SOME FACTS ABOUT RIEMANN SURFACES

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Exercise 0.1. Suppose that a Riemann surface Γ is given in $\mathbb{CP}^2$ by the equation

$$\sum\nolimits_{i+j\leq 4} a_{ij}x^iy^jz^{i+j-4},$$

and this curve in nonsingular in $\mathbb{CP}^2$.
To compute its genus we can proceed in several ways.

(1) Consider the morphism

$$\phi:\Gamma\rightarrow\mathbb{P}^{2*},\ p\mapsto T_p\Gamma,$$

where $T_p\Gamma$ is the tangent of Γ in p. The curve $\Gamma^*=\phi(\Gamma)$ is a plane curve, called the dual curve of Γ. If $\Gamma=Z(F)$ with $F\in\mathbb{C}[x,y,z]_d$ the the tangent line in $p=[x_p:y_p:z_p]\in\Gamma$ is given by

$$T_p\Gamma=Z(x\frac{\partial F}{\partial x}(p)+y\frac{\partial F}{\partial y}(p)+z\frac{\partial F}{\partial z}(p)),$$

so the morphism ϕ is given explicitly by

$$\phi(p)=[\frac{\partial F}{\partial x}(p):\frac{\partial F}{\partial y}(p):\frac{\partial F}{\partial z}(p)].$$

Let $R=Z(\alpha_0\xi_0+\alpha_1\xi_1+\alpha_2\xi_2)\subseteq\mathbb{P}^{2*}$ be a line and consider the intersection $R\cdot\Gamma^*$ i.e. the points $p\in\mathbb{P}^2$ such that $F(p)=0$ and $\alpha_0\frac{\partial F}{\partial x}(p)+\alpha_1\frac{\partial F}{\partial y}(p)+\alpha_2\frac{\partial F}{\partial z}(p)=0$. Since these points are the complete intersection of a curve of degree d and a curve of degree $(d-1)$, we deduce that the intersection consists of $d(d-1)$ points counted with multiplicity. Then the dual curve has degree $deg(\Gamma^*)=d(d-1)$.
Let $O\in\mathbb{P}^2$ be a point that does not lie on Γ and let L be a line such that $O\notin L$. Consider the projection

$$\phi:\Gamma\rightarrow L,\ p\mapsto<O,p>\cap L.$$

Since $deg(\Gamma)=d$ the inverse image $\pi^{-1}(q)$, $q\in L$ consists of d distinct points. The branch points are those that lie on a tangent line of Γ that passes through O i.e. the points in the intersection $O^*\cdot\Gamma^*$, and we know that these points are $d(d-1)$ and of ramification index 2. Then the morphism $\pi:\Gamma\rightarrow L\cong\mathbb{P}^1$ has degree $deg(\pi)=d$ and the degree of its ramification divisor is $deg(R_\pi)=d(d-1)$. By Riemann-Hurwitz formula we have

$$2g_\Gamma-2=deg(\pi)(2g_{\mathbb{P}^1}-2)+deg(R).$$

Substituting we have $2g_\Gamma=-2d+d(d-1)+2=d^2-3d+2=(d-1)(d-2)$, so

$$g_\Gamma=\frac{1}{2}(d-1)(d-2).$$

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(2) From another viewpoint we know that if $X \subseteq \mathbb{P}^n$ is a smooth hypersurface of degree d, then its canonical sheaf is given by $\omega_X = \mathcal{O}_X(d-n-1)$. In partical for a curve $\Gamma \subseteq \mathbb{P}^2$ we find

$$\omega_\Gamma = \mathcal{O}_\Gamma(d-3).$$

Then the degree of the canonical divisor is $deg(K_\Gamma) = deg(\Gamma)(d-3) = d(d-3)$. By Riemann-Roch theorem we know that $deg(K_\Gamma) = 2g_\Gamma - 2$, equaling the two expression we find again

$$g_\Gamma = \frac{1}{2}(d-1)(d-2).$$

(3) The projection $\pi : \Gamma \rightarrow \mathbb{CP}^1$, $(x,y) \mapsto x$, realizes Γ as a 4-sheet covering of the Riemann sphere ramified at 4 points with index of ramification 3, so the ramification divisor has degree $deg(R_\pi) = 12$. By Riemann-Hurwitz formula

$$2g_\Gamma - 2 = deg(\pi)(2g_{\mathbb{CP}^1} - 2) + deg(R_\pi),$$

we have $g_\Gamma = 3$.

