, Y be smooth complex projective curves and $f : X \to Y$ a surjective morphism. Denote by g_X the genus of X and by g_Y the genus of Y. Then*

$$2g_X - 2 = (2g_Y - 2)\deg(f) + \deg(R_f)$$

where R_f is the ramification divisor of f in (0.2).

Proof. By Lemma 0.4 there is a meromorphic 1-form ω on Y. Locally we may represent the morphism f ad $v = u^\nu$ and the 1-form ω as $\omega = g(v)dv$. Hence

$$f^*\omega = \nu g(u^\nu)u^{\nu-1}du$$

and so

$$\nu_{f^*\omega}(x) = \nu_f(x)\nu_\omega(f(x)) + \nu_f(x) - 1$$

where $x \in X$ is a fixed point. Therefore,

$$\operatorname{div}(f^*\omega) = \sum_{x\in X} \nu_f(x)\nu_\omega(f(x))x + R_f = \sum_{y\in Y} \nu_\omega(y) \sum_{x \mid f(x)=y} \nu_f(x)x + R_f.$$

Finally, to conclude it is enough to note that $\deg(\operatorname{div}(f^*\omega)) = 2g_X - 2$ and $\sum_{y\in Y} \nu_\omega(y) \sum_{x \mid f(x)=y} \nu_f(x)x$ has degree $\deg(\operatorname{div}(\omega)) \deg(f) = (2g_Y - 2)\deg(f)$. □

Corollary 0.6. *Let $f : X \to Y$ be a surjective morphism of smooth projective complex curves. Then $g_X \geq g_Y$.*

Proof. If $g_Y = 0$ then clearly $g_X \geq g_Y$. If $g_Y \geq 1$ then Theorem 0.5 yields that

$$g_X = g_Y + (\deg(f) - 1)(g_Y - 1) + \frac{1}{2}\deg(R_f).$$

Since $\deg(f) \geq 1$ and $\deg(R_f) \geq 0$ we get that $g_X \geq g_Y$, and $g_X = g_Y$ if and only if either $\deg(f) = 1$ or $g_Y = 1$ and $\deg(R_f) = 0$. □

Corollary 0.7. *Let X be a smooth complex projective curve. Then X is unirational if and only if it is rational.*

Proof. If X is rational then it is in particular unirational. Assume that X is unirational. Then the is a dominant rational map $\mathbb{P}^1 \dashrightarrow X$ and since $\mathbb{P}^1$ is a smooth curve such rational map extends to a surjective morphism $\mathbb{P}^1 \to X$. So, Corollary 0.6 yields that $g_X \leq g_{\mathbb{P}^1} = 0$. Therefore, $g_X = 0$ and hence X is rational. □

Dual curves, inflection points and multiple tangents. Let $X \subset \mathbb{P}^2$ be a smooth curve of degree d. For $p \in X$ the tangent line T_pX determines a point of $\mathbb{P}^{2*}$. The image of the morphism

$$\begin{array}{cccc} \psi: & X & \longrightarrow & \mathbb{P}^{2*} \\ & p & \mapsto & (T_pX)^* \end{array}$$

is the dual curve X^* of X. Note that $\psi(p) := [f_x(p) : f_y(p) : f_z(p)]$, where x, y, z are homogeneous coordinates on $\mathbb{P}^2$, $X = \{f = 0\}$ and $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_z = \frac{\partial f}{\partial z}$. Consider the dual coordinates r, s, t on $\mathbb{P}^{2*}$ and a general line $R = \{\alpha r + \beta s + \gamma t = 0\} \subset \mathbb{P}^{2*}$. Then $X^* \cap R = \{\psi(p) \mid \alpha f_x(p) + \beta f_y(p) + \gamma f_z(p) = f(p) = 0\}$ and so

$$\deg(X^*) = d(d-1).$$

Let $L \subset \mathbb{P}^2$ be a line not tangent to X, and consider the morphism

$$\begin{array}{cccc} \phi: & X & \longrightarrow & L \\ & p & \mapsto & T_pX \cap L. \end{array}$$

Let $q \in L$ be a general point. The fibers $\phi^{-1}(q)$ consists of the points $p \in X$ such that T_pX passes through q. Dually these corresponds to the points in the intersection $q^* \cap X^*$. Hence $\deg(\phi) = d(d-1)$. Note that $q \in L \cap X$ if and only if q^* is tangent to X^*. Furthermore, q is a branch point of ϕ if and only if q^* intersects X^* with multiplicity at least two at some point $b \in p^* \cap X^*$. This can happen if and only if either q^* is tangent to X^* at b or b is a singular point of X^*. The first case occurs if and only if $q \in L \cap X$. Let us analyze the second situation.

