RATIONALITY AND UNIRATIONALITY PROBLEMS

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Abstract. These notes are an introduction to rationality problems in algebraic geometry, with emphasis on the relation between rationality, stable rationality, unirationality, and rational connectedness. We begin with the classical Lüroth problem and explain why the three notions coincide for curves, and for smooth projective surfaces in characteristic zero, but separate in higher dimension. We discuss fundamental examples such as cubic threefolds, Artin–Mumford threefolds, quartic threefolds, and special cubic fourfolds. We then describe modern methods for proving stable irrationality, including the decomposition of the diagonal, unramified Brauer groups, and specialization techniques. The second part of the notes focuses on unirationality questions for Fano varieties, conic bundles, del Pezzo surfaces, quadric bundles, weighted complete intersections, and low degree hypersurfaces in high dimension.

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Introduction

One of the oldest and most natural questions in algebraic geometry is whether the solutions of a system of polynomial equations can be described by rational functions. In geometric language, this asks whether an algebraic variety is birational to projective space, or at least dominated by projective space. These two possibilities lead to the notions of rationality and unirationality.

A variety is rational if its general points can be parametrized by independent rational parameters without redundancy. It is unirational if such a parametrization exists with possible redundancy, that is, if there is a dominant rational map from a projective space. Between these two notions lies stable rationality: a variety is stably rational if it becomes rational after multiplying by some projective space. Thus we have the chain X rational implies X stably rational, and X stably rational implies X unirational.

The central difficulty is that none of the reverse implications holds in general. Understanding when they fail, and how to detect their failure, is one of the main themes of modern birational geometry.

The starting point is the classical Lüroth problem. For curves, Lüroth's theorem says that every unirational curve is rational [Lur75]. Thus, for curves, rationality, stable rationality, and unirationality coincide. Over the complex numbers, this is also visible topologically: a smooth projective curve is rational exactly when its associated Riemann surface is a sphere, or equivalently when its genus is zero. From the arithmetic point of

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view, genus zero curves with a rational point have many rational points, while curves of genus at least two have only finitely many rational points over number fields by Faltings' theorem [Fal83].

For surfaces over an algebraically closed field of characteristic zero, unirationality still implies rationality. The key input is Castelnuovo's rationality criterion: a smooth projective surface is rational if and only if its irregularity and second plurigenus vanish. A dominant rational map from the projective plane forces the relevant differential forms to vanish, and Castelnuovo's criterion applies. In positive characteristic, however, purely inseparable phenomena break this argument. Zariski surfaces give examples of unirational surfaces which are not rational.

Starting from dimension three, the situation changes completely even over the complex numbers. The smooth cubic threefold is unirational but not rational by the theorem of Clemens and Griffiths [CG72]. Their obstruction is the intermediate Jacobian. Artin and Mumford constructed unirational threefolds which are not even stably rational, using torsion in cohomology, or equivalently a nontrivial Brauer class [AM72]. Iskovskikh and Manin proved that smooth quartic threefolds are not rational by developing the method of birational rigidity [IM71]. These examples show that rationality, stable rationality, and unirationality are genuinely different in dimension three.

A major development of the last decade is the emergence of powerful methods to prove stable irrationality. The key idea is that stable rationality forces strong conditions on zero-cycles. More precisely, a stably rational smooth projective variety is universally CH_0-trivial, or equivalently admits a decomposition of the diagonal. This point of view goes back to Bloch and Srinivas [BS83], and was brought to the center of rationality questions by Voisin [Voi15]. Together with specialization techniques, this method allows one to prove that very general varieties in a family are not stably rational by degenerating them to special singular fibers with computable obstructions. This strategy has been developed by Colliot-Thélène–Pirutka, Totaro, Schreieder, Hassett–Pirutka–Tschinkel, Auel–Böhning–Pirutka, Nicaise–Ottem, and others.

The second guiding theme of these notes is the relation between unirationality and rational connectedness. A complex variety is rationally connected if two general points can be joined by a rational curve. Every unirational variety is rationally connected, but the converse is one of the most basic open problems in the area. For curves and surfaces over $\mathbb{C}$, rational connectedness and unirationality coincide. In dimension at least three, no example is known of a complex rationally connected variety which is not unirational. At the same time, there is no known birational invariant capable of separating the two notions in general.

The minimal model program explains why Fano varieties, del Pezzo fibrations, and conic bundles are the natural testing ground for these questions. Smooth Fano varieties are rationally connected, and Mori fiber spaces organize the birational geometry of uniruled varieties. We therefore discuss the known rationality and unirationality results for smooth Fano threefolds of Picard rank one, with special attention to the still mysterious cases such as quartic threefolds and certain weighted hypersurfaces.

We then turn to conic bundles and del Pezzo surfaces. For conic bundle surfaces over $\mathbb{P}^1$, the number of singular fibers controls the geometry through the formula $K_S^2 = 8 - \delta_S$. The theorem of Kollár–Mella shows that conic bundles with at most seven singular fibers are unirational as soon as they have a rational point [KM17]. The next case, $\delta_S = 8$, is a boundary case: it is the first case outside the del Pezzo range and already exhibits a subtle separation between rationality and unirationality. We also discuss degree 1 and degree 2 del Pezzo surfaces, where the general unirationality problem over arbitrary fields remains open.

Cubic hypersurfaces form another central thread. Kollár proved that a smooth cubic hypersurface of dimension at least two is unirational if and only if it has a rational point [Kol02]. Rationality is much subtler. Cubic fourfolds provide one of the most important open problems in higher-dimensional algebraic geometry. We review Hassett's Noether–Lefschetz divisors, the role of associated $K3$ surfaces, and explicit rationality constructions for special cubic fourfolds in $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, $\mathcal{C}_{38}$, and $\mathcal{C}_{42}$.

Finally, we discuss several recent and ongoing directions. These include unirationality results for quadric bundles and quartic or quintic hypersurfaces, log del Pezzo weighted complete intersections, rationality of complex conic bundle threefolds, and low degree complete intersections in high dimension. The notes end with a collection of open problems, emphasizing that many basic questions remain unresolved: whether there are rational odd-dimensional cubic hypersurfaces, whether all complex rationally connected varieties are unirational, and whether conic bundle threefolds over rational surfaces are always unirational.

The aim is not to give a complete survey, but to provide a guided path through the main ideas. The emphasis is on constructions, examples, and geometric intuition: rationality problems are often solved not by writing down parametrizations directly, but by finding the right rational curves, multisections, degenerations, and birational invariants.

1. Rationality, stable rationality and unirationality

Before giving the formal definitions, let us recall the geometric intuition behind the three notions that will be used throughout these notes.

Rationality is the strongest one: a variety is rational if, up to birational equivalence, it is just projective space. In other words, outside lower-dimensional subsets, its points can be parametrized by n independent parameters.

Stable rationality is a weaker notion. It is the birational analogue of the biregular notion of stable isomorphism: instead of asking whether two varieties become isomorphic after taking products with auxiliary factors, one asks whether they become birational after taking products with projective spaces. In particular, a variety X is stably rational if, after adding a number of projective parameters, it becomes birational to projective space. That is, X is stably birational to a point.

This point of view is closely related to the philosophy of cancellation problems, such as the Zariski cancellation problem, where one asks whether an isomorphism $X \times \mathbb{A}^n \cong Y \times \mathbb{A}^n$ forces $X \cong Y$.

Unirationality is weaker still. A variety is unirational if it can be dominated by a projective space, that is, if its general points admit a rational parametrization, possibly with redundancy. Thus rationality gives a birational parametrization, whereas unirationality only gives a dominant one.

Definition 1.1. Let X be an n-dimensional integral variety over a field k.

- X is *rational* over k if it is birational to $\mathbb{P}^n_k$. That is, in terms of function fields, X is rational over k if $k(X) \cong k(x_1, \dots, x_n)$.
- X is *unirational* over k if there exists a dominant rational map $\mathbb{P}^N_k \dashrightarrow X$ for some N. That is, the function field $k(X)$ embeds into a purely transcendental extension of k.
- X is *stably rational* over k if there exists a nonnegative integer m such that $X \times \mathbb{P}^m_k$ is rational. That is, in terms of function fields, there exist indeterminates $y_1, \dots, y_m$ such that $k(X)(y_1, \dots, y_m) \cong k(x_1, \dots, x_{n+m})$.

Assume that X is rational over a field k. By definition, there exists a birational equivalence $X \dashrightarrow \mathbb{P}^n_k$. Hence, for every $m \geq 0$, the product $X \times \mathbb{P}^m_k$ is birational to $\mathbb{P}^n_k \times \mathbb{P}^m_k$, which is rational. Thus X is stably rational.

Now assume that X is stably rational. Then there exists a nonnegative integer m such that $X \times \mathbb{P}^m_k$ is rational. Hence there is a birational map $\mathbb{P}^{n+m}_k \dashrightarrow X \times \mathbb{P}^m_k$. Composing it with the natural projection $\pi\colon X \times \mathbb{P}^m_k \to X$ we obtain a dominant rational map $\mathbb{P}^{n+m}_k \dashrightarrow X$. This shows that X is unirational. Therefore:

$$X \text{ rational } \Rightarrow X \text{ stably rational } \Rightarrow X \text{ unirational.}$$

Remark 1.2 (Zariski cancellation)**.** The preceding notions are birational in nature. They should be compared with a biregular question known as the Zariski cancellation problem. In one of its standard forms, this problem asks whether an isomorphism

$$X \times \mathbb{A}^1_k \cong Y \times \mathbb{A}^1_k$$

forces $X \cong Y$. In this generality the answer is negative: there exist non-isomorphic affine varieties whose cylinders are isomorphic. A particularly important special case is the cancellation problem for affine spaces: if

$$X \times \mathbb{A}^1_k \cong \mathbb{A}^{n+1}_k,$$

must X be isomorphic to $\mathbb{A}^n_k$? This is known to have an affirmative answer in low dimension, for instance for affine surfaces over an algebraically closed field [CML08]. However, in positive characteristic the answer is negative in dimension at least 3. In characteristic zero, the corresponding problem for $\mathbb{A}^n$, $n \geq 3$, remains open.

The birational analogue of the special cancellation problem is precisely the question whether stable rationality implies rationality. In contrast with the biregular problem for affine spaces, this implication is known to fail: there exist smooth projective varieties which are stably rational but not rational [BCTSSD85].

1.2. **The Lüroth problem and its variations.** The classical Lüroth problem asks whether unirationality is actually the same as rationality. More explicitly, if a variety X is dominated by a projective space, $\mathbb{P}^N \dashrightarrow X$, must X be birational to a projective space?

For curves the answer is affirmative. This is precisely Lüroth's theorem: every intermediate field

$$k \subset K \subset k(t)$$

is itself purely transcendental over k, that is, $K \cong k(u)$ for some rational function u [Lur75]. Geometrically, this means that if a curve is dominated by $\mathbb{P}^1$, then its function field is a subfield of $k(t)$, hence it is again the function field of a projective line. Therefore, for curves, unirationality implies rationality. Since we already know that rational $\Rightarrow$ stably rational $\Rightarrow$ unirational, we obtain that, for curves, the three notions are equivalent.

For smooth projective surfaces over an algebraically closed field of characteristic zero, the answer is again affirmative. The key input is Castelnuovo's rationality criterion, which says that a smooth projective surface S is rational if and only if

$$q(S) = h^1(S, \mathcal{O}_S) = 0 \qquad \text{and} \qquad P_2(S) = h^0(S, 2K_S) = 0$$

[Bea96]. Now assume that S is unirational. Then there exists a dominant rational map $\mathbb{P}^2 \dashrightarrow S$. The projective plane has no nonzero holomorphic 1-forms and no nonzero pluricanonical forms. Since differential forms pull back under dominant rational maps, the existence of such a parametrization forces the corresponding invariants of S to vanish. Thus $q(S) = P_2(S) = 0$, and Castelnuovo's criterion gives that S is rational. Hence, in characteristic zero, the three notions are equivalent also for smooth projective surfaces.

In positive characteristic the situation changes. The reason is that dominant maps may be purely inseparable. Informally, this means that a surface can be covered by the projective plane through phenomena involving p-th powers, without being birationally equivalent to the projective plane. In such a situation, the pullback of differential forms may lose information, since in characteristic p one has $d(x^p) = 0$. Thus the characteristic-zero argument using differential forms no longer proves rationality.

Indeed, there exist unirational surfaces in positive characteristic which are not rational. Classical examples are provided by Zariski surfaces, and more recent constructions give unirational non-rational Zariski surfaces in every positive characteristic [Mit14b].

Let us briefly explain how such examples are constructed. Let k be an algebraically closed field of characteristic $p > 0$. A Zariski surface is, birationally, a surface obtained from an equation of the form

$$z^p = f(x, y),$$

where $f(x, y) \in k(x, y)$ is not a p-th power. That is, its function field is a purely inseparable extension of degree p of the rational function field $k(x, y)$.

The reason why this gives a unirational surface is special to characteristic p. Since k is algebraically closed, after writing $x = u^p$ and $y = v^p$, one has

$$f(u^p, v^p) = g(u, v)^p$$

for a suitable rational function $g(u, v)$. Hence the formula $(u, v) \longmapsto \big(u^p, v^p, g(u, v)\big)$ defines a dominant rational map $\mathbb{P}^2 \dashrightarrow S$. This map is purely inseparable of degree p. Thus S is unirational, but not because it has a birational parametrization in the usual sense: the parametrization factors through the Frobenius phenomenon.

This is precisely why these surfaces can fail to be rational. In characteristic zero, a dominant rational map from $\mathbb{P}^2$ to a smooth projective surface forces the vanishing of the birational differential invariants used in Castelnuovo's criterion. In characteristic p, however, the differential of a p-th power vanishes: $d(u^p) = 0$.

Thus a purely inseparable parametrization may destroy differential information. After compactifying the affine equation $z^p = f(x, y)$ and resolving the singularities, one can obtain smooth projective surfaces which are dominated by $\mathbb{P}^2$ but still have birational invariants incompatible with rationality. These are the classical Zariski surfaces introduced by Zariski [Zar58] and studied systematically by Blass and Lang [BL87]. More recent constructions show that Zariski's question about such surfaces has a negative answer in every positive characteristic [Mit14a].

Therefore, in positive characteristic, unirationality does not imply rationality for surfaces. On the other hand, for smooth projective surfaces, stable rationality still implies rationality by the Castelnuovo–Zariski criterion [Lan81]. Thus, for surfaces in positive characteristic, the failure is specifically unirational $\not\Rightarrow$ rational, while rationality and stable rationality remain equivalent in the smooth projective case.

Starting from dimension three, the Lüroth problem has a negative answer even over the complex numbers. In fact, there are several different ways in which the implications rational $\Rightarrow$ stably rational $\Rightarrow$ unirational can fail to be reversed.

First, stable rationality is not the same as rationality in dimension three. Beauville, Colliot-Thélène, Sansuc and Swinnerton-Dyer constructed smooth complex projective threefolds which are stably rational but not rational [BCTSSD85]. In their examples, after taking the product with a projective space one obtains a rational variety, but the original threefold still carries a birational invariant which prevents it from being rational. The starting point is the following affine threefold over $\mathbb{C}$: $X \subset \mathbb{A}^4_{x,t,y,z}$ defined by an equation of the form

$$y^2 - \delta(t) z^2 = P(x, t),$$

where $P(x, t) = x^3 + p(t)x + q(t)$ is an irreducible cubic polynomial in x, depending on the parameter t, and $\delta(t) = 4p(t)^3 + 27q(t)^2$ is, up to sign, its discriminant. The projection $X \longrightarrow \mathbb{A}^2_{x,t}$ makes X into a conic bundle: over a general point (x, t), the equation is a smooth conic in the variables y, z.