In our case the curve has degree $deg(\Gamma) = 4$ and it is smooth, consider as instance the degree 4 Fermat curve $x^4 + y^4 + z^4 = 0$. By the genus formula we have $g_\Gamma = 3$. Intuitively the map π gives rise to a 4-sheet covering of $\mathbb{P}^1$, ramified on 6 double points. Then we have 4 copies of the Riemann sphere, and we have to cut the spheres on three segments connecting the ramifications points. Gluing the sphere together we obtain a torus with 3 handles.

Suppose that the curve is of the form

$$\Gamma = Z(x^4 + y^4 + z^4).$$

On its affine part $\Gamma_0 = Z(x^4+y^4+1)$, $\omega_1 = \frac{dx}{y^3}$ is a holomorphic differential. In fact since $y^4 = -(x^4+1)$ the function y has zeros on the points $P_j = (\xi_j, 0)$, $j = 1, ..., 4$ such that $\xi_j^4 + 1 = 0$, and since it is a holomorphic function it has a pole of order 4 in the points at infinity. Then its divisor is $div(y) = \sum_{j=1}^4 P_j - P_1^\infty - ... - P_4^\infty$. Now consider dx, in a neighborhood of P_k we can choose $\tau = (x - \xi_k)^{\frac{1}{4}}$ as local parameter, and we have $dx = 4\tau^3 d\tau$, so dx has zero of order 3 in any P_j. In a neighborhood of the points at infinity we choose the local parameter $u = x^{-1}$, then $dx = -u^{-2}du$ and dx has a pole of order 2 in any point at infinity. We conclude that $div(dx) = 3\sum_{j=1}^4 P_j - 2P_1^\infty - ... - 2P_4^\infty$, and $div(\frac{dx}{y^3}) = 3\sum_{j=1}^4 P_j - 2(P_1^\infty + ... + P_4^\infty) - 3(\sum_{j=1}^4 P_j - P_1^\infty - ... - P_4^\infty) = P_1^\infty + ... + P_4^\infty$. The divisor of $\frac{dx}{y^3}$ is positive, so the differential is holomorphic. Similarly $\frac{dx}{y^2}, \frac{xdx}{y^3}$ are holomorphic differentials, and since $g = 3$, $\{\frac{dx}{y^3}, \frac{dx}{y^2}, \frac{xdx}{y^3}\}$ is a basis of the canonical linear system. The canonical map has the form

$$\phi_K : \Gamma \rightarrow \mathbb{P}^2,\ (x,y) \mapsto [\frac{1}{y} : 1 : \frac{x}{y}] = [1 : y : x],$$

and in homogeneous coordinates $\phi_K(x,y,z) = [z : y : x]$. Clearly the canonical map is the identity modulo an automorphism of $\mathbb{P}^2$.

Now we want to prove that Γ is a non hyperelliptic surface. Suppose that there exists a morphism $f : \Gamma \rightarrow \mathbb{P}^1$ of degree 2. This morphism correspond to an effective divisor D on Γ with $deg(D) = 2$ and $h^0(D) = 2$. By Riemann-Roch theorem on D we have

$$h^0(D) - h^0(K - D) = deg(D) - g + 1 = 0.$$

Then $h^0(D) = h^0(K - D)$ i.e. D is in the base locus of the canonical linear system $|K|$. A contradiction since for $g \geq 2$ the linear system $|K|$ has no base points. In fact $dim|K| = g - 1$ and since Γ is not rational we have $dim|P| = 0$ for any $P \in \Gamma$, and by Riemann-Roch we find $dim|K - P| = 2g - 3 - g + 1 = g - 2$. This means exactly that $|K|$ has no base points.