If T_pX is a k-tangent of X that is T_pX is tangent to X at k distinct points then X^* has an ordinary singularity of multiplicity k at $(T_pX)^*$. To see this consider the curve

$$X = \{yz^{2k-1} - (x - a_1z)^2 \dots (a - a_kz)^2 = 0\} \subset \mathbb{P}^2.$$

Note that $y = 0$ is tangent to X at $[a_1 : 0 : 1], \dots, [a_k : 0 : 1]$. The tangent cone of X^* at $[0 : 1 : 0]$ is given by $\{(a_1r + t)\dots(a_kr + t) = 0\}$, and hence X^* has an ordinary singularity of multiplicity k at $[0 : 1 : 0]$. In particular, since X^* has finitely many singular points X has finitely many k-tangents.

Hence, when q^* passes through an ordinary singularity of X^* the fiber $\phi^{-1}(q)$ consists of $d(d-1)$ distinct points.

Now, assume that X has an inflection point of order k at p. We may assume that $p = [1 : 0 : 0]$ and that locally X is given by $yz^{k-1} - x^k = 0$. Then X^* is given by

$$X^* = \{(-k)^k st^{k-1} - (k-1)^{k-1} r^k = 0\} \subset \mathbb{P}^{2*}.$$

Note that X has a cusp of order $k-1$ at $[0 : 1 : 0]$. Hence, when q^* passes though a cusp of order $k-1$ of X^* there is an inflection tangent of order k of X through q and so q is a branch point of ϕ.

We conclude that $p \in X$ is a ramification point of ϕ if and only if either $p \in X \cap L$ or p is an inflection point of X.

The inflection points of X can be computed by mean of the Hessian matrix

$$H_f = \begin{pmatrix} f_{x,x} & f_{x,y} & f_{x,z} \\ f_{x,y} & f_{y,y} & f_{y,z} \\ f_{x,z} & f_{y,z} & f_{z,z} \end{pmatrix}$$

of the second partial derivatives of f. Indeed the inflection points are given by the intersection $X \cap \{\det(H_f) = 0\}$. Therefore, X has $3d(d-2)$ inflection points. Summing-up ϕ is ramified at the $3d(d-2)$ inflection points of X plus the d intersection points in $X \cap L$.

Let $p \in \mathbb{P}^2$ be a point not lying on X nor on an inflection tangent or on a k-tangent of X, and L a line not passing through p. Such a point exists since X has finitely many inflection tangents and k-tangents.

Consider the linear projection $\pi : X \to L$ from p. Let $q \in L$ be a point. The line $\langle q, p\rangle$ intersects X in d points counted with multiplicity which are exactly d distinct points if and only if $\langle q, p\rangle$ is not tangent to X. Furthermore, since p does not lie on an inflection tangent nor on a k-tangent of X if $\langle q, p\rangle$ is tangent to X at at point $p_0 \in X$ then $\langle q, p\rangle$ intersects X at p_0 with multiplicity exactly two. Therefore, the ramification index of π at p_0 is two. Therefore, the degree of the ramification divisor R_π of π is equal to the number of tangents of X through p. By Theorem 0.5 we have

$$2g_X - 2 = -2d + \deg(R_\pi)$$

and since $g_X = \frac{1}{2}(d-1)(d-2)$ we get that $\deg(R_\pi) = d(d-1)$. Hence, there are $d(d-1)$ tangents of X through p. This proves again that $\deg(X^*) = d(d-1)$.

Now, assume that $p \in X$ but again p does not lie on an inflection tangent nor on a k-tangent of X. Since X is smooth the projection from p extends to a morphism $\pi : X \to L$. In this case $\deg(\pi) = d-1$ since we are projecting from a point of X. Again by Theorem 0.5 and the genus formula for plane curves we get that $\deg(R_\pi) = (d+1)(d-2)$. So, not counting the tangent at p, there are $(d+1)(d-2)$ tangents of X through p.

Assume that $d \geq 2$ and X^* has only ordinary singularities of multiplicity two or cusps of multiplicity two. Since $\psi : X \to X^*$ is birational we have that $g_X = g_{X^*}$. On the other hand, since X^* has only ordinary nodes and ordinary cusps as singularities we have that

$$g_{X^*} = \frac{1}{2}(d(d-1)-2)(d(d-1)-2) - n_{X^*} - c_{X^*}$$

where n_{X^*} and c_{X^*} are respectively the number of nodes and cusps of X^*. Now, c_{X^*} is equal to the number of inflection points of X which we know to be $3d(d-2)$. Finally, from

$$\frac{1}{2}(d-1)(d-2) = g_X = g_{X^*} = \frac{1}{2}(d(d-1)-2)(d(d-1)-2) - n_{X^*} - 3d(d-2)$$

we get that $n_{X^*} = \frac{1}{2}d(d-2)(d-3)(d+3)$ which is also the number of bitangents of X.

For instance, for $d = 3$ we get that a general plane cubic curve has 9 inflection points on does not have bitangents. For $d = 4$ we get that a general plane quartic curve has 24 inflection points and 28 bitangents.

Alex Massarenti, Dipartimento di Matematica e Informatica, Università di Ferrara, Via Machiavelli 30, 44121 Ferrara, Italy

Email address: alex.massarenti@unife.it