There is another useful way to look at the same variety. If one regards t as a parameter and works over the field $K = \mathbb{C}(t)$, then the generic fiber is a surface over K given by $y^2 - \delta z^2 = P(x)$.

This is a Châtelet-type surface. Over the algebraic closure $\overline{K}$, the element δ becomes a square, say $\delta = d^2$. Hence the conic $y^2 - \delta z^2 = P(x)w^2$ has the point at infinity $[y : z : w] = [d : 1 : 0]$. Thus the conic bundle acquires a rational section over $\overline{K}$, and consequently the surface becomes rational over $\overline{K}$. Over K, however, this point at infinity need not be defined. If δ is not a square in K, the two points at infinity are conjugate over K, and the conic bundle may have no rational section over K. In this way the surface can be geometrically rational without being rational over K. The obstruction is arithmetic: it is detected by the Galois action on the geometric Picard group, or equivalently by a nontrivial Brauer class [BCTSSD85, CTSSD87]. Thus the construction produces a complex threefold by spreading out, over the t-line, a family of surfaces which are geometrically rational but not rational over their ground field.

Thus stable rationality should not be thought of as a minor technical variant of rationality: in dimension at least three it is genuinely weaker.

There are also fundamental examples of unirational varieties which are not rational.

1.2.1. *The smooth cubic threefold.* Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold over $\mathbb{C}$. Let us first explain why X is unirational. Since X is a smooth cubic threefold over an algebraically closed field, it contains lines. Fix a line $\ell \subset X$. For a point $p \in \ell$, consider the projective tangent hyperplane $T_pX \subset \mathbb{P}^4$. A line $L \subset T_pX$ passing through p is tangent to X at p. Therefore, if L is not contained in X, the intersection cycle of L with X has the form $L \cap X = 2p + q$, where $q \in X$ is the residual point.

Thus we get a rational map

$$\Phi \colon U \dashrightarrow X,$$

where U parametrizes pairs (p, L), with $p \in \ell$ and $L \subset T_pX$ a line through p, by sending (p, L) to the residual point q. The variety U is rational: it is birational to a projective bundle over $\ell \simeq \mathbb{P}^1$, with fiber the space of lines through p inside the hyperplane T_pX. Hence U is rational of dimension three.

The map Φ is dominant. Indeed, for a general point $q \in X$, one asks for points $p \in \ell$ such that the line $\overline{pq}$ is tangent to X at p. This tangency condition is a nontrivial algebraic condition on $p \in \ell$, and for general q it has finitely many solutions. Therefore the residual-point construction dominates X. This gives a dominant rational map from a rational threefold to X, hence X is unirational.

There is also another very useful description. Blowing up the line ℓ, one obtains $\mathrm{Bl}_\ell(X) \longrightarrow \mathbb{P}^2$ by projecting from ℓ. A point of $\mathbb{P}^2$ corresponds to a plane $\Pi \subset \mathbb{P}^4$ containing ℓ, and $\Pi \cap X = \ell + C_\Pi$, where C_Π is a residual conic. Thus the blow-up of X along ℓ is a conic bundle over $\mathbb{P}^2$. This conic bundle structure is central in the study of the intermediate Jacobian of X.

Clemens and Griffiths proved that X is not rational [CG72]. Their obstruction is the intermediate Jacobian $J(X)$, a principally polarized abelian variety associated with the Hodge structure on $H^3(X, \mathbb{C})$. The point is that the intermediate Jacobian behaves like a birational invariant for smooth projective threefolds, up to adding Jacobians of curves.

More precisely, if a smooth projective threefold is rational, then it can be obtained birationally from $\mathbb{P}^3$ by a sequence of blow-ups and blow-downs. Blowing up points does not change the intermediate Jacobian, while blowing up smooth curves adds the Jacobians of those curves. Since $J(\mathbb{P}^3) = 0$, the intermediate Jacobian of a rational threefold must be a product of Jacobians of smooth curves.

Clemens and Griffiths show that this does not happen for a smooth cubic threefold. They study the Fano surface $F(X)$ of lines contained in X and prove that $J(X)$ is closely related to the Albanese variety of $F(X)$. Then they analyze the principal polarization on $J(X)$, especially its theta divisor, and show that $J(X)$ cannot be written as a product of Jacobians of curves. Hence X cannot be rational.

1.2.2. *The Artin–Mumford example.* Artin and Mumford constructed a unirational threefold which is not only non-rational, but not even stably rational [AM72]. Their example is a quartic double solid.

The construction starts from a web of quadrics in $\mathbb{P}^3$. Concretely, take a symmetric 4×4 matrix

$$A(u) = u_0 A_0 + u_1 A_1 + u_2 A_2 + u_3 A_3$$

whose entries are linear forms in homogeneous coordinates $[u_0 : u_1 : u_2 : u_3] \in \mathbb{P}^3$. For each point $u \in \mathbb{P}^3$, the matrix $A(u)$ defines a quadric surface in another copy of $\mathbb{P}^3$. The discriminant of this web is the quartic surface $B = \{\det A(u) = 0\} \subset \mathbb{P}^3$.

This is a quartic symmetroid. For a general choice of the web, B has exactly ten ordinary double points; these correspond to the quadrics in the web whose rank drops by at least two.

Now take the double cover $Y \longrightarrow \mathbb{P}^3$ branched along B. That is, Y is given by an equation $w^2 = \det A(u)$. Since B is singular, the double cover Y is also singular. Let $\widetilde{Y} \longrightarrow Y$ be a resolution of singularities. This smooth projective threefold $\widetilde{Y}$ is the Artin–Mumford example.

Let us explain why it is unirational. Choose one of the nodes $o \in B$. Consider the lines in $\mathbb{P}^3$ passing through o. They are parametrized by a plane $\mathbb{P}^2$.

For a general such line L, the intersection $L \cap B$ contains the node o with multiplicity two, since o is an ordinary double point of B. Since B has degree four, the residual intersection consists of two further points. The inverse image of L in the double cover Y is therefore a double cover of $L \simeq \mathbb{P}^1$ branched at two points. Such a double cover is again a rational curve.

Thus the lines through o give a family of rational curves on Y, parametrized by $\mathbb{P}^2$. After resolving the singularities, this gives a conic bundle structure $\widetilde{Y} \dashrightarrow \mathbb{P}^2$.

Moreover, in the Artin–Mumford construction this conic bundle has a rational multisection. After pulling back to this rational multisection, the conic bundle acquires a rational point on the generic fiber, and hence becomes birational to a $\mathbb{P}^1$-bundle over a rational surface. Therefore $\widetilde{Y}$ is dominated by a rational threefold. This proves that $\widetilde{Y}$ is unirational.

The non-rationality is proved using torsion in cohomology. Artin and Mumford show that $\operatorname{Tors} H^3(\widetilde{Y}, \mathbb{Z}) \neq 0$. In fact, there is a nonzero 2-torsion class. That is, one can express the obstruction as a nontrivial element of the Brauer group $\operatorname{Br}(\widetilde{Y})$.

For a smooth projective complex threefold X, the torsion subgroup of $H^3(X, \mathbb{Z})$ is a birational invariant. On the other hand, $H^3(\mathbb{P}^3, \mathbb{Z}) = 0$. More generally, rational varieties have no such torsion obstruction. Hence the existence of nonzero torsion in $H^3(\widetilde{Y}, \mathbb{Z})$ proves that $\widetilde{Y}$ is not rational.

The same argument proves more: $\widetilde{Y}$ is not stably rational. Indeed, if $\widetilde{Y}$ were stably rational, then for some m one would have $\widetilde{Y} \times \mathbb{P}^m$ rational. But the Künneth formula shows that taking the product with projective space does not kill the torsion in H^3. Therefore

$$\operatorname{Tors} H^3(\widetilde{Y} \times \mathbb{P}^m, \mathbb{Z}) = \operatorname{Tors} H^3(\widetilde{Y}, \mathbb{Z}) \neq 0,$$

which is impossible for a rational variety. Thus the Artin–Mumford example gives $\widetilde{Y}$ unirational but not stably rational. In particular, it is not rational.

1.2.3. *Quartic threefolds and the theorem of Iskovskikh–Manin.* The third fundamental family comes from quartic hypersurfaces $V \subset \mathbb{P}^4$. Iskovskikh and Manin proved that every smooth quartic threefold is not rational [IM71]. Together with earlier examples of smooth unirational quartic threefolds constructed by Segre [Seg60], this gives another class of counterexamples to the Lüroth problem.

Let us first explain the unirational examples. Segre's construction, that we will reproduce in detail later, is based on triple tangent lines.

Let $V \subset \mathbb{P}^4$ be a smooth quartic threefold. A line $L \subset \mathbb{P}^4$ is called a triple tangent to V at a point $x \in V$ if either $L \subset V$, or $L \cap V = 3x + y$ as a zero-dimensional cycle. In the second case, the point $y \in V$ is called the fourth point of intersection.

Now fix a good point $x \in V$. The tangent hyperplane section $C(x) = V \cap T_xV$ is a quartic surface in $T_xV \simeq \mathbb{P}^3$, singular at x. Under suitable genericity assumptions, x is an ordinary double point of $C(x)$, and the triple tangent lines through x are precisely the generators of the quadratic tangent cone of $C(x)$ at x. These generators form a rational curve.

Thus, by varying x, one obtains a family of triple tangent lines. Sending a triple tangent line to its fourth point of intersection with V gives a rational map to V.

The key point in Segre's construction is to find inside V a rational surface $F \subset V$ such that the family of triple tangents over F is itself rational and the fourth-point map dominates V. In the explicit example of Segre, F is a singular rational hyperplane section of V. Over a general point of F, the relevant triple tangent directions form two rationally varying choices; that is, the natural double section splits into two rational sections. This makes the total parameter space rational.

Therefore one obtains a dominant rational map $\mathbb{P}^3 \dashrightarrow V$. In the explicit example written down in [IM71, Section 9], the resulting unirational parametrization has degree 24. Hence these special smooth quartic threefolds are unirational.

We now explain why they are not rational. Iskovskikh and Manin prove a much stronger theorem: if $V, V' \subset \mathbb{P}^4$ are smooth quartic threefolds and $\varphi\colon V \dashrightarrow V'$ is a birational map, then φ is induced by a projective isomorphism. In particular, $\operatorname{Bir}(V) = \operatorname{Aut}(V)$, and this group is finite.

The proof is the origin of the method of birational rigidity. The idea is as follows. Suppose that V were rational, or more generally that there existed a nontrivial birational map from V to another Mori fiber space. Pull back hyperplane sections through such a birational map. This produces a mobile linear system $\mathcal{M} \subset |nH|$ on V, where H is the hyperplane class. If the birational map is not an isomorphism, then the Noether–Fano inequalities force $\mathcal{M}$ to have a maximal singularity: roughly speaking, a point, curve, or infinitely near subvariety along which the multiplicity of $\mathcal{M}$ is too large.

The rest of the argument shows that such a maximal singularity cannot exist on a smooth quartic threefold. One intersects divisors from $\mathcal{M}$, estimates multiplicities along the possible centers, and obtains contradictions with the positivity of intersection numbers on V. Thus every birational self-map of V is actually regular, and every birational map between smooth quartic threefolds is projective.

This proves that a smooth quartic threefold is not rational. Indeed, if V were rational, then $\mathrm{Bir}(V)$ would be isomorphic to the Cremona group $\mathrm{Bir}(\mathbb{P}^3)$, which is huge and contains many non-regular transformations. This is incompatible with the rigidity theorem, which says that $\mathrm{Bir}(V) = \mathrm{Aut}(V)$ is finite.

Therefore Segre's unirational quartic threefolds are unirational but not rational.

It is worth stressing that the Iskovskikh–Manin theorem proves non-rationality, not stable irrationality. Stable rationality of quartic threefolds is a subtler question. Much later, using degeneration methods and the decomposition of the diagonal, Colliot-Thélène and Pirutka proved that a very general quartic threefold is not stably rational [CTP16].

These examples show that, in dimension three, rationality, stable rationality and unirationality are genuinely distinct phenomena. The problem is no longer merely to construct rational parametrizations, but to understand which birational invariants survive such parametrizations.

2. Recent methods for stable irrationality

The examples discussed above show that rationality and unirationality are different in dimension at least three. However, many classical methods prove only non-rationality. For instance, the theorem of Iskovskikh–Manin proves that a smooth quartic threefold is not rational, but it does not decide whether it may become rational after taking the product with some projective space.

The modern theory of stable irrationality is based on invariants which are stable birational invariants. The guiding principle is the following:

$$X \text{ stably rational} \quad\Longrightarrow\quad X \text{ has very simple } CH_0\text{-theory.}$$

This principle is made precise by the decomposition of the diagonal.

Decomposition of the diagonal. Let X be a smooth projective variety of dimension n over a field k. Assume, for simplicity, that X has a zero-cycle of degree 1. We say that X admits an integral Chow-theoretic decomposition of the diagonal if there exist a zero-cycle x of degree 1 on X, a proper closed subset $D \subsetneq X$, and a cycle $Z \in CH_n(X \times X)$ supported on $D \times X$, such that

$$[\Delta_X] = [X \times x] + Z \qquad \text{in } CH_n(X \times X).$$

The meaning of this formula is intuitive: the diagonal correspondence, which controls the identity map on the Chow groups and on cohomology, is forced to be supported away from a dense open subset of X. In other words, all zero-cycles on X, after extension of the ground field, are controlled by a lower-dimensional part of X.

This is closely related to universal CH_0-triviality. A smooth projective variety X is called universally CH_0-trivial if, for every field extension L/k, the degree map

$$\deg\colon CH_0(X_L) \longrightarrow \mathbb{Z}$$

is an isomorphism. For smooth projective varieties with a zero-cycle of degree 1, universal CH_0-triviality is equivalent to the existence of a Chow-theoretic decomposition of the diagonal [BS83].

Stable rationality implies universal CH_0-triviality. Indeed, projective space has universally trivial CH_0, and this property is invariant under stable birational equivalence among smooth projective varieties. Therefore:

$$X \text{ stably rational} \Longrightarrow X \text{ universally } CH_0\text{-trivial} \Longleftrightarrow X \text{ admits a decomposition of the diagonal.}$$

Consequently, to prove that X is not stably rational, it is enough to prove that X does not admit such a decomposition.

This point of view goes back to Bloch and Srinivas [BS83], but its power in stable rationality questions was brought to the foreground by Voisin [Voi15]. The key advantage is that the decomposition of the diagonal behaves well in degenerating families.

The Brauer group and unramified classes. One concrete way of obstructing universal CH_0-triviality is through the Brauer group. Assume in this subsection that k is a perfect field of characteristic different from two.

The Brauer group $\mathrm{Br}(k)$ is the group of equivalence classes of central simple k-algebras, where

$$A \sim B \quad \text{if} \quad A \otimes_k M_m(k) \cong B \otimes_k M_n(k) \quad \text{for some } m, n \geq 1.$$

Given $a, b \in k^\times$, the quaternion algebra $(a, b)_k$ is the k-algebra generated by i, j with

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji.$$

It is a 4-dimensional central simple algebra over k, and it defines a class in $\mathrm{Br}(k)[2]$.

Let W be an integral variety over k, with function field $k(W)$. For $a, b \in k(W)^\times$, we write $(a, b) \in \mathrm{Br}(k(W))[2]$ for the Brauer class of the quaternion algebra $(a, b)_{k(W)}$. That is, this is the Brauer class of the conic

$$C_{(a,b)} = \{ax^2 + by^2 - z^2 = 0\}$$

over $k(W)$. Let $D \subset W$ be a prime divisor. Assume that W is smooth at the generic point of D, and let v_D be the associated discrete divisorial valuation. There are residue maps

$$\partial^1_D \colon k(W)^\times / k(W)^{\times 2} \longrightarrow \mathbb{Z}/2, \qquad \partial^1_D(a) \equiv v_D(a) \pmod 2,$$

and

$$\partial^2_D \colon \mathrm{Br}(k(W))[2] \longrightarrow k(D)^\times / k(D)^{\times 2}.$$

For a quaternion class one has the explicit formula

$$\partial^2_D(a, b) = (-1)^{v_D(a) v_D(b)} \, \overline{\frac{a^{v_D(b)}}{b^{v_D(a)}}},$$

where the bar denotes the image in the residue field $k(D)$.