Now let Γ be any non hyperelliptic surface of genus $g = 3$. The canonical divisor K of Γ has degree $deg(K) = 2g - 2 = 4$ and dimension $h^0(K) = 3$. Furthermore since Γ is non hyperelliptic of genus $g = 3 > 2$, then $|K|$ is very ample. We conclude that $|K|$ induces an embedding (the canonical embedding)

$$\phi_K : \Gamma \to \mathbb{P}(H^0(K)^*) \cong \mathbb{P}^2,$$

and so any non hyperelliptic Riemann surface of genus $g = 3$ can be realized as a smooth quartic curve in $\mathbb{P}^2$.

Exercise 0.2. Let $B = (B_{jk})$ be a symmetric $g \times g$ matrix with negative-definite real part. A Riemann theta function is defined by

$$\theta(z) = \theta(z|B) = \sum\nolimits_{N\in\mathbb{Z}^g} exp(\frac{1}{2} < BN, N > + < N, z >),$$

where

$$< BN, N >= \sum\nolimits_{j,k=1}^{g} B_{jk}N_jN_k, \; < N, z >= \sum\nolimits_{j=1}^{g} N_jz_j.$$

In particular for $g = 1$ we obtain

$$\theta(z) = \sum\nolimits_{n=-\infty}^{\infty} exp(\frac{bn^2}{2} + nz),$$

where b is a complex number such that $\mathfrak{RE}(b) < 0$.
Let Γ be a compact Riemann surface of genus g and let $v = (v_1, ..., v_g)$ be the normalized basis of holomorphic differentials with respect to a canonical homology basis. Let B the corresponding period matrix. Suppose $\theta(e, B) = 0$, where $e \in \mathbb{C}^g$, and consider the function

$$f(P) = \theta(\int_{P_1}^{P} v - e, B)\theta(\int_{P_2}^{P} v + e, B), \; P_1 \neq P_2,$$

where we assume that $\theta(\int_{P_1}^{P} v - e, B)$ and $\theta(\int_{P_2}^{P} v + e, B)$ are not identically zero.
For $P = P_1$ we have $f(P_1) = \theta(-e) = \theta(e) = 0$ since θ is a even function, and for $P = P_2$, $f(P_2) = \theta(e) = 0$.
We know that the function $\theta(\int_{P_1}^{P} v - e, B)$ has g zeros on Γ, so $f(P)$ has $2g$ zeros on Γ.
We have that

$$div(\theta(\int_{P_1}^{P} v - e)) = P_1 + D, \; div(\theta(\int_{P_2}^{P} v + e)) = P_2 + D^{'},$$

where D, $D^{'}$ are positive divisor of degree $g-1$. Now $e = D - \Delta$ and $-e = D^{'} - \Delta$, where $D + D^{'} = 2\Delta = K$. So

$$div(\theta(\int_{P_1}^{P} v - e, B)\theta(\int_{P_2}^{P} v + e, B)) = P_1 + P_2 + H(P),$$

for some holomorphic differential $H(P)$ independent from P_1 and P_2.
Consider now the case $P_1 = P_2$, we can write

$$\theta(\int_{P_1}^{P} v - e)\theta(\int_{P_1}^{P} v + e) = (P - P_1)^2 H(P)^2$$

we denote $F(P) = \theta(\int_{P_1}^{P} v - e)\theta(\int_{P_1}^{P} v + e)$ and then

$$H(P)^2 = \frac{F(P)}{(P - P_1)^2}.$$

The Taylor expansion of F in a neighborhood of P_1 is in the form

$$F(P) = F(P_1) + \frac{\partial F}{\partial P}(P_1)(P - P_1) + \frac{1}{2}\frac{\partial^2 F}{\partial P^2}(P_1)(P - P_1)^2 + \frac{1}{6}\frac{\partial^3 F}{\partial P^3}(P_1)(P - P_1)^3 + ...,$$

but P_1 is a double zero of F, so $F(P) = \frac{1}{2}\frac{\partial^2 F}{\partial P^2}(P_1)(P - P_1)^2 + \frac{1}{6}\frac{\partial^3 F}{\partial P^3}(P_1)(P - P_1)^3 + ...$, and