A Brauer class $\alpha \in \mathrm{Br}(k(W))$ is called unramified if $\partial_D(\alpha) = 0$ for every divisorial valuation D. The group of such classes is the unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(k(W)/k)$. If W is smooth and projective, then $\mathrm{Br}_{\mathrm{nr}}(k(W)/k) = \mathrm{Br}(W)$.

This is important since the unramified Brauer group is a stable birational invariant of smooth projective varieties. In particular, over an algebraically closed field, $\mathrm{Br}(\mathbb{P}^n) = 0$.

Thus, if X is smooth projective and $\mathrm{Br}(X) \neq 0$, then X cannot be stably rational. For conic bundles this obstruction is especially concrete. Let

$$\mathcal{Q} = \{a(t)x^2 + b(t)y^2 - z^2 = 0\} \longrightarrow \mathbb{P}^1$$

be a conic bundle over k. Its generic fiber is a conic over $k(t)$, and its Brauer class is $(a(t), b(t)) \in \mathrm{Br}(k(t))[2]$. The conic bundle has a rational section if and only if this class is zero in $\mathrm{Br}(k(t))$.

The Faddeev exact sequence implies that the kernel of the residue map

$$\mathrm{Br}(k(t))[2] \longrightarrow \bigoplus_{p \in \mathbb{P}^1} k(p)^\times / k(p)^{\times 2}$$

is precisely $\mathrm{Br}(k)[2]$. In particular, if $(a(t), b(t)) = 0$ in $\mathrm{Br}(k(t))$, then all its residues vanish: $\partial^2_p(a(t), b(t)) = 0$ for every $p \in \mathbb{P}^1$. Therefore, to prove that the generic conic has no rational point, it is enough to find a point p such that $\partial^2_p(a(t), b(t)) \neq 0$.

This simple residue calculation is the prototype for many higher-dimensional arguments. In conic and quadric bundle constructions, one often produces a nonzero unramified Brauer class on a special fiber. Since such a class obstructs universal CH_0-triviality, it also obstructs stable rationality. For background on Brauer groups, residues, and their use in rationality problems we refer to [CTS21].

Specialization and the Voisin method. The modern breakthrough is the specialization method. Suppose that we have a flat family $\mathcal{X} \longrightarrow B$ whose very general fiber is smooth. The idea is to specialize the very general fiber to a singular fiber X_0 which is easier to study. One then takes a resolution of singularities $\widetilde{X}_0 \longrightarrow X_0$.

If the resolution morphism is universally CH_0-trivial, and if $\widetilde{X}_0$ is not universally CH_0-trivial, then the very general fiber of the family is not stably rational.

The reason is the following. If the very general fiber were stably rational, then it would admit a decomposition of the diagonal. The specialization theorem says that such a decomposition specializes to the special fiber. If the singularities are mild enough, and the resolution is universally CH_0-trivial, the decomposition lifts to $\widetilde{X}_0$. Therefore $\widetilde{X}_0$ would be universally CH_0-trivial. So, if we can show that $\widetilde{X}_0$ has a nontrivial stable birational invariant, such as a nonzero unramified Brauer class, we get a contradiction.

This is the central strategy introduced by Voisin and then developed by Colliot-Thélène–Pirutka, Totaro, Schreieder, Hassett–Pirutka–Tschinkel, Auel–Böhning–Pirutka, and others.

Voisin: quartic double solids and decomposition of the diagonal. Voisin used the decomposition of the diagonal to prove new stable irrationality results for unirational threefolds [Voi15]. One of the main examples is the quartic double solid. This is a double cover $X \longrightarrow \mathbb{P}^3$ branched along a quartic surface $B \subset \mathbb{P}^3$. That is, X is given by an equation

$$w^2 = f_4(x_0, x_1, x_2, x_3).$$

Quartic double solids are Fano threefolds and are often unirational. Classically, their rationality was subtle: the intermediate Jacobian method does not always apply as directly as for cubic threefolds. Voisin's point was to use the decomposition of the diagonal instead.

She proved that certain very general quartic double solids, including cases with ordinary double points in the branch surface, do not admit a Chow-theoretic decomposition of the diagonal. Consequently, they are not stably rational.

The proof has two complementary aspects. On one hand, the decomposition of the diagonal translates stable rationality into a statement about zero-cycles. On the other hand, for these double solids one can detect the failure of such a decomposition through cohomological invariants, especially the absence of a universal codimension 2 cycle parametrized by the intermediate Jacobian. Thus the obstruction is not merely that the threefold is non-rational; it is strong enough to exclude stable rationality.

Colliot-Thélène–Pirutka: very general quartic threefolds. A decisive application of the specialization method was given by Colliot-Thélène and Pirutka [CTP16]. They proved that a very general quartic threefold $X_4 \subset \mathbb{P}^4$ is not stably rational.

This is striking since Iskovskikh–Manin had already proved that smooth quartic threefolds are not rational, but their method does not address stable rationality. Colliot-Thélène and Pirutka prove something stronger for the very general quartic: it cannot become rational even after multiplying by a projective space.

The structure of the proof is as follows. One degenerates a smooth quartic threefold to a carefully chosen singular quartic X_0. This special fiber is chosen so that it admits a resolution

$$\widetilde{X}_0 \longrightarrow X_0$$

with two key properties:

- the resolution morphism is universally CH_0-trivial;
- the smooth variety $\widetilde{X}_0$ has a nontrivial unramified Brauer group.

The second property implies that $\widetilde{X}_0$ is not universally CH_0-trivial. Hence $\widetilde{X}_0$ has no decomposition of the diagonal. By specialization, the very general smooth quartic threefold also has no decomposition of the diagonal. Therefore it is not stably rational.

So the proof does not try to understand the birational maps of a general quartic threefold directly. Instead, it degenerates the quartic to a special fiber where a Brauer class can be computed explicitly.

Kollár and Totaro: hypersurfaces in many dimensions. Kollár's paper [Kol95] is an important precursor to the modern theory. He proved non-rationality results for very general hypersurfaces by degenerating them to varieties in positive characteristic. The central idea is that in characteristic p, differential forms can behave in ways that produce birational obstructions. In particular, one can construct special degenerations carrying differential forms that would be incompatible with ruledness or rationality.

Kollár's result is about non-rationality, not stable irrationality. Totaro later adapted and strengthened this circle of ideas using the decomposition of the diagonal [Tot16]. He proved that, for $n \geq 3$ and

$$d \geq 2 \left\lceil \frac{n+2}{3} \right\rceil,$$

a very general complex hypersurface $X_d \subset \mathbb{P}^{n+1}$ of degree d is not stably rational.

This result includes the theorem of Colliot-Thélène–Pirutka on quartic threefolds and goes further. For example, it proves that very general quartic fourfolds are not stably rational. This was especially notable since, before these methods, even the rationality of very general quartic fourfolds was not known.

At a conceptual level, Totaro's proof combines three ingredients:

- degeneration of hypersurfaces to special varieties in characteristic p;
- differential forms on the special fiber, following Kollár's philosophy;
- the decomposition of the diagonal, which upgrades the obstruction from non-rationality to non-stable-rationality.

The point is that stable rationality would force strong restrictions on differential forms and on CH_0-groups after specialization. Totaro constructs degenerations where these restrictions are violated.

Schreieder: hypersurfaces of small slope. Schreieder pushed the method much further [Sch19]. He proved that, over an uncountable field of characteristic different from two, a very general hypersurface of dimension $N > 2$ and degree

$$d \geq \log_2(N) + 2$$

is not stably rational.

This is much stronger asymptotically than Totaro's bound. Indeed, Totaro's bound is linear in the dimension, while Schreieder's bound is logarithmic.

The basic strategy is again degeneration, but with more elaborate special fibers. Schreieder constructs singular hypersurfaces with controlled resolutions and nontrivial stable birational invariants. These invariants can be detected through unramified cohomology and the failure of the decomposition of the diagonal.

The important conceptual point is that the obstruction is not tied only to high degree. Even hypersurfaces whose degree is very small compared with their dimension can fail to be stably rational. This gives evidence for the expectation that stable rationality is rare in high-dimensional families.

Quadric surface bundles and unramified Brauer classes. Another major source of examples comes from quadric surface bundles. These are morphisms $X \longrightarrow S$ whose general fiber is a smooth quadric surface. The most important case for the recent theory is when $S = \mathbb{P}^2$ and X is a smooth projective fourfold.

Hassett, Pirutka and Tschinkel studied the stable rationality of quadric surface bundles over rational surfaces [HPT18]. They exhibited families of smooth projective complex fourfolds containing both rational and non-stably-rational fibers.

The obstruction again comes from the Brauer group. A quadric surface over a field has two rulings after passing to an algebraic closure. The Galois action on these rulings gives a natural discriminant double cover. From this data one can often construct a nontrivial Brauer class on the total space, or on a degeneration of the total space.

The proof follows the now-standard specialization pattern:

- degenerate the quadric surface bundle to a singular model;
- resolve the singularities in a universally CH_0-trivial way;
- compute a nonzero unramified Brauer class on the resolution;
- conclude that the very general member is not stably rational.

Auel, Böhning and Pirutka developed this viewpoint further [ABP18]. They proved, among other things, that a very general hypersurface of bidegree $(2,3)$ in $\mathbb{P}^2 \times \mathbb{P}^3$ is not stably rational. This variety has two natural interpretations: projection to $\mathbb{P}^2$ makes it a cubic surface bundle, while projection to $\mathbb{P}^3$ makes it a conic bundle. The same variety can therefore be studied from two different fibration structures.

Their proof again uses the degeneration method for CH_0-groups, together with explicit Brauer computations. The philosophy is that the degeneration reveals an unramified Brauer class which is invisible on a very general smooth fiber but obstructs stable rationality after specialization.

Nicaise–Ottem: tropical degenerations and motivic obstructions. Nicaise and Ottem introduced a different but related approach using tropical degenerations and motivic obstructions [NO22]. Their work builds on the motivic specialization method of Nicaise and Shinder [NS19].

The idea is to study a degeneration of a variety by looking at the combinatorics of its special fiber. Tropical geometry provides a way to organize this combinatorial information. Instead of producing only one special fiber with one explicit Brauer class, the method studies degenerations whose components and intersections are controlled by a tropical complex.

The motivic obstruction lives in the Grothendieck ring of varieties, modulo the class of the affine line. Stable rational varieties have a very restricted class in this ring. Therefore, if the motivic specialization of a degeneration violates this restriction, the very general fiber cannot be stably rational.

Using this method, Nicaise and Ottem prove new stable irrationality results for complete intersections and hypersurfaces. In particular, they show that very general quartic fivefolds are stably irrational, and also that very general complete intersections of a quadric and a cubic in $\mathbb{P}^6$ are stably irrational [NO22].

The advantage of this method is that it can handle degenerations that are too complicated for a direct Brauer group computation. The geometry is encoded tropically, and the stable birational obstruction is extracted motivically.

The modern picture can be summarized as follows.

- Classical methods such as the intermediate Jacobian and birational rigidity prove non-rationality, but usually not non-stable-rationality.
- Stable irrationality is detected by stable birational invariants.
- The decomposition of the diagonal translates stable rationality into a strong condition on CH_0-groups.
- Nontrivial unramified Brauer classes and higher unramified cohomology obstruct this condition.
- Degeneration methods allow one to prove stable irrationality for very general members of a family by studying one carefully chosen singular special fiber.
- Tropical and motivic methods extend this strategy to more complicated degenerations.

Thus, after Voisin's work, the problem of stable rationality became much more accessible.

3. Rationality and unirationality: topology, geometry, and arithmetic

In this section we pause for a broader discussion of the notions of rationality and unirationality, with special emphasis on curves. The purpose is not to develop the full birational theory of algebraic varieties, but rather to explain why curves provide the cleanest testing ground for the general philosophy.

The guiding principle is simple: rational parametrizations are algebraic shadows of simple geometry.

For curves over the complex numbers, this simplicity is measured by topology: a rational curve is a sphere. From the differential-geometric point of view, the sphere has positive curvature. From the arithmetic point of view, rational curves usually have many rational points, provided they have at least one. As soon as the genus becomes positive, all three pictures change.

3.0. **The topological viewpoint over $\mathbb{C}$.** Let C be a smooth projective complex curve. Then C is a compact Riemann surface, hence a compact oriented topological surface. Such a surface is classified by its genus g, the number of handles:

$$g = 0 \text{ is a sphere}, \qquad g = 1 \text{ is a torus}, \qquad g \geq 2 \text{ has at least two handles.}$$

For smooth projective curves, birational geometry and topology are very closely related. Indeed, a birational map between smooth projective curves is automatically regular and is an isomorphism outside finitely many points; by properness and smoothness, it extends to an isomorphism. Thus, for curves, birational equivalence is essentially the same as isomorphism of compact Riemann surfaces [Har77, Chapter I].

If C is rational, then it is birational to $\mathbb{P}^1$. Since smooth projective curves do not have indeterminacy in codimension one, this means that C is isomorphic to $\mathbb{P}^1$. But $\mathbb{P}^1(\mathbb{C})$ is the Riemann sphere, which has genus 0.

Conversely, every compact Riemann surface of genus 0 is isomorphic to the Riemann sphere. Algebraically, this says that a smooth projective complex curve of genus 0 is isomorphic to $\mathbb{P}^1$. Then

$$C \text{ is rational} \iff g(C) = 0.$$

The same conclusion can be seen directly from maps. Suppose that there is a non constant morphism $f\colon \mathbb{P}^1 \to C$. By the Riemann–Hurwitz formula,

$$2g(\mathbb{P}^1) - 2 = \deg(f)\big(2g(C) - 2\big) + R,$$

where $R \geq 0$ is the total ramification. Since $g(\mathbb{P}^1) = 0$, this becomes $-2 = \deg(f)\big(2g(C) - 2\big) + R$.

If $g(C) = 1$, the right-hand side is nonnegative. If $g(C) \geq 2$, it is even positive. Both cases are impossible. Therefore a curve dominated by $\mathbb{P}^1$ must have genus 0.

For smooth projective complex curves, the following are equivalent:

$$C \text{ rational} \iff C \text{ stably rational} \iff C \text{ unirational} \iff g(C) = 0.$$

Remark 3.1. Stable rationality adds nothing for complex curves. Indeed, if $C \times \mathbb{P}^m$ were rational, then all birational invariants of $C \times \mathbb{P}^m$ would have to agree with those of projective space. But the space of holomorphic one-forms has dimension $h^0(C, \Omega^1_C) = g(C)$, and this invariant survives after taking the product with projective space. Thus stable rationality forces $g(C) = 0$, hence rationality.

3.1. **The differential-geometric viewpoint.** The topological classification of curves has a differential-geometric refinement. By the uniformization theorem, every compact Riemann surface admits a constant-curvature metric of one of three types:

Genus	Model geometry	Curvature
0	sphere	$K > 0$
1	Euclidean plane modulo a lattice	$K = 0$
≥ 2	hyperbolic plane modulo a group	$K < 0$.

Thus a rational complex curve is not only algebraically simple: it is also geometrically the most positively curved case. A genus one curve is flat, and a curve of genus at least two is hyperbolic.