$$H(P_1)^2 = \frac{1}{2}\frac{\partial^2 F}{\partial P^2}(P_1).$$

We have that $\frac{\partial^2 F}{\partial P^2}(P_1) = 2(\sum_{k=1}^{g} \frac{\partial \theta}{\partial z_k}(e) v_k(P_1))^2$. We conclude that

$$H(P_1)^2 = (\sum_{k=1}^{g} \frac{\partial \theta}{\partial z_k}(e) v_k(P_1))^2, \ \forall \ P_1 \in \Gamma,$$

and

$$H(P) = \sum_{k=1}^{g} \frac{\partial \theta}{\partial z_k}(e) v_k(P_1).$$

Consider now the Riemann Surface Γ given by the equation

$$y^2 = (x - u_1)(x - u_2)(x - u_3)(x - u_4)(x - u_5)(x - u_6),$$

and set $Q_i = (u_i, 0)$. The Q_i are the branch points of the surfaces, clearly Γ is a hyperelliptic surfaces of genus $g = 2$.
Let $\Delta = 2K$ be the Riemann divisor. The vector of Riemann constant k_{Q_6} with base point Q_6 takes the form

$$k_{Q_6} = (\frac{1}{2}, 0) + (\frac{1}{2}, \frac{1}{2})B,$$

where B is the period matrix. Consider the Abel map with base point Q_6,

$$\mathcal{A} : \Gamma \to J(\Gamma), \ P \mapsto (\int_{Q_6}^{P} \omega_1, \int_{Q_6}^{P} \omega_2),$$

where $\{\omega_1 = \frac{dx}{y}, \omega_2 = \frac{xdx}{y}\}$ is a basis of the holomorphic differentials.
Consider the divisor $D = Q_1 + Q_3 + Q_5 - P_{\infty^+} - P_{\infty^-}$. If b_3 is a loop through Q_5 and Q_6, in the homology basis $\{a_1, a_2, b_1, b_2\}$ we have $b_3 = -b_2$. Then

$$\int_{Q_6}^{Q_1} \omega_1 = \frac{1}{2}(\int_{b_3} \omega_1 + \int_{a_2} \omega_1 + \int_{b_2} \omega_1) = \frac{1}{2}\int_{a_2} \omega_1,$$

similarly we have $\int_{Q_6}^{Q_3} \omega_1 = \frac{1}{2} + \frac{1}{2}\int_{b_1} \omega_1$ and $\int_{Q_6}^{Q_5} \omega_1 = \frac{1}{2}\int_{b_2} \omega_1$. Computing the same integrals on ω_2 we find

$$\mathcal{A}(D) = k_{Q_6},$$

so D is a representative for the Riemann divisor Δ.
We can see the previous fact from another viewpoint. Let $\pi : \Gamma \to \mathbb{CP}^1$, $(x, y) \mapsto x$, be the projection. We know that

$$K_\Gamma = R_\pi + \pi^* K_{\mathbb{CP}^1},$$

where R_π is the branch point divisor, in our case $R_\pi = Q_1+Q_2+Q_3+Q_4+Q_5+Q_6$. Since the canonical divisor of $\mathbb{CP}^1$ is $K_{\mathbb{CP}^1} = -2P_\infty$ we have

$$K_\Gamma = Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + Q_6 - 2P_{\infty^+} - 2P_{\infty^-},$$

and since $Q_1 \cong Q_2$, $Q_3 \cong Q_4$, $Q_5 \cong Q_6$ on Γ we find

$$K_\Gamma = 2Q_1 + 2Q_3 + 2Q_5 - 2P_{\infty^+} - 2P_{\infty^-}.$$

Finally since $\Delta = 2K$ we see that

$$\Delta = Q_1 + Q_3 + Q_5 - P_{\infty^+} - P_{\infty^-}.$$

Let $e = \mathcal{A}(Q_k) + k_{Q_6}$, $k = 1, 3, 5, 6$. Consider $e = q + pB$. For $k = 6$ clearly $e = k_{Q_6}$ is a half integer odd characteristic. For $k = 5$ we have

$$\mathcal{A}(Q_5) + K_{Q_6} = (\frac{1}{2} + \frac{1}{2}\int_{b_1} \omega_1, \frac{1}{2}\int_{b_1} \omega_2) = (\frac{1}{2}, 0) + (\frac{1}{2}, 0)B.$$

Note that $< q, p >=< (\frac{1}{2}, 0), (\frac{1}{2}, 0) >= \frac{1}{4} = \xi$ satisfies $4\xi \equiv 1\ mod(2)$, then (q, p) is an half integer odd characteristic. Similarly one has that (q, p) is an half integer odd characteristic for any $k = 1, 3, 5, 6$.