This trichotomy matches the Gauss–Bonnet formula. If C is a compact Riemann surface of genus g, then

$$\int_C K\,dA = 2\pi\chi(C) = 2\pi(2-2g).$$

Therefore:

$$g=0 \Rightarrow \int_C K\,dA > 0, \qquad g=1 \Rightarrow \int_C K\,dA = 0, \qquad g\geq 2 \Rightarrow \int_C K\,dA < 0.$$

Remark 3.2. A rational parametrization of a curve is a nonconstant map from $\mathbb{P}^1$, hence from the sphere. But a curve of genus 1 or at least 2 has too much intrinsic rigidity to be covered by the sphere. Algebraically this is Lüroth's theorem or Riemann–Hurwitz; geometrically it reflects the fact that flat or negatively curved surfaces cannot be parametrized by the positively curved sphere in a finite algebraic way.

3.2. **The arithmetic viewpoint.** Over a non-algebraically closed field, for instance over $\mathbb{Q}$, there is an additional arithmetic issue: a genus zero curve need not have a rational point.

Let C be a smooth projective curve of genus 0 over a field k. Then

$$C \cong \mathbb{P}^1_k \text{ if and only if } C(k) \neq \varnothing.$$

Indeed, if $C \cong \mathbb{P}^1_k$, then C certainly has k-rational points. Conversely, suppose that $P \in C(k)$. The complete linear system associated with P gives a degree one parameter on C. Concretely, in the case of a plane conic, one draws all lines through the rational point P. A general such line meets the conic in P and in one further point. This gives a rational parametrization of the conic by the pencil of lines through P, which is a copy of $\mathbb{P}^1_k$.

Remark 3.3. The equation

$$X^2 + Y^2 = Z^2$$

defines a smooth conic in $\mathbb{P}^2$. It has the rational point $[1:0:1]$. Therefore it is rational over $\mathbb{Q}$. Drawing lines of slope t through $[1:0:1]$ gives the classical parametrization of Pythagorean triples:

$$X = 1-t^2, \qquad Y = 2t, \qquad Z = 1+t^2.$$

Thus the abundance of integer solutions for $n=2$ is the arithmetic manifestation of the fact that the Fermat conic is a rational curve.

For curves of positive genus, rational points behave very differently.

- If $g=0$, then a rational point gives a rational parametrization. Rational points, when they exist, usually come in infinite families.
- If $g=1$, then a rational point turns the curve into an elliptic curve. The set of rational points is no longer parametrized by one rational parameter; instead, it forms a finitely generated abelian group by the Mordell–Weil theorem [Mor22].
- If $g\geq 2$, then rational points are expected to be sparse. This is made precise by Faltings' theorem: a smooth projective curve of genus at least 2 over a number field has only finitely many rational points [Fal83].

Remark 3.4. This explains why the Fermat curves change nature as n grows. The curve

$$F_n: \quad X^n + Y^n = Z^n$$

has genus

$$g(F_n) = \frac{(n-1)(n-2)}{2}.$$

For $n=2$, this is genus 0, and the curve is rational. For $n=3$, this is genus 1, so the curve is an elliptic curve. For $n\geq 4$, the genus is at least 3, so Faltings' theorem implies that $F_n(\mathbb{Q})$ is finite. Fermat's Last Theorem is much stronger in this special family: it says that all rational points with $XYZ \neq 0$ are absent.

3.4. **A dictionary for curves.** For smooth projective curves over $\mathbb{C}$, the algebraic, topological, differential-geometric, and arithmetic pictures fit together as follows:

g	algebraic type	topology	geometry	arithmetic over number fields
0	rational if a point exists	sphere	$K>0$	many points if one exists
1	elliptic if a point exists	torus	$K=0$	finitely generated group
≥ 2	general type	many handles	$K<0$	finitely many points.

The first row is the rational world. The second row is the elliptic world, where points are not parametrized but organized by a group law. The third row is the hyperbolic world, where curves are rigid and rational points are finite over number fields.

3.4. **Why higher dimensions are harder.** The clean equivalence rational $\iff$ unirational $\iff$ $g = 0$ is special to curves. In higher dimensions there is no single integer like the genus which controls all birational phenomena. A variety may be dominated by projective space and still fail to be rational. This is why the higher-dimensional Lüroth problem is much subtler.

The curve case remains the guiding model. It teaches us to look for invariants which cannot exist on projective space but do exist on the variety under study. For curves, the invariant is the genus, or equivalently holomorphic differentials. In dimension three and higher, the corresponding obstructions become more sophisticated: intermediate Jacobians, torsion in cohomology, Brauer groups, and decompositions of the diagonal.

Nevertheless, there are some useful guiding principles. The first one is that rational and unirational varieties should be viewed as varieties with many rational curves. Indeed, if X is unirational, then there exists a dominant rational map $\mathbb{P}^N \dashrightarrow X$. Since projective space is covered by lines, the variety X is covered by images of rational curves. Thus unirational varieties are, in particular, uniruled. Over a field of characteristic zero, smooth projective uniruled varieties have negative Kodaira dimension: $\kappa(X) = -\infty$. So a variety of general type, for which $\kappa(X) = \dim X$, cannot be unirational. This gives the first broad dichotomy: general type varieties are far from rational, whereas Fano and rationally connected varieties are the natural candidates for rationality problems.

For instance, projective space is Fano, smooth cubic threefolds and quartic threefolds are Fano, and many conic bundles and del Pezzo fibrations are rationally connected. These are precisely the kinds of varieties for which rationality, stable rationality and unirationality become subtle questions.

There is also an arithmetic version of the same philosophy. Suppose that X is a variety over a number field k. If X is rational over k, and has a k-rational point in the open set where the parametrization is defined, then its rational points are expected to be abundant. More generally, if X is unirational over k, then the image of $\mathbb{P}^N(k)$

is often a large set of rational points on X. Thus unirationality tends to force arithmetic largeness.

The opposite philosophy is expressed by the Bombieri–Lang conjecture. In one of its standard forms, it predicts that if X is a variety of general type over a number field k, then the set of rational points $X(k)$ is not Zariski dense in X [Lan97, BG06]. In other words, varieties of general type should have few rational points. This conjecture generalizes the behavior of curves. If C is a smooth projective curve, then

$$C \text{ is of general type} \quad \iff \quad g(C) \geq 2.$$

In this case, Faltings' theorem says that $C(K)$ is finite for every number field k [Fal83]. Thus, for curves, the Bombieri–Lang philosophy is a direct generalization of Mordell's conjecture.

Let us briefly recall the meaning of the main geometric classes appearing here. A smooth projective variety X is called *Fano* if its anticanonical divisor $-K_X$ is ample. This means, roughly speaking, that X has many rational curves and should be thought of as positively curved from the algebro-geometric point of view. Projective space is the basic example.

A variety X is called *rationally connected* if two general points of X can be joined by a rational curve. Every smooth Fano variety over a field of characteristic zero is rationally connected, but the converse is not true in general. Rational connectedness is a weaker and more flexible notion than being Fano.

At the opposite end, a smooth projective variety is said to be of *general type* if its canonical divisor K_X is big. This means that X has many pluricanonical forms and behaves, in a broad sense, like a negatively curved space. For curves, this condition is exactly the same as having genus at least 2.

Between these two extremes lie *Calabi–Yau type* varieties. In this informal table, this means varieties whose canonical divisor is numerically trivial, or at least very close to being trivial: $K_X \equiv 0$. They are neither positively curved like Fano varieties nor negatively curved like varieties of general type.

The differential-geometric expression *Ricci-flat* refers to a metric whose Ricci curvature is zero.

The best way to understand Ricci curvature is to compare it with the more familiar Gaussian curvature of a real surface.

Let S be a smooth real surface with a Riemannian metric. At a point $p \in S$, the Gaussian curvature $K(p)$ measures how the surface bends near p. If S is embedded in $\mathbb{R}^3$, and if $k_1(p)$ and $k_2(p)$ are the two principal curvatures, then $K(p) = k_1(p)k_2(p)$.

Thus positive Gaussian curvature corresponds to sphere-like behavior, zero Gaussian curvature to flat behavior, and negative Gaussian curvature to saddle-like behavior.

Ricci curvature is a higher-dimensional analogue of this idea. In real dimension greater than 2, there is no single Gaussian curvature at a point. Instead, every two-dimensional plane $\Pi \subset T_pX$ has its own sectional curvature. Given a tangent direction $v \in T_pX$, the Ricci curvature $\mathrm{Ric}(v, v)$

is obtained by averaging, or more precisely summing, the sectional curvatures of the two-dimensional planes containing v. In this sense, Ricci curvature measures the average curvature seen in the directions transverse to v.

That is, Ricci curvature measures how volumes of small geodesic balls differ from the Euclidean case. Positive Ricci curvature means that small balls tend to have smaller volume than Euclidean balls, while negative Ricci curvature means that they tend to have larger volume. A Ricci-flat metric is a metric with $\mathrm{Ric} = 0$, so it has no average volume distortion, even though the full curvature tensor may still be nonzero.

In real dimension 2, Ricci curvature contains exactly the same information as Gaussian curvature. More precisely, if g is the Riemannian metric on a surface, then $\mathrm{Ric} = Kg$.

Thus, for real surfaces, Ricci-flat is the same as flat, that is, the same as having $K = 0$. In higher dimension this is no longer true: Ricci curvature is only an averaged part of the full curvature tensor, and a Ricci-flat manifold need not be flat.

Over the complex numbers, Yau's theorem says that a compact Kähler manifold with trivial canonical class admits such a metric. Thus Calabi–Yau varieties are the algebro-geometric source of Ricci-flat geometry.

So the higher-dimensional picture can be summarized as follows:

Geometric type	**Curvature/Topology**	**Birational expectation**	**Arithmetic expectation**
Fano/rationally connected	positive curvature	possible rationality or unirationality	many rational points
Calabi–Yau type	Ricci-flat geometry	not unirational	subtle arithmetic
General type	negative curvature	not unirational	rational points not Zariski dense.

This table should not be interpreted as a theorem in every entry. Rather, it is a map of the landscape. Rationality problems usually live on the left side of the table, among Fano and rationally connected varieties. Varieties of general type, by contrast, are expected to be both geometrically rigid and arithmetically sparse.

4. Quadrics

Let $Q \subset \mathbb{P}^{n+1}_k$ be an irreducible quadric hypersurface defined over a field k. If Q contains a k-rational point, then Q is rational. In fact, the rationality of Q is realized by projecting from that rational point, which yields a birational map between Q and $\mathbb{P}^n_k$.

Consider the quadric

$$Q = \{x_0x_1 - x_2x_3 = 0\} \subset \mathbb{P}^3$$

and the point $p = [1 : 0 : 0 : 0] \in Q$. The projection from p is given by the rational map

$$\pi_p : Q \dashrightarrow \mathbb{P}^2, \quad [x_0 : x_1 : x_2 : x_3] \mapsto [x_1 : x_2 : x_3].$$

This map is well-defined on the open subset of Q where $(x_1, x_2, x_3) \neq (0, 0, 0)$. The inverse of π_p is the rational map

$$\pi_p^{-1} : \mathbb{P}^2 \dashrightarrow Q, \quad [y_0 : y_1 : y_2] \mapsto [y_1y_2 : y_0^2 : y_0y_1 : y_0y_2]$$

so π_p^{-1} is defined by the conics through $[0 : 1 : 0]$ and $[0 : 0 : 1]$.

5. Cubics

The unirationality problem for cubic hypersurfaces is completely settled: a smooth cubic hypersurface of dimension at least two is unirational if and only if it has a rational point [Kol02]. On the other hand, the rationality problem for cubics is much more subtle and is widely open in many cases.

In even dimension there are many smooth rational cubic hypersurfaces. The simplest example is given by a cubic $X^{2n} \subset \mathbb{P}^{2n+1}$ containing two skew n-planes P_1, P_2 defined over the base field. Given two general points $p_1 \in P_1$, $p_2 \in P_2$, the line $\overline{p_1p_2}$ meets X^{2n} in three points. Since two of them already lie on $P_1 \cup P_2 \subset X^{2n}$, the third point defines a rational map

$$P_1 \times P_2 \dashrightarrow X^{2n}.$$

This map is birational. Thus the presence of two skew middle-dimensional linear spaces gives a very concrete rationality construction.

A particularly interesting case, also from the point of view of number theory, is that of Fermat cubics. Even-dimensional Fermat cubics contain many n-planes, but any two of those defined over the base field intersect. However, they contain several pairs of skew n-planes defined over a quadratic extension of the base field and conjugate under the Galois action. This observation is the key point in the following result.

Theorem 5.1. [Mas26, Theorem 1.1] *Let k be a field with $\mathrm{char}(k) \neq 3$. For every $n \geq 1$, the Fermat cubic hypersurface*

$$X^{2n} = \{x_0^3 + \cdots + x_{2n+1}^3 = 0\} \subset \mathbb{P}^{2n+1}$$

is rational over k. More precisely, for $n \geq 2$ there exists a birational parametrization $\mathbb{P}^{2n} \dashrightarrow X^{2n}$ given by homogeneous polynomials of degree four. For $n = 1$, there is a birational parametrization $\mathbb{P}^2 \dashrightarrow X^2$ given by homogeneous polynomials of degree three.

When $\mathrm{char}(k) = 3$, the Fermat equation becomes $x_0^3 + \cdots + x_{2n+1}^3 = (x_0 + \cdots + x_{2n+1})^3$, so the hypersurface is non-reduced. This explains why the characteristic 3 case has to be excluded.

Example 5.2. The theorem gives, in particular, an explicit birational parametrization of the Fermat cubic fourfold

$$X^4 = \{x_0^3 + \cdots + x_5^3 = 0\} \subset \mathbb{P}^5.$$

One obtains a map $\mathbb{P}^4 \dashrightarrow X^4$ defined by homogeneous quartic polynomials. Geometrically, this map is constructed from a pair of conjugate skew planes contained in X^4 over a quadratic extension of the base field. Although the two planes are not individually defined over k, the construction is Galois invariant, and hence descends to a birational parametrization over k. The following is a birational parametrization of the Fermat cubic 4-fold:

$$f : \mathbb{P}^4 \dashrightarrow X^4 \subset \mathbb{P}^5, \quad u = [u_0 : \cdots : u_4] \to [f_0(u) : \cdots : f_5(u)]$$

where

$$\begin{aligned}
f_0 &= -u_0^3u_1 - u_1^4 - 3u_0^3u_2 - 6u_1^2u_2^2 - 9u_2^4 - u_1u_3^3 - 3u_2u_3^3 + 3u_1u_3^2u_4 - 3u_2u_3^2u_4 - 3u_1u_3u_4^2 - 9u_2u_3u_4^2 + 9u_1u_4^3 - 9u_2u_4^3;\\
f_1 &= u_0^3u_1 + u_1^4 - 3u_0^3u_2 + 6u_1^2u_2^2 + 9u_2^4 + u_1u_3^3 - 3u_2u_3^3 + 3u_1u_3^2u_4 + 3u_2u_3^2u_4 + 3u_1u_3u_4^2 - 9u_2u_3u_4^2 + 9u_1u_4^3 + 9u_2u_4^3;\\
f_2 &= -u_0^3u_3 - u_1^3u_3 + 3u_1^2u_2u_3 - 3u_1u_2^2u_3 + 9u_2^3u_3 - u_3^4 - 3u_0^3u_4 - 3u_1^3u_4 - 3u_1^2u_2u_4 - 9u_1u_2^2u_4 - 9u_2^3u_4 - 6u_3^2u_4^2 - 9u_4^4;\\
f_3 &= u_0^3u_3 + u_1^3u_3 + 3u_1^2u_2u_3 + 3u_1u_2^2u_3 + 9u_2^3u_3 + u_3^4 - 3u_0^3u_4 - 3u_1^3u_4 + 3u_1^2u_2u_4 - 9u_1u_2^2u_4 + 9u_2^3u_4 + 6u_3^2u_4^2 + 9u_4^4;\\
f_4 &= -u_0^4 - u_0u_1^3 + 3u_0u_1^2u_2 - 3u_0u_1u_2^2 + 9u_0u_2^3 - u_0u_3^3 + 3u_0u_3^2u_4 - 3u_0u_3u_4^2 + 9u_0u_4^3;\\
f_5 &= u_0^4 + u_0u_1^3 + 3u_0u_1^2u_2 + 3u_0u_1u_2^2 + 9u_0u_2^3 + u_0u_3^3 + 3u_0u_3^2u_4 + 3u_0u_3u_4^2 + 9u_0u_4^3.
\end{aligned}$$

5.2. **Cubic fourfolds.** Let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold over $\mathbb{C}$. The rationality problem for cubic fourfolds is one of the central open problems in higher-dimensional birational geometry.

Let $h \in H^2(X, \mathbb{Z})$ be the hyperplane class. Voisin proved that if X is very general, then $H^{2,2}(X, \mathbb{Z}) = \mathbb{Z}h^2$ [Voi86]. Thus a very general cubic fourfold has no algebraic surface classes other than the class forced by the hyperplane section.

A cubic fourfold is called *special* if it contains an algebraic surface whose class is not proportional to h^2. More precisely, assume that X contains a two-dimensional integral effective cycle S. Consider the rank two lattice

$$K = \langle h^2, S \rangle \subset H^{2,2}(X, \mathbb{Z}).$$

The intersection form on K is

$$\begin{pmatrix} h^4 & h^2 \cdot S \\ h^2 \cdot S & S^2 \end{pmatrix} = \begin{pmatrix} 3 & \deg(S) \\ \deg(S) & S^2 \end{pmatrix}.$$

The discriminant of K is $d(K) = 3S^2 - \deg(S)^2$. Hassett defined the Noether–Lefschetz divisors

$$\mathcal{C}_d = \{[X] \in \mathcal{C} \mid \text{there exists a saturated rank two lattice } K \subset H^{2,2}(X, \mathbb{Z}), h^2 \in K,\ d(K) = d\}.$$

The loci $\mathcal{C}_d$ are either empty or divisors in the moduli space $\mathcal{C}$ of smooth cubic fourfolds. Moreover by [Has99, Has00],

$$\mathcal{C}_d \neq \emptyset \quad \iff \quad d > 6 \quad \text{and} \quad d \equiv 0, 2 \pmod 6.$$

If $X \in \mathcal{C}_d$ is very general, then $H^{2,2}(X, \mathbb{Z}) = K$. When $S \subset X$ is smooth, its self-intersection in X can be computed from the double point formula:

$$S^2 = 6h^2 \cdot S + 3h \cdot K_S + K_S^2 - \chi(S).$$

For example, if $S \subset X$ is a plane, then $S^2 = 3$, and therefore $d(K) = 3 \cdot 3 - 2^2 = 8$. Thus $\mathcal{C}_8$ is the divisor parametrizing cubic fourfolds containing a plane.

If $S \subset \mathbb{P}^5$ is a quintic del Pezzo surface, then $S^2 = 13$, and hence $d(K) = 3 \cdot 13 - 5^2 = 14$. Thus $\mathcal{C}_{14}$ is the divisor parametrizing cubic fourfolds containing a quintic del Pezzo surface.

An even integer $d > 6$ is called *admissible* if it is not divisible by 4, by 9, or by any odd prime number congruent to 2 modulo 3. Hassett proved that d is admissible if and only if a very general cubic fourfold in $\mathcal{C}_d$ has an associated polarized $K3$ surface in the Hodge-theoretic sense [Has99]. Kuznetsov's conjecture predicts that a smooth cubic fourfold X is rational if and only if its Kuznetsov component is equivalent to the derived category of a $K3$ surface [Kuz10]. For a very general cubic in $\mathcal{C}_d$, this condition is expected precisely when d is admissible [AT14]. Several Hassett divisors have concrete geometric descriptions:

d	Geometric description
8	Cubics containing a plane
12	Cubics containing a cubic scroll
14	Cubics containing a quintic del Pezzo surface or a quartic scroll
20	Cubics containing a Veronese surface.

The first admissible values are

$$14,\ 26,\ 38,\ 42,\ 62.$$

The rationality of a general cubic fourfold in $\mathcal{C}_{14}$ was proved classically by Morin and Fano [Mor40, Fan43]. It was later extended to every smooth cubic fourfold in $\mathcal{C}_{14}$ by Bolognesi, Russo and Staglianò [BRS19], and also follows from the specialization theorem of Kontsevich and Tschinkel [KT19]. Russo and Staglianò then proved rationality for cubic fourfolds in $\mathcal{C}_{26}$, $\mathcal{C}_{38}$, and $\mathcal{C}_{42}$ [RS19, RS23].

5.2. **Congruences of secant curves.** The constructions for $\mathcal{C}_{14}$, $\mathcal{C}_{26}$, $\mathcal{C}_{38}$, and $\mathcal{C}_{42}$ follow a common geometric principle. Let $S \subset \mathbb{P}^5$ be a surface, and suppose that there is a four-dimensional rational family $\mathcal{H}$ of rational curves of degree e such that a general curve $C \in \mathcal{H}$ is $(3e-1)$-secant to S. This means that $\operatorname{length}(C \cap S) = 3e-1$.

Assume moreover that through a general point of $\mathbb{P}^5$ there passes a unique curve of the family $\mathcal{H}$.

Now let $X \subset \mathbb{P}^5$ be a cubic fourfold containing S. Since C has degree e, we have $C \cdot X = 3e$. But $3e-1$ of these intersection points already lie on $S \subset X$. Therefore there is one residual point $q \in C \cap X$. The map $\mathcal{H} \dashrightarrow X$, $C \longmapsto q$, is birational. This gives a rationality construction for X, provided $\mathcal{H}$ is rational.

This explains why the geometry of secant curves to a special surface $S \subset X$ can imply the rationality of the cubic fourfold X.

Cubics in $\mathcal{C}_{14}$. Let $S \subset \mathbb{P}^5$ be a quintic del Pezzo surface, and let $X \subset \mathbb{P}^5$ be a smooth cubic fourfold containing S. The ideal of S is generated by five quadrics. Thus the linear system of quadrics through S defines a rational map $\varphi\colon X \dashrightarrow \mathbb{P}^4$. This map is birational.

The geometric reason is that S is an OADP surface, meaning a surface with one apparent double point: through a general point of $\mathbb{P}^5$ there passes a unique line which is 2-secant to S. The quadrics containing S contract this 2-secant line. Since X is a cubic and contains S, such a line meets X in exactly one residual point outside S. This residual point is what makes the inverse map $\mathbb{P}^4 \dashrightarrow X$ birational.

There is also a second construction of rational cubic fourfolds in $\mathcal{C}_{14}$, closer to the later constructions of Russo and Staglianò. One considers a rational surface $S' \subset \mathbb{P}^5$ obtained by projecting to $\mathbb{P}^5$ the image of $\mathbb{P}^2$ by the linear system of quartic curves passing through eight general points. This surface admits a four-dimensional rational congruence of 5-secant conics. Through a general point of $\mathbb{P}^5$ there passes a unique conic of the congruence.

If $X \subset \mathbb{P}^5$ is a general cubic fourfold containing S', then a conic C of the congruence satisfies $C \cdot X = 6$. Since C is 5-secant to $S' \subset X$, there remains one residual intersection point. Sending C to this residual point gives a birational map $\mathcal{H} \dashrightarrow X$, where $\mathcal{H}$ is the rational fourfold parametrizing the congruence.

Cubics in $\mathcal{C}_{26}$. For $d = 26$, Russo and Staglianò consider a rational surface $S \subset \mathbb{P}^5$ of degree 7 with one node. It is obtained by starting from a smooth del Pezzo surface $S_7 \subset \mathbb{P}^7$ of degree 7, and projecting it from a line which meets the secant variety of S_7 transversely in one point. This projection creates exactly one ordinary double point. The key property of this surface is the existence of a rational four-dimensional congruence of 5-secant conics. Through a general point of $\mathbb{P}^5$ there passes a unique conic of this family. Let $X \subset \mathbb{P}^5$ be a cubic fourfold containing S. For a conic C in the congruence, $C \cdot X = 6$. Since C is 5-secant to S, the sixth point of intersection is residual. The assignment $C \longmapsto$ the residual point of $C \cap X$ defines a birational map from the rational parameter space of the congruence to X. Hence the cubic fourfold X is rational.

Cubics in $\mathcal{C}_{38}$. For $d = 38$, the surface used by Russo and Staglianò is a smooth rational surface $S \subset \mathbb{P}^5$ of degree 10 and sectional genus 6. It is birational to the image of $\mathbb{P}^2$ by the linear system of plane curves of degree 10 with ten general triple base points.

Again, the crucial point is not merely the existence of the surface, but the existence of a special congruence of secant curves. The surface S admits a rational four-dimensional family of 5-secant conics, and through a general point of $\mathbb{P}^5$ there passes exactly one conic of the family.

If $X \subset \mathbb{P}^5$ is a cubic fourfold containing S, then every conic C of the congruence meets X in six points, counted with multiplicity. Five of them are forced to lie on S. The sixth point is residual, and it defines a birational map $\mathcal{H} \dashrightarrow X$. Since $\mathcal{H}$ is rational, this proves the rationality of X.

Cubics in $\mathcal{C}_{42}$. The case $d = 42$ was the first open admissible case after 14, 26, and 38. Russo and Staglianò proved that cubic fourfolds in $\mathcal{C}_{42}$ are rational by developing the method of trisecant flops [RS23].

The relevant surface is a rational surface $S \subset \mathbb{P}^5$ of degree 9 and sectional genus 2, with five ordinary double points. A general cubic fourfold in $\mathcal{C}_{42}$ contains such a surface. The surface S may be viewed as a non-minimal birational model of the $K3$ surface associated with the cubic fourfold.

The key geometric property is the existence of a congruence of 8-secant twisted cubics. More precisely, there is a rational four-dimensional family $\mathcal{H}$ of twisted cubic curves $C \subset \mathbb{P}^5$ such that a general $C \in \mathcal{H}$ meets S in a zero-dimensional scheme of length 8. Moreover, through a general point of $\mathbb{P}^5$ there passes a unique twisted cubic of the congruence.

Now let $X \subset \mathbb{P}^5$ be a cubic fourfold containing S. Since a twisted cubic has degree 3, we have $C \cdot X = 9$. Eight of these intersection points already lie on $S \subset X$. Therefore there is one residual point $q \in C \cap X$. The assignment $C \longmapsto q$ defines a birational map $\mathcal{H} \dashrightarrow X$. Since $\mathcal{H}$ is rational, this proves that X is rational. The term "trisecant flop" refers to the birational operation underlying this construction. Instead of only using explicit parametrizations, Russo and Staglianò interpret the rationality map through Mori theory. One starts from a rational Fano fourfold containing a birational model of the associated $K3$ surface, and performs a flop along curves which are trisecant to this surface. The resulting birational model is the cubic fourfold. In this way, the rationality of the cubic is connected directly with the geometry of its associated $K3$ surface.

Recent developments. The rationality problem for cubic fourfolds remains one of the most active questions in the area. The constructions above give many rational special cubic fourfolds. On the opposite side, one expects a very general cubic fourfold to be irrational, but this is still one of the major open problems.

A recent preprint of Katzarkov, Kontsevich, Pantev and Yu proposes a new obstruction to rationality using Hodge theory and quantum multiplication [KKPY25]. They introduce birational invariants called *Hodge atoms*. These invariants combine classical Hodge theory with rational Gromov–Witten invariants. Roughly speaking, quantum multiplication produces additional structure on cohomology, and the authors extract from this structure pieces which behave additively under blow-ups. Since birational maps between smooth projective varieties can be resolved by sequences of blow-ups and blow-downs, such additive invariants can be used to obstruct rationality.

In particular, they claim that the Hodge atoms of a very general cubic fourfold are incompatible with those of a rational variety. If confirmed, this would prove that a very general cubic fourfold is not rational. This would be a major advance, since the classical Hodge-theoretic obstructions used for cubic threefolds do not directly rule out rationality for cubic fourfolds.

Another recent preprint of Engel, de Gaay Fortman and Schreieder concerns stable rationality and the integral Hodge conjecture for abelian varieties [EdGFS25]. Their result is especially relevant for cubic threefolds. Recall that the non-rationality of a smooth cubic threefold is proved by studying its intermediate Jacobian. If a cubic threefold were stably rational, then its intermediate Jacobian would have to satisfy very strong algebraicity properties; in particular, certain minimal cohomology classes would have to be represented by algebraic curves.

Engel, de Gaay Fortman and Schreieder prove that, on the intermediate Jacobian of a very general cubic threefold, every curve class is an even multiple of the minimal class. Thus the minimal class itself is not algebraic. This gives an obstruction to stable rationality. Their method is quite different from the classical Clemens–Griffiths argument: it uses tropical geometry, multivariable Mumford constructions, monodromy, and the combinatorics of matroids. The role of matroids is to control the possible limiting behavior of algebraic cycles under degeneration.

Thus the recent picture has two complementary directions. On one side, explicit rationality constructions for special cubic fourfolds use special surfaces, congruences of secant curves, and birational transformations such as trisecant flops. On the other side, new obstructions to rationality and stable rationality come from refined Hodge-theoretic, quantum, and tropical methods.

6. Unirationality vs rational connection in higher dimension

The problem of determining whether a variety is rational or unirational is in general very hard and unirationality is very poorly understood. Also due to this difficulty, unirationality has gradually been replaced by the notion of rational connection.

A variety X, over an algebraically closed field of characteristic zero, is rationally connected if two general points of X can be joined by a rational curve. We refer to [Ara05] for a comprehensive survey on the subject. If X is unirational we may choose a dominant rational map $f : \mathbb{P}^n \dashrightarrow X$. If $x_1, x_2 \in X$ are general point and $y_1 \in f^{-1}(x_1), y_2 \in f^{-1}(x_2)$, the image $f(\langle y_1, y_2 \rangle)$ of the line through y_1, y_2 yields a rational curve through x_1, x_2. Hence X is rationally connected. Summing-up:

$$X \text{ rational} \Rightarrow X \text{ stably rational} \Rightarrow X \text{ unirational} \Rightarrow X \text{ rationally connected.}$$

- Smooth Fano varieties are rationally connected [Cam92], [KMM92].
- If $X \dashrightarrow Y$ is a rational fibration, with rational connected general fiber, over a rationally connected variety Y then X is rationally connected [GHS03].

For curves and surfaces the notions of unirationality and rational connection are equivalent. A crucial open problem consists in establishing whether this holds also in higher dimension.

Nowadays, thanks to the minimal model program [Mor88, BCHM10], the birational geometry of uniruled varieties can be organized in terms of Mori fiber spaces.

More precisely, if X is a smooth projective uniruled variety over an algebraically closed field of characteristic zero, then X is birational to a projective variety X' with terminal $\mathbb{Q}$-factorial singularities admitting a contraction $\pi\colon X' \longrightarrow Y$ with relative Picard number one, where $\dim Y < \dim X'$, and such that $-K_{X'}$ is π-ample. Such a morphism is called a Mori fiber space.

When $\dim X = 3$, there are three possibilities:

- Y is a point, and X' is a terminal $\mathbb{Q}$-factorial Fano threefold of Picard rank one;
- Y is a curve, and $\pi\colon X' \to Y$ is a del Pezzo fibration;
- Y is a surface, and $\pi\colon X' \to Y$ is a conic bundle.

Remark 6.1. The output X' of the minimal model program is not necessarily smooth. In the first case, when Y is a point, X' is a Fano threefold of Picard rank one with terminal singularities. Thus the classification of smooth Fano threefolds of Picard rank one does not, by itself, classify all possible outputs of the MMP.

For unirationality this distinction matters. Unirationality is a birational property, so X' is unirational if and only if any smooth projective resolution of X' is unirational. However, a resolution of a terminal Fano threefold is usually no longer Fano. Therefore the smooth classification is not a formal substitute for the singular one.