Now consider the function $f(P)$ with $P_1 = Q_6$, $P_2 = Q_5$,

$$f(P) = \theta(\int_{Q_6}^P v - e, B)\theta(\int_{Q_5}^P v + e, B).$$

We know that f vanishes in Q_5, Q_6 and in other $2g - 2 = 2$ points that are the zeros of a holomorphic differentials.
Recall that For a non special divisor $D = P_1 + ... + P_g$ of degree g the function $F(P) = \theta(A_{P_0}(P) - A_{P_0}(D) - K_{P_0})$ has on Γ exactly g zeros $P = P_1, ..., P = P_g$. In our case $e = \mathcal{A}(Q_k)+k_{Q_6}$ and $\theta(\int_{Q_6}^P v-e) = \theta(\int_{Q_6}^P v-\mathcal{A}(Q_k)-k_{Q_6})$ has two zeros Q_k, Q_6, and $\theta(\int_{Q_5}^P v - e) = \theta(\int_{Q_5}^P v - \mathcal{A}(Q_k) - k_{Q_5})$ has two zeros Q_k, Q_5. Then the differential $H(P) = \sum_{k=1}^g \frac{\partial\theta}{\partial z_k}(e)v_k(P)$, is a holomorphic differentials with a double zero in Q_k, so of the form

$$\sum\nolimits_{k=1}^2 \frac{\partial\theta}{\partial z_k}(e)v_k(P) = (x - u_k)^2\frac{dx}{y}.$$

Then $\nu^2 = \sum_{k=1}^2 \frac{\partial\theta}{\partial z_k}(e)v_k(P)$ is a holomorphic differential form on Γ, hence it is a holomorphic section of the canonical line bundle $\omega_\Gamma = \mathcal{O}_\Gamma(K_\Gamma)$. Since $\Delta = 2K_\Gamma$ we conclude that ν is a holomorphic section of the Riemann line bundle $\mathcal{O}_\Gamma(\Delta)$ associated to the Riemann divisor.

Exercise 0.3. Consider the curve

$$\Gamma = \{(z, w) \in \mathbb{C}^2 \mid w^5 = \prod\nolimits_{k=1}^5(z - z_k)\},$$

and let $Q_k = (z_k, 0)$ be its branch points.
The map $\pi : \Gamma \to \mathbb{P}^1$ is a 5-sheet covering of the Riemann Sphere, the covering is

ramified on each Q_k and the index of ramification is 4. So the ramification divisor has degree $deg(R) = 4 \times 5 = 20$. By Riemann-Hurwitz formula we have

$$g_\Gamma = \frac{deg(R)}{2} - deg(\pi) + 1,$$

and in our case $g_\Gamma = 10 - 5 + 1 = 6$.

The compactification of Γ in $\mathbb{P}^2$ is given by the equation

$$W^5 = \prod_{k=1}^{5} (Z - Xz_k),$$

where $w = \frac{W}{X}$ and $z = \frac{Z}{X}$. In this way Γ is realized as a smooth curve of degree 5 in $\mathbb{P}^2$, and using the genus formula we recover $g_\Gamma = \frac{1}{2}(5-1)(5-2) = 6$.

Definition 0.4. A point P of a Riemann surface Γ of genus g is called a Weierstrass point if $l(kP) > 1$ for some $k \leq g$. Where $l(kP)$ denotes the dimension of the linear system $|kP|$.