In these notes we use the smooth classification for a different reason: it gives a clean and classical testing ground for the relation between rational connectedness and unirationality. Smooth Fano threefolds of Picard rank one are rationally connected and completely classified. They therefore provide a natural list of basic examples where one can ask which rationally connected varieties are actually unirational.

Fano 3-folds of Picard rank one. We summarize the state of the art on the rationality and unirationality of the 17 (up to deformation) Fano 3-folds of Picard rank one. In the following table:

- $\mathbb{G}(h,n) \subset \mathbb{P}^N$ is the Grassmannian of h-planes in $\mathbb{P}^n$, in its Plücker embedding in $\mathbb{P}^N$ with $N = \binom{n+1}{h+1} - 1$, and $\mathcal{U}$ is the universal bundle on $\mathbb{G}(h,n)$.
- $\mathbb{S}^{10}$ is the Spinor variety, parametrizing one of the families of 5-dimensional subspaces contained in a 10-dimensional quadric, in its minimal embedding in $\mathbb{P}^{15}$.
- $\mathbb{LG}(2,5)$ is the 6-dimensional Lagrangian Grassmannian, parametrizing 3-planes in a 6-dimensional symplectic vector space, in its minimal embedding in $\mathbb{P}^{13}$.
- $G_2\mathbb{G}(1,6)$ is the unique 5-dimensional homogeneous Fano variety associated to the exceptional Lie group G_2, in its minimal embedding in $\mathbb{P}^{13}$. Let $V \cong \mathbb{C}^7$ be the standard 7-dimensional representation of G_2, which carries a nondegenerate, G_2-invariant 3-form $\varphi \in \bigwedge^3 V^*$. Then $G_2\mathbb{G}(1,6)$ can be described as the subvariety of the usual Grassmannian $\mathbb{G}(1,6)$ defined as the locus of 2-planes $W \subset V$ on which the 3-form φ vanishes.

  That is, $G_2\mathbb{G}(1,6)$ is isomorphic to the *adjoint variety* of G_2, which is the unique closed orbit in the projectivization $\mathbb{P}(\mathfrak{g}_2)$ of the 14-dimensional Lie algebra $\mathfrak{g}_2$ of G_2.

X	Rational	Unirational
Degree 6 hypersurface in $\mathbb{P}(1,1,1,1,3)$, alternatively double cover of $\mathbb{P}^3$ branched along a divisor of degree 6	no	unknown
(a) Quartic $3-$fold in $\mathbb{P}^4$ (b) Double cover of a quadric $3-$fold branched along a divior of degree 8	no	some are
Complete intersection of quadric and cubic in $\mathbb{P}^5$	no	yes
Complete intersection of three quadrics in $\mathbb{P}^6$	no	yes
Gushel–Mukai 3-folds (a) Section of Plücker embedding of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ with two hyperplanes and a quadric (b) Double cover of section of Plücker embedding of $\mathbb{G}(1,4) \subset \mathbb{P}^9$ with three hyperplanes branched along an anticanonical divisor	general one is not	yes
Codimension 7 linear section of the Spinor variety $\mathbb{S}^{10} \subset \mathbb{P}^{15}$	yes	yes
Section of Plücker embedding of $\mathbb{G}(1,5)$ by a codimension 5 subspace	no	yes
Section of Plücker embedding of $\mathbb{LG}(2,5)$ by a codimension 3 subspace	yes	yes
Section of Plücker embedding of $G_2\mathbb{G}(1,6)$ by a codimension 2 subspace	yes	yes
Zero locus of a section of $(\bigwedge^2 \mathcal{U}^\vee)^{\oplus 3}$ on $\mathbb{G}(2,6)$	yes	yes
Hypersurface of degree 6 in $\mathbb{P}(1,1,1,2,3)$	no	unknown
Double cover of $\mathbb{P}^3$ branched along a smooth quartic surface	no	yes
Cubic 3-fold in $\mathbb{P}^4$	no	yes
Complete intersection of three quadrics in $\mathbb{P}^5$	yes	yes
Section of Plücker embedding of $\mathbb{G}(1,4)$ by a codimension 3 subspace	yes	yes
Quadric 3-fold in $\mathbb{P}^4$	yes	yes
Projective space $\mathbb{P}^3$	yes	yes

As stated in the table, there are smooth unirational quartic threefolds $X \subset \mathbb{P}^4$, although it is still unknown whether the general smooth quartic threefold is unirational. The first example is due to B. Segre:

$$x_0^4 + x_0x_4^3 + x_1^4 - 6x_1^2x_2^2 + x_2^4 + x_3^4 + x_3^3x_4 = 0 \subset \mathbb{P}^4.$$

This example is unirational but not rational, by the theorem of Iskovskikh–Manin on smooth quartic threefolds.

The geometric reason for the unirationality is not merely the presence of a monoidal surface, but the presence of a rational surface with separable asymptotics. Let $X_4 \subset \mathbb{P}^{m+1}$, $m \geq 3$, be a quartic hypersurface and let $S_0 \subset X_4$ be a rational surface. For a general point $R \in S_0$, consider the tangent cone $C_R(X_4)$ and intersect it with a fixed hyperplane H_0. This gives a quadric

$$Q_R = C_R(X_4) \cap H_0.$$

Varying R, one obtains a quadric bundle

$$p\colon X^1 \longrightarrow S_0, \qquad X^1 = \{(R, P') \mid R \in S_0,\ P' \in Q_R\}.$$

If $m = 3$, namely for quartic threefolds $X_4 \subset \mathbb{P}^4$, this is a conic bundle over S_0.

The existence of a rational surface $S_0 \subset X_4$ is not by itself enough to prove unirationality in dimensions 3 and 4, because one cannot in general apply Segre's theorem to obtain a rational section of the conic or quadric bundle. The crucial additional condition is that S_0 has separable asymptotics.

Recall the definition. Let $F_n \subset \mathbb{P}^3$ be an irreducible surface of degree $n \geq 3$, not a developable ruled surface, and let x be a general point of F_n. If P_x is the tangent plane and Q_x is the polar quadric of F_n at x, then

$$C_x := Q_x \cap P_x$$

is a conic with a double point at x. Over $K(x)$, or after a quadratic extension, this conic splits as

$$C_x = \ell'_x + \ell''_x,$$

where ℓ'_x, ℓ''_x are the two asymptotic tangent lines to F_n at x. Let $\Sigma \subset G(1,3)$ be the congruence swept out by these asymptotic lines. One says that F_n has separable asymptotics if

$$\Sigma = \Sigma' \cup \Sigma''$$

is reducible, over an algebraic extension of the ground field. Equivalently, the two families of asymptotic directions are algebraically separated. In that case the corresponding asymptotic lines have contact at least three with the surface:

$$\operatorname{mult}_x(\ell'_x \cap F_n) \geq 3, \qquad \operatorname{mult}_x(\ell''_x \cap F_n) \geq 3.$$

If $S_0 \subset X_4$ has separable asymptotics, then the two families of asymptotic directions give two rational sections of the bundle

$$p\colon X^1 \to S_0$$

after a possible algebraic extension of the ground field. Since S_0 is rational, this implies that X^1 is rational. The fourth-point construction then gives a dominant rational map

$$X^1 \dashrightarrow X_4.$$

Indeed, a point of X^1 corresponds to a line through $R \in S_0$ having intersection multiplicity at least 3 with X_4 at R; the residual fourth point of intersection with X_4 defines the rational map. Hence X_4 is unirational. Thus the general mechanism is:

rational surface with separable asymptotics inside $X_4 \Rightarrow$ rational section of $X^1 \to S_0 \Rightarrow X_4$ unirational.

Several sources of such surfaces are studied in [Mar00]. Ruled non-developable surfaces have separable asymptotics; in degree 4 they contain a triple line, and they give smooth examples in $\mathbb{P}^5$, but not in $\mathbb{P}^4$. Also, if $X_4 \subset \mathbb{P}^5$ contains a Hirzebruch surface, in particular a plane, then the ruling gives a rational section of the quadric bundle, and X_4 is unirational. More generally, one may look for rational surfaces containing a pencil of algebraic asymptotics.

For quartic threefolds in $\mathbb{P}^4$, the explicit large family studied in detail is obtained from monoidal quartic surfaces with separable asymptotics. A quartic monoid is a quartic surface $F_4 \subset \mathbb{P}^3$ with a triple point as its unique singularity. If the triple point is $T = (0:0:0:1)$, then F_4 has equation $x_3\, b(x_0, x_1, x_2) - a(x_0, x_1, x_2) = 0$, where a and b are homogeneous forms of degrees 4 and 3, without common factors.

Among quartic monoids, the condition of separable asymptotics can be characterized by the Hessian: F_4 has separable asymptotics if and only if, at the generic point x, the Hessian equation satisfies $h(x) = \ell(x)^2$ over $K(x)$ or over a quadratic extension of $K(x)$. Predonzan classified the non-ruled quartic monoids with separable asymptotics over an algebraically closed field of characteristic zero. Up to projective equivalence there are six canonical types:

$$\begin{array}{rl}
\text{I} & X_0X_1X_2X_3 + X_0^4 + X_1^4 + X_2^4 - 2(X_0^2X_1^2 + X_1^2X_2^2 + X_2^2X_0^2) = 0,\\
\text{II} & X_0X_1X_2X_3 + (X_1^2 - X_2^2)^2 = 0,\\
\text{III} & X_1^2X_2X_3 + X_0^2X_1^2 + X_2^4 = 0,\\
\text{IV} & X_1^2X_2X_3 + X_0^4 = 0,\\
\text{V} & X_0^3X_3 + X_1X_2(X_1^2 - X_2^2) = 0,\\
\text{VI} & X_0^3X_3 + X_1^2X_2^2 = 0.
\end{array}$$

The most general case is type I, called the tetrahedral surface:

$$W = X_0X_1X_2X_3 + X_0^4 + X_1^4 + X_2^4 - 2(X_0^2X_1^2 + X_1^2X_2^2 + X_2^2X_0^2) = 0.$$

It is called tetrahedral because its cubic tangent cone at the triple point splits into three planes. More explicitly, in coordinates $(x_0 : x_1 : x_2 : x_3)$ with triple point $T = (0 : 0 : 0 : 1)$, the most general case has cubic tangent cone $x_0x_1x_2 = 0$. The surface also has six double points, which are the vertices of a complete plane quadrilateral; the diagonal triangle of this quadrilateral is the intersection of its plane with the cubic tangent cone. If $a_0, a_1, a_2 \neq 0$, one can write

$$\begin{aligned}
L_1 &= \sqrt{a_0}x_0 + \sqrt{a_1}x_1 - \sqrt{a_2}x_2,\\
L_2 &= \sqrt{a_0}x_0 - \sqrt{a_1}x_1 - \sqrt{a_2}x_2,\\
L_3 &= \sqrt{a_0}x_0 + \sqrt{a_1}x_1 + \sqrt{a_2}x_2,\\
L_4 &= \sqrt{a_0}x_0 - \sqrt{a_1}x_1 + \sqrt{a_2}x_2,
\end{aligned}$$

and $a(x_0, x_1, x_2) = L_1L_2L_3L_4$. Thus

$$a(x_0, x_1, x_2) = a_0^2x_0^4 + a_1^2x_1^4 + a_2^2x_2^4 - 2(a_0a_1x_0^2x_1^2 + a_1a_2x_1^2x_2^2 + a_0a_2x_0^2x_2^2),$$

and the corresponding quartic monoid is

$$x_0x_1x_2x_3 + a_0^2x_0^4 + a_1^2x_1^4 + a_2^2x_2^4 - 2(a_0a_1x_0^2x_1^2 + a_1a_2x_1^2x_2^2 + a_0a_2x_0^2x_2^2) = 0.$$

After the change of coordinates

$$X_0 = \sqrt{a_0}x_0, \qquad X_1 = \sqrt{a_1}x_1, \qquad X_2 = \sqrt{a_2}x_2, \qquad X_3 = \frac{x_3}{\sqrt{a_0a_1a_2}},$$

one obtains the canonical tetrahedral form above.

We now count the corresponding family of quartic hypersurfaces. Let $\mathbb{P}^r = \mathbb{P}^r_K$, and let $\mathcal{F}$ be the family of tetrahedral surfaces $S \subset \mathbb{P}^3 \subset \mathbb{P}^r$. Marchisio considers the incidence

$$I = \{(S, X) \in \mathcal{F} \times |\mathcal{O}_{\mathbb{P}^r}(4)| \mid S \subset X\}.$$

The family of tetrahedral surfaces in a fixed $\mathbb{P}^3$ has dimension 15, since all such surfaces are projectively equivalent and $\dim \mathrm{PGL}_4 = 15$. For a fixed tetrahedral surface

$$S = \{W(x_0, x_1, x_2, x_3) = 0\} \subset \{x_4 = \cdots = x_r = 0\} \simeq \mathbb{P}^3,$$

a quartic hypersurface $X \subset \mathbb{P}^r$ containing S has equation

$$\sum_{i=4}^{r} x_if_i(x_0, \dots, x_r) + \lambda W(x_0, x_1, x_2, x_3) = 0,$$

where the f_i are arbitrary cubic forms and $\lambda \in K^*$.

The resulting algebraic system is irreducible for $r \leq 7$, and its dimension is $\binom{r+4}{4} + 4r - 32$. For $r = 4$, this gives $\binom{8}{4} + 16 - 32 = 70 + 16 - 32 = 54$. Since the full space of quartic threefolds in $\mathbb{P}^4$ is $|\mathcal{O}_{\mathbb{P}^4}(4)| \simeq \mathbb{P}^{69}$, the tetrahedral, hence Segre–Marchisio, locus of smooth unirational quartic threefolds has dimension 54 inside $\mathbb{P}^{69}$. Equivalently, it has codimension $69 - 54 = 15$.

For comparison, the same formula for $r = 5$ gives $\binom{9}{4} + 20 - 32 = 126 + 20 - 32 = 114$, so one obtains a 114-dimensional family of smooth unirational quartic fourfolds in $\mathbb{P}^5$, inside the full parameter space $\mathbb{P}^{125}$.

We stress, however, that the preceding count should not be interpreted as a classification of all quartic threefolds containing rational surfaces with separable asymptotics. It only concerns the explicit family obtained from tetrahedral quartic monoids, i.e. from the most general type in Predonzan's classification of non-ruled quartic monoids with separable asymptotics [Pre60b, Mar00, Mar06]. In particular, the conclusion is that a general quartic threefold in $\mathbb{P}^4$ does not contain a tetrahedral monoidal quartic surface of the above type. It does not show that the general quartic threefold contains no rational surface with separable asymptotics at all.

Indeed, the class of rational surfaces with separable asymptotics is larger than the class of quartic monoids classified by Predonzan. Besides the monoidal examples, one has ruled non-developable surfaces, rational surfaces with a pencil of algebraic asymptotics, and other possible surfaces of the kind considered by Segre [Seg45, Seg60, Mar00]. Marchisio explicitly presents the problem of determining all such surfaces as open or at least not settled by the known classification [Mar00]. Thus the implication

$$S_0 \subset X_4 \text{ rational with separable asymptotics} \quad \Longrightarrow \quad X_4 \text{ unirational}$$

is a sufficient criterion, but no converse or classification of all quartics satisfying this criterion is known.

There is also a small historical distinction worth recording. The tetrahedral surface used above is the most general monoidal type in Predonzan's classification, namely type I. Segre's original smooth unirational quartic threefold contains instead a monoidal surface belonging to another projective type in that list, namely type V [Pre60a]. Thus the word "Segre" refers to the geometric method based on separable asymptotics and to the first explicit unirational quartic threefold, whereas the 54-dimensional family counted above is more precisely the tetrahedral, or Predonzan–Marchisio, family [Pre60b, Mar06].

Consequently, the following is the precise statement supplied by the dimension count. Let $\mathcal{T}_4 \subset |\mathcal{O}_{\mathbb{P}^4}(4)| \simeq \mathbb{P}^{69}$ be the locus of quartic threefolds containing a tetrahedral quartic monoid. Then $\dim \mathcal{T}_4 = 54$, $\operatorname{codim}_{\mathbb{P}^{69}}(\mathcal{T}_4) = 15$. In particular, the general quartic threefold does not belong to $\mathcal{T}_4$. However, this does not exclude the existence on the general quartic threefold of some other rational surface with separable asymptotics. Such an exclusion would be much stronger: if the general quartic threefold contained such a surface, the Segre–Predonzan construction would imply the unirationality of the general quartic threefold. This remains one of the classical open problems on quartic hypersurfaces in low dimension [Mar00, Mas23b]. What is presently known may be summarized as follows:

$$\begin{array}{c} \text{quartic monoids with separable asymptotics} \\ \text{classified by Predonzan} \end{array} \quad \subsetneq \quad \begin{array}{c} \text{rational surfaces with separable asymptotics} \\ \text{not classified in general} \end{array}$$

and the 54-dimensional locus above only comes from the left-hand side. Hence the general quartic threefold is known not to contain a tetrahedral quartic monoid, but it is not known whether it contains no rational surface with separable asymptotics.

At present, no larger family of smooth quartic threefolds in $\mathbb{P}^4$ obtained from non-monoidal rational surfaces with separable asymptotics seems to be known. The 54-dimensional family constructed by Marchisio comes from tetrahedral quartic monoids, namely from the most general case in Predonzan's classification of monoidal quartic surfaces with separable asymptotics. Predonzan's classification does not cover arbitrary rational surfaces with separable asymptotics, and Marchisio explicitly leaves the determination of such surfaces as a further problem. Thus the number 54 should be understood as the dimension of the known tetrahedral Predonzan–Marchisio family, not as an upper bound for all possible quartics containing rational surfaces with separable asymptotics. No classification is known which would exclude the existence of further, possibly higher dimensional, loci.

Conic bundles and del Pezzo surfaces. A conic bundle over a field k is a smooth, projective, geometrically irreducible surface S, together with a morphism $\pi\colon S \longrightarrow B$ to a smooth projective curve B, such that the geometric fibers of π are plane conics, either smooth conics or pairs of lines. A conic bundle is called minimal if it cannot be obtained from another conic bundle by blowing up points on fibers.

If S is unirational, then the base curve B is dominated by a projective space. By Lüroth's theorem, this forces $B \simeq \mathbb{P}^1$. Thus, for unirationality questions, the main case is $\pi\colon S \longrightarrow \mathbb{P}^1$.

A basic invariant of a minimal conic bundle is its discriminant δ_S, the number of singular geometric fibers. The canonical self-intersection is related to the discriminant by

$$K_S^2 = 8 - \delta_S.$$

Thus conic bundles with small discriminant are close to del Pezzo surfaces. In particular, the range $\delta_S \leq 7$ corresponds to $K_S^2 \geq 1$. This is the range where the surface is expected to have many rational curves.

Del Pezzo surfaces. Let S be a del Pezzo surface over k. Recall that the degree of S is $d = K_S^2$. Over an algebraic closure, S is isomorphic either to $\mathbb{P}^1 \times \mathbb{P}^1$, when $d = 8$, or to the blow-up of $\mathbb{P}^2$ at $9 - d$ points in general position.

Over a non-algebraically closed field the situation is subtler. The surface may be geometrically rational without being rational over k. The following table summarizes the general picture. The last two columns should be read under the assumption that $S(k) \neq \emptyset$ [Poo17].

K_S^2	k-point automatic?	k-rational if $S(k) \neq \emptyset$?	k-unirational if $S(k) \neq \emptyset$?
9	no	yes	yes
8	no	yes in the rational cases	yes
7	yes	yes	yes
6	no	yes	yes
5	yes	yes	yes
4	no	not in general	yes
3	no	not in general	yes
2	no	not in general	known in many cases
1	yes	not in general	known in many cases

For degrees $d \geq 3$, a del Pezzo surface with a rational point is unirational; this is due to classical work of Segre and Manin, with later refinements by Kollár and others. For degree 2, the problem is harder. For degree 1, one has to distinguish the general del Pezzo surface from those carrying an additional conic bundle structure.

6.1.1. *Degree* 1 *del Pezzo surfaces and conic bundles.* A del Pezzo surface of degree 1 always has a distinguished rational point: the unique base point of the anticanonical pencil $|-K_S|$.

However, the existence of this point does not automatically make S rational or unirational over the ground field.

Let S be a del Pezzo surface of degree 1 over a field k. By definition, $K_S^2 = 1$ and the anticanonical divisor $-K_S$ is ample. The first important difference with del Pezzo surfaces of higher degree is that the anticanonical linear system $|-K_S|$ does not give a morphism. It is a pencil, and it has a unique base point, which we denote by $p^* \in S$.

The correct object to consider is the anticanonical ring

$$R(S, -K_S) = \bigoplus_{m \geq 0} H^0(S, -mK_S).$$

For a degree 1 del Pezzo surface this graded ring is generated by two sections of degree 1, one section of degree 2, and one section of degree 3. Thus we may choose weighted homogeneous coordinates x, y, z, w with weights

$$\deg(x) = \deg(y) = 1, \qquad \deg(z) = 2, \qquad \deg(w) = 3.$$

These generators define an embedding $S \subset \mathbb{P}(1,1,2,3)$. Since the weighted projective space $\mathbb{P}(1,1,2,3)$ has dimension 3, the surface S is cut out by one weighted homogeneous equation. The degree of this equation is 6. Therefore, after choosing suitable coordinates, S can be written as

$$S = \{F_6(x,y,z,w) = 0\} \subset \mathbb{P}(1,1,2,3),$$

where F_6 is weighted homogeneous of degree 6.

If $\operatorname{char}(k) \neq 2, 3$, one can put the equation in Weierstrass form:

$$w^2 = z^3 + f_4(x,y)z + f_6(x,y),$$

where f_4 and f_6 are binary forms of degrees 4 and 6, respectively. Indeed, w^2, z^3, $f_4(x,y)z$, and $f_6(x,y)$ all have weighted degree 6.

This model also explains the geometry of the anticanonical system. The pencil $|-K_S|$ is generated by the two weight-one coordinates x and y. Its members are the curves $\{\lambda x + \mu y = 0\} \cap S$, and they all pass through the unique base point $p^* = [0:0:1:1] \in S$ after a suitable normalization of the equation.

On the other hand, the system $|-2K_S|$ is base-point-free. It is generated by x^2, xy, y^2, z, and gives a morphism $S \longrightarrow \mathbb{P}^3$. The image is the quadric cone $Q = \{U_0U_2 = U_1^2\} \subset \mathbb{P}^3$, where $[U_0 : U_1 : U_2 : U_3] = [x^2 : xy : y^2 : z]$. The morphism $S \longrightarrow Q$ has degree 2. The corresponding involution is the *Bertini involution*; in the Weierstrass equation it is simply $(x,y,z,w) \longmapsto (x,y,z,-w)$. Thus a degree 1 del Pezzo surface should be thought of as a double cover of a quadric cone, or equivalently as a sextic hypersurface in the weighted projective space $\mathbb{P}(1,1,2,3)$.

A particularly important case occurs when S also carries a conic bundle structure $\pi\colon S \longrightarrow \mathbb{P}^1$. Then $K_S^2 = 1$ is equivalent to saying that the conic bundle has $\delta_S = 7$ singular fibers.

Kollár and Mella proved that, over a field of characteristic different from 2, conic bundles with at most 7 singular fibers are unirational as soon as they have a rational point [KM17]. In particular, a degree 1 del Pezzo surface carrying a k-defined conic bundle structure is unirational.

Let us explain the idea. Enriques' criterion says that a surface with a pencil of rational curves $\pi\colon S \to \mathbb{P}^1$ is unirational if and only if it has a rational multisection. Thus the problem is not to parametrize all of S directly, but rather to construct a rational curve on S which maps nontrivially to the base.

For degree 1 conic bundles, the anticanonical system $|-K_S|$ is a pencil of genus one curves with a unique base point p^*. The Bertini involution associated with $|-2K_S|$ interacts with the conic bundle structure. In the general case, the Bertini involution sends a suitable fiber to a rational multisection of degree 8. By Enriques' criterion, this proves unirationality.

The remaining special cases are handled by elementary transformations of the conic bundle and by symmetric-product arguments. The conclusion is that the whole range $\delta_S \leq 7$ is unirational, provided a rational point exists.

Theorem 6.2 (Kollár–Mella)**.** *Let k be a field of characteristic different from 2, and let $\pi\colon S \to \mathbb{P}^1$ be a minimal conic bundle with at most 7 singular fibers. Then $S(k) \neq \emptyset \iff S$ is unirational over k.*

In terms of the degree K_S^2, this says that minimal conic bundles with $K_S^2 \geq 1$ are unirational as soon as they have a rational point.

6.2.2. *Degree 2 del Pezzo surfaces.* A del Pezzo surface of degree 2 has a very concrete description. The anticanonical linear system defines a double cover $\kappa\colon S \longrightarrow \mathbb{P}^2$.

If $\mathrm{char}(k) \neq 2$, then S can be written as

$$w^2 = f_4(x, y, z)$$

inside the weighted projective space $\mathbb{P}(1, 1, 1, 2)$, where the branch curve $B = \{f_4 = 0\} \subset \mathbb{P}^2$ is a smooth plane quartic.

The unirationality problem for degree 2 del Pezzo surfaces is more delicate than for degrees $d \geq 3$. The basic strategy is to construct rational curves on S. Once one has a nonconstant morphism $\mathbb{P}^1 \longrightarrow S$, one can often spread this construction and obtain a dominant rational map $\mathbb{P}^2 \dashrightarrow S$.

Over an arbitrary field, the unirationality of degree 2 del Pezzo surfaces with a rational point is still not completely understood. The results of Salgado–Testa– Várilly-Alvarado should be viewed as effective sufficient criteria, not as a complete answer over the ground field [STVA14].

Let $\kappa\colon S \longrightarrow \mathbb{P}^2$ be the anticanonical double cover, let $R \subset S$ be its ramification divisor, and let $B \subset \mathbb{P}^2$ be the branch quartic. If $\mathrm{char}(k) \neq 2$, then B is a smooth plane quartic and S can be written as

$$w^2 = f_4(x, y, z) \subset \mathbb{P}(1, 1, 1, 2).$$

The basic idea is the following. Suppose that $p \in S(k)$ is a rational point. Manin's construction tries to produce a rational curve on S passing through p. Once such a rational curve exists, one can repeat the construction along the points of this curve and obtain a dominant rational map $\mathbb{P}^2 \dashrightarrow S$.

Thus, for degree 2 del Pezzo surfaces, finding one rational curve is often enough to prove unirationality.

Salgado–Testa–Várilly-Alvarado refine Manin's construction and clarify exactly which points have to be avoided. A point $p \in S$ is called a generalized Eckardt point if it lies on four exceptional curves. Their main criterion over an arbitrary field is:

$$p \in S(k), \qquad p \notin R, \qquad p \text{ not a generalized Eckardt point} \quad \Longrightarrow \quad S \text{ is unirational over } k.$$

Let us explain why. Blow up S at p: $\beta\colon \widetilde{S} \longrightarrow S$, and let E be the exceptional divisor. One then considers the linear system $|-2K_{\widetilde{S}} - E|$. Under the assumptions above, this system contains a divisor with a rational component. Pushing this component down to S, one obtains a rational curve defined over k. The existence of such a rational curve implies the unirationality of S over k [STVA14].

There is also a useful criterion involving several rational points. If $S(k)$ contains eight points $p_1, \ldots, p_8$ whose images $\kappa(p_1), \ldots, \kappa(p_8) \in \mathbb{P}^2$ are distinct and do not lie on the branch quartic B, then at least one of these points is good for the above construction, or lies on an exceptional curve defined over k. In both cases S is unirational over k [STVA14].

The ramification case is more delicate. If $p \in R(k)$, there is a unique member of $|-K_S|$ which is singular at p. This curve is called the *spine* of S at p. When the spine is geometrically integral and rational, it gives a rational curve on S, and hence again implies unirationality. Over finite fields, this observation is one of the key tools used to reduce the problem to a finite list of exceptional cases.

A different but related criterion was later formulated by Festi and van Luijk. Assume $\mathrm{char}(k) \neq 2$. Let $P \in S(k)$, and let $C \subset \mathbb{P}^2$ be a geometrically integral plane curve of degree $d \geq 2$ such that $\kappa(P)$ is a point of multiplicity $d-1$ on C. Suppose moreover that C meets the branch curve B with even multiplicity everywhere. Then the double cover $\kappa^{-1}(C) \longrightarrow C$ is very close to being unramified. If $\kappa(P) \notin B$, this produces a rational curve on S defined over k, and hence S is unirational over k. If instead $\kappa(P) \in B$, and $\kappa(P)$ is an ordinary singular point of C with $d = 3$ or 4, then the construction gives a rational curve after an extension of degree at most 2. Consequently S becomes unirational after a quadratic extension [FvL16].

In particular, Salgado–Testa–Várilly-Alvarado prove that for every degree 2 del Pezzo surface S over a field k, there exists a quadratic extension k'/k such that $S_{k'}$ is unirational [STVA14]. This does not mean that S is always unirational over k; the general question over the ground field remains open.

Over finite fields the situation is now completely settled. Salgado–Testa– Várilly-Alvarado proved that every degree 2 del Pezzo surface over a finite field is unirational, except possibly three explicit surfaces over $\mathbb{F}_3$ and $\mathbb{F}_9$ [STVA14]. Festi and van Luijk then found rational curves on those three remaining surfaces and proved that every degree 2 del Pezzo surface over a finite field is unirational [FvL16].

6.2. **A general strategy for fibrations in quadrics and cubics.** A recurring difficulty in unirationality problems is that direct parametrizations are rarely available. A useful substitute is the following principle, inspired by Enriques' criterion for conic bundles.

Let $f\colon X \longrightarrow Y$ be a fibration over a unirational variety Y. Suppose that there exists a unirational subvariety $Z \subset X$ such that $f|_Z\colon Z \longrightarrow Y$ is dominant. Then, after base change to Z, the fibration acquires a rational section: $X_Z = X \times_Y Z \longrightarrow Z$.

If the generic fiber of f is a quadric, this section gives a rational point on the generic quadric, and projection from that point makes the generic fiber rational. If the generic fiber is a cubic hypersurface of dimension at least two, Kollár's theorem says that the presence of a point implies unirationality, provided the cubic is not a cone [Kol02]. Hence, in both cases, the total space X is unirational.

Thus the problem becomes: find a unirational multisection of f.

This is the main idea behind the recent unirationality results for quadric bundles, quartic hypersurfaces, and quintic hypersurfaces.

6.2. **Quadric bundles.** Let $\pi\colon Q^h \longrightarrow \mathbb{P}^{n-h}$ be an h-fold quadric bundle. Its discriminant $D_{Q^h} \subset \mathbb{P}^{n-h}$ parametrizes the singular quadrics in the fibration, and we write $\delta_{Q^h} := \deg D_{Q^h}$.

For quadric bundles over $\mathbb{P}^1$, the positivity of the anticanonical class is controlled by the discriminant. If $\pi\colon Q^{n-1} \longrightarrow \mathbb{P}^1$ is a quadric bundle, then

$$(-K_{Q^{n-1}})^n = (n-1)^{n-1}(4n - \delta_{Q^{n-1}}).$$

Thus $(-K_{Q^{n-1}})^n > 0$ is equivalent to $\delta_{Q^{n-1}} < 4n$.

The main result of [Mas23a] says that many such quadric bundles are unirational.