Suppose that z is a local parameter for a Riemann Surface Γ of genus g in a neighborhood of a point $P_0 \in \Gamma$, such that $z(P_0) = 0$. Assume that locally the basis of holomorphic differentials has the form $\omega_j = \alpha_j(z)dz$, $j = 1, ..., g$. Consider the determinant

$$W(z) = det \begin{pmatrix} \alpha_1(z) & \alpha_1^{'}(z) & \dots & \alpha_1^{(g-1)}(z) \\ \vdots & \ddots & \ddots & \vdots \\ \alpha_g(z) & \alpha_g^{'}(z) & \dots & \alpha_1^{(g-1)}(z) \end{pmatrix}.$$

Then the point P_0 is a Weierstrass point if and only if $W(z(P_0)) = W(0) = 0$. Let us define the weight of a Weierstrass point P_0 as the multiplicity of zero of $W(z)$ at this point. The total weight of all Weierstrass points on a Riemann surface Γ of genus g is equal to $(g-1)g(g+1)$.
In our case a basis of the holomorphic differentials is given by

$$\{\frac{dz}{w^4}, \frac{dz}{w^3}, \frac{dz}{w^2}, \frac{zdz}{w^4}, \frac{zdz}{w^3}, \frac{z^2dz}{w^4}\}.$$

Computing the zeros of $W(z)$ we found that the Weierstrass points are exactly the Q_k each of weight 42. The total weight is $42 \times 5 = 210 = (g-1)g(g+1)$.

Consider now the abelian differential on Γ, and the basis of $H^1(\Gamma)$ given above. Let $\pi : \Gamma \rightarrow \mathbb{CP}^1$, $(z, w) \mapsto z$, be the projection and $\{P_1^\infty, ..., P_5^\infty\} = \pi^{-1}(\infty)$, $Q_k = (z_k, 0)$, $k = 1, ..., 5$ be the branch points, and $\overline{Q_k} = (0, \xi_k)$, $k = 1, ..., 5$ be the points on Γ with $z = 0$. Then we have:

- $div(z) = \overline{Q_1} + ... + \overline{Q_5} - P_1^\infty - ... - P_5^\infty$,
- $div(w) = Q_1 + ... + Q_5 - P_1^\infty - ... - P_5^\infty$,
- $div(dz) = 4Q_1 + ... + 4Q_5 - (P_1^\infty)^2 - ... - (P_5^\infty)^2$.

The divisors of the holomorphic differentials are given by:

- $div(\frac{dz}{w^4}) = (P_1^\infty)^2 + ... + (P_5^\infty)^2$,
- $div(\frac{dz}{w^3}) = Q_1 + ... + Q_5 + P_1^\infty + ... + P_5^\infty$,
- $div(\frac{dz}{w^2}) = Q_1^2 + ... + Q_5^2$,

- $div(\frac{zdz}{w^4}) = \overline{Q_1} + ... + \overline{Q_5} + P_1^{\infty} + ... + P_5^{\infty}$,
- $div(\frac{zdz}{w^3}) = Q_1 + ... + Q_5 + \overline{Q_1} + ... + \overline{Q_5}$,
- $div(\frac{z^2dz}{w^4}) = \overline{Q_1}^2 + ... + \overline{Q_5}^2$.

Recall that a divisor D on Γ is said to be special when $l(K-D) = 0$ i.e.

$$\Omega(D) = \{\omega \mid \omega \in \Omega^1_{\Gamma},\ div(\omega) \geq D\} = 0,$$

in other words when there are not holomorphic differentials vanishing on D.
Consider now the divisors supported on the branch points. Clearly there are no holomorphic differentials vanishing on the divisors of the type $D = Q_i + 2Q_j + 3Q_k, i \neq j \neq k$, since the maximum order of Q_k as zero of a holomorphic differentials in 2. On the other hand for any other divisors of degree 6 supported on the branch points we have only three possibilities:

$$D = 2Q_i + 2Q_j + 2Q_k,\ D = 2Q_i + 2Q_j + Q_k + Q_l,\ D = 2Q_i + Q_j + Q_k + Q_l + Q_m.$$

Clearly for any divisor D of the previous type we can find a holomorphic differentials vanishing on D. We conclude that the divisors of the form $D = Q_i + 2Q_j + 3Q_k$ are not special, and that any other divisor of degree 6 supported on the branch points is special.

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