Theorem 6.3. *Let $\pi\colon Q^{n-1} \longrightarrow \mathbb{P}^1$ be a general quadric bundle over a number field. If $(-K_{Q^{n-1}})^n > 0$ and the discriminant degree $\delta_{Q^{n-1}}$ is odd, then Q^{n-1} is unirational.*

The same strategy also gives results over arbitrary infinite fields, under additional generality and point-existence hypotheses. For instance, if Q^{n-1} has a rational point and $n \leq 5$, or if the discriminant is sufficiently small, then the same conclusion holds.

Let us explain the method. By the Birkhoff–Grothendieck theorem, every quadric bundle over $\mathbb{P}^1$ is a hypersurface in a splitting projective bundle. This allows one to write the equation in Cox coordinates. Inside the quadric bundle one then constructs special subvarieties, usually obtained by setting some fiber coordinates equal to zero. These subvarieties are themselves lower-dimensional quadric bundles. If one of them is unirational and dominates the base, then Enriques' criterion gives the unirationality of the original quadric bundle.

A second step passes from bases of higher dimension to the case of $\mathbb{P}^1$. Given a quadric bundle over $\mathbb{P}^{n-h}$, one restricts it to a general line in the base. The resulting quadric bundle over $\mathbb{P}^1$ controls a multisection of the original fibration. This reduction gives, for example, the following result.

Theorem 6.4. *Let $\pi\colon Q^2 \longrightarrow \mathbb{P}^2$ be a smooth complex quadric surface bundle. If $\delta_{Q^2} \leq 12$, then Q^2 is unirational.*

Combining this with stable irrationality results for quadric surface bundles gives new examples of varieties which are unirational but not stably rational. In particular, a very general quadric surface bundle over $\mathbb{P}^2$ with $10 \leq \delta_{Q^2} \leq 12$ is unirational but not stably rational.

6.4. **Quartics, quintics, and fibrations in cubics.** The same philosophy applies to certain hypersurfaces. Suppose that a hypersurface contains a linear space, or has high multiplicity along a linear space. Blowing up that linear space often turns the hypersurface into a fibration whose fibers are quadrics or cubics. Then one can try to find unirational multisections and apply the criterion above.

One of the main results of [Mas23b] concerns quartic hypersurfaces. Let $X_4 \subset \mathbb{P}^{n+1}$ be a quartic hypersurface over an infinite field. If X_4 contains an h-plane $\Lambda \subset X_4$ with

$$h \geq \max\{2, \dim(\Lambda \cap \mathrm{Sing}(X_4)) + 2\},$$

and X_4 is not a cone, then X_4 is unirational. There is also a second case: if X_4 has double points along an h-plane with $h \geq 3$, has a smooth k-rational point outside that plane, and is otherwise general, then X_4 is unirational.

The proof is geometric. Projection from the linear subspace Λ, after resolving the indeterminacies by a blow-up, produces a fibration in lower-degree hypersurfaces. The exceptional divisor is a divisor in a product of projective spaces, often of bidegree $(2,2)$ or $(3,2)$.

These divisors carry natural quadric bundle structures. Once one constructs a unirational subvariety dominating the base, the fibration criterion gives unirationality.

The same paper also gives a density result for quartic threefolds. If $X_4 \subset \mathbb{P}^4$ is a quartic threefold over a number field, with double points along a plane and with a rational point outside that plane, then the set $X_4(k)$ is Zariski dense, under the stated generality assumptions.

Finally, this method gives new unirationality results for quintic hypersurfaces. For example, if $X_5 \subset \mathbb{P}^{n+1}$ has multiplicity three along an $(n-1)$-plane and is otherwise general, then in several cases X_5 is unirational. The idea is again to replace X_5 birationally by a divisor of bidegree $(3,2)$

in a product of projective spaces. This divisor is a fibration in quadrics, and the unirationality follows by constructing a suitable unirational multisection.

This produces examples of unirational varieties in situations where stable rationality is known to fail. For instance, very general divisors of bidegree $(3,2)$ in $\mathbb{P}^3 \times \mathbb{P}^2$ are not stably rational by the work of Auel–Böhning–Pirutka, while the unirationality criterion above proves that they are unirational.

7. The watershed case $\delta_S = 8$

The theorem of Kollár–Mella settles the unirationality of conic bundle surfaces $\pi\colon S \longrightarrow \mathbb{P}^1$ with $\delta_S \leq 7$, provided that S has a rational point. Since $K_S^2 = 8 - \delta_S$, this is exactly the range $K_S^2 \geq 1$.

In other words, it is the del Pezzo range, or the range of conic bundles which are still closely related to del Pezzo surfaces.

The next case is $\delta_S = 8$. Then $K_S^2 = 0$. This is a genuine boundary case. It is the first case outside the del Pezzo world, and for this reason it behaves as a watershed between the classical unirationality results for low discriminant conic bundles and the much more mysterious behavior expected for large discriminant.

Geometrically, the difference is the following. When $\delta_S \leq 7$, positivity of $-K_S$ is still strong enough to force the existence of rational curves and rational multisections. When $\delta_S = 8$, this positivity disappears: $K_S^2 = 0$.

Thus one can no longer expect the same del Pezzo-type arguments to work. At the same time, the discriminant is still small enough that explicit geometry is possible.

This boundary case is addressed in [CGM25]. Let

$$S \subset \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1}(a_0) \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \mathcal{O}_{\mathbb{P}^1}(a_2)\big)$$

be a splitting conic bundle over $\mathbb{P}^1$. It is given by an equation

$$\sum_{0 \leq i \leq j \leq 2} \sigma_{ij}(x_0, x_1) y_i y_j = 0,$$

where the σ_{ij} are binary forms of suitable degrees. The discriminant degree is

$$\delta_S = d_{00} + d_{11} + d_{22}.$$

For $\delta_S = 8$, there are exactly four possible splitting types. They are:

(a_0, a_1, a_2)	$(d_{00}, d_{01}, d_{02}, d_{11}, d_{12}, d_{22})$
$(2,1,1)$	$(4,3,3,2,2,2)$
$(2,2,0)$	$(4,4,2,4,2,0)$
$(3,1,0)$	$(6,4,3,2,1,0)$
$(4,0,0)$	$(8,4,4,0,0,0)$.

Each of these families has parameter space of dimension 21.

The main point of [CGM25] is that, although $\delta_S = 8$ lies beyond the del Pezzo range, many such conic bundles are still unirational. More precisely, over an infinite perfect C_1-field of characteristic different from 2, the authors prove that in each of the four families above there is a Zariski dense set of minimal conic bundles which are unirational. Since these bundles are minimal and have discriminant degree 8, they are not k-rational. Thus one obtains many examples of surfaces which are k-unirational but not k-rational.

The constructions are explicit and follow the same general philosophy as Enriques' criterion. One does not try to parametrize the whole surface directly. Instead, one constructs rational multisections of $\pi\colon S \to \mathbb{P}^1$.

Once such a multisection is found, the conic bundle becomes rational after base change to that multisection, and this gives a dominant rational map from a rational surface to S.

The four families require different constructions.

- For type $(2,1,1)$, the unirationality is obtained from rational 2-sections cut out by special tangent or bitangent conditions.
- For type $(2,2,0)$, the authors again construct rational 2-sections, now using prescribed tangent conditions and the action of the automorphism group of the ambient projective bundle.
- For type $(3,1,0)$, a general member is unirational, and in fact the unirational locus is Zariski open. This is the strongest openness statement among the four cases.
- For type $(4,0,0)$, the equation can be put into a diagonal-type normal form. This family is especially interesting since it contains both rational and non-rational behavior on dense loci. Non-rationality is detected using residues of 2-torsion Brauer classes.

Thus $\delta_S = 8$ is a watershed in two senses. First, it is the first case not covered by the del Pezzo range $\delta_S \leq 7$. Second, it is the first case where rationality and unirationality visibly separate in large families: the same moduli space contains dense loci of rational conic bundles and dense loci of minimal non-rational but unirational conic bundles.

The four splitting types $(2,1,1)$, $(2,2,0)$, $(3,1,0)$, $(4,0,0)$ belong to the same deformation component of projective bundles over $\mathbb{P}^1$. The authors construct explicit one-parameter degenerations changing the splitting type:

$$(2,1,1) \rightsquigarrow (2,2,0) \rightsquigarrow (3,1,0) \rightsquigarrow (4,0,0).$$

This explains how the unirationality constructions in the four families are related by specialization. In particular, the boundary case $\delta_S = 8$ is not a collection of isolated examples, but a coherent deformation-theoretic picture.

8. Log del Pezzo weighted complete intersections

A smooth del Pezzo surface is a smooth projective surface S such that $-K_S$ is ample. A log del Pezzo surface is the singular analogue: it is a normal projective surface with quotient singularities and ample anticanonical divisor. Thus log del Pezzo surfaces are still Fano surfaces, but one allows mild singularities. This is natural from the point of view of the minimal model program, and also from differential geometry, where quotient singularities correspond to orbifold metrics.

Weighted projective spaces provide a useful source of such surfaces. Let $X \subset \mathbb{P}(a_0, \dots, a_N)$ be a quasi-smooth well-formed weighted complete intersection of multidegree $(d_1, \dots, d_c)$. By adjunction, its canonical class is

$$K_X \sim \mathcal{O}_X\left(\sum_i d_i - \sum_j a_j\right).$$

Thus X is Fano precisely when the amplitude $\alpha = \sum_j a_j - \sum_i d_i$ is positive. When $\alpha = 1$, we have $-K_X \sim \mathcal{O}_X(1)$. This means that the embedding is anticanonical.

This generalizes the classical picture. Smooth del Pezzo surfaces are embedded by their anticanonical linear systems, for example cubic surfaces in $\mathbb{P}^3$ or degree 2 del Pezzo surfaces as double covers of $\mathbb{P}^2$. Log del Pezzo weighted complete intersections are similar objects, but the anticanonical ring may have generators of different weights. The weighted ambient space records these generators.

Johnson and Kollár first studied the hypersurface case in weighted projective 3-spaces [JK01]. They classified anticanonically embedded quasi-smooth log del Pezzo hypersurfaces $X_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$, with $d = a_0 + a_1 + a_2 + a_3 - 1$. Apart from sporadic cases, they found one infinite family:

$$X_{8k+4} \subset \mathbb{P}(2, 2k+1, 2k+1, 4k+1).$$

These surfaces are log del Pezzo surfaces with quotient singularities. They are not ordinary smooth del Pezzo surfaces, but they play the same role in the orbifold category.

Kim and Park then considered complete intersections of codimension 2 in weighted projective 4-spaces [KP15]. They classified quasi-smooth well-formed complete intersection log del Pezzo surfaces of amplitude 1. There are 38 sporadic families and three infinite series. In the notation $(a_0, a_1, a_2, a_3, a_4; d_1, d_2)$, the infinite series are:

Weights	**Multidegree**
$(1, 1, n+1, n+1, 2n+1)$	$(2n+2, 2n+2)$
$(1, 2, 2n+1, 2n+1, 4n+1)$	$(4n+2, 4n+3)$
$(2, 2n+1, 2n+1, 4n+1, 6n+1)$	$(6n+3, 8n+2)$,

where $n \geq 1$.

The lists of Johnson–Kollár and Kim–Park are complete under the standard anticanonical assumptions. More precisely, they classify all quasi-smooth well-formed log del Pezzo weighted complete intersection surfaces of amplitude 1, not obtained as intersections with linear cones.

Indeed, Johnson–Kollár classify the hypersurface case $X_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ with $d = a_0+a_1+a_2+a_3-1$, that is, the anticanonically embedded case [JK01]. Kim–Park then classify the codimension 2 case in $\mathbb{P}(a_0, \ldots, a_4)$ with amplitude 1 [KP15]. Since a quasi-smooth weighted complete intersection log del Pezzo surface of amplitude 1 has codimension at most 2, these two lists together give the complete classification in amplitude 1.

This should not be confused with a classification of all log del Pezzo surfaces, or even of all weighted complete intersection log del Pezzo surfaces in arbitrary amplitude. The result is specifically about the quasi-smooth well-formed anticanonical case $-K_X \sim \mathcal{O}_X(1)$.

The reason these families are interesting for unirationality is that they are very explicit. Their equations are weighted homogeneous, their singularities are controlled by quasi-smoothness, and their anticanonical geometry is visible from the coordinates. Thus one can try to construct rational curves, pencils, or rational multisections directly from the weighted model.

Theorem 8.1 (Gugiatti–Massarenti [GM26])**.** *Let k be an infinite field, and let X be a quasi-smooth well-formed log del Pezzo weighted complete intersection belonging to one of the infinite families above. Assume that $X(k) \neq \emptyset$ and that X is otherwise general in its family. Then X is unirational over k.*

Thus Theorem 8.1 shows that, at least for the infinite weighted complete intersection families, the classical expectation for del Pezzo surfaces persists in the singular weighted setting: the presence of a rational point, together with generality, forces unirationality.

9. Rationality of complex conic bundle threefolds

Let $\pi\colon X \to S$ be a standard conic bundle over a rational complex surface. Thus X is a smooth projective threefold, the general fiber is a smooth conic, and the relative Picard number is one. The discriminant curve $\Delta \subset S$ records the points over which the conic degenerates.

The basic rationality question is: when is X rational? If π has a rational section, then X is birational to $S \times \mathbb{P}^1$, hence rational. But in general the generic conic over $\mathbb{C}(S)$ need not have a rational point. The obstruction is encoded by the discriminant and by a natural double cover $\widetilde{\Delta} \to \Delta$, whose fiber over a general point of Δ parametrizes the two components of the reducible conic.

Over $S = \mathbb{P}^2$, the degree of Δ is a first measure of complexity. If $\deg \Delta \leq 4$, then X is rational. Indeed, a general pencil of lines on $\mathbb{P}^2$ cuts the discriminant in at most four points, and after passing to the generic member of the pencil one obtains a surface conic bundle with small discriminant, hence rational. This gives a rational fibration by rational surfaces.

The first subtle case is $\deg \Delta = 5$. Here the discriminant is a plane quintic, so its arithmetic genus is 6. The double cover $\widetilde{\Delta} \to \Delta$ determines a theta-characteristic on Δ. The parity of this theta-characteristic decides rationality:

Theta-characteristic	Birational type
even	X is rational
odd	X is birational to a smooth cubic threefold, hence not rational.

This is the exceptional borderline case in the rationality theory of conic bundle threefolds.

The two sides of the dichotomy have concrete models. In the odd case, one starts with a smooth cubic threefold $Y \subset \mathbb{P}^4$ and blows up a line $\ell \subset Y$. Projection from ℓ gives a conic bundle over $\mathbb{P}^2$ with discriminant of degree 5. Since Y is not rational by Clemens–Griffiths, this conic bundle is not rational. In the even case, one can obtain X as the blow-up of $\mathbb{P}^3$ along a smooth curve $\Gamma \subset \mathbb{P}^3$ of degree 7 and genus 5; the linear system of cubic surfaces through Γ gives a conic bundle over $\mathbb{P}^2$, again with discriminant of degree 5, but now X is rational.

For a general rational base surface S, Prokhorov's survey follows the Shokurov–Sarkisov viewpoint. The expected criterion says that X should be rational precisely when $|2K_S + \Delta| = \emptyset$, except that when $p_a(\Delta) = 6$ one must also require the vanishing of the Griffiths component of the intermediate Jacobian. The necessity is known over minimal rational bases, and the degree 5 plane case is exactly the case where this extra intermediate-Jacobian condition cannot be ignored [Pro18].

Thus the moral is the following. Low degree discriminant usually forces rationality. Degree 5 is the first genuinely Hodge-theoretic case: the same numerical data allow both rational and non-rational conic bundles, and the difference is detected by the Prym/intermediate Jacobian attached to $\widetilde{\Delta} \to \Delta$.

10. Low degree complete intersections in high dimension

We collect here some general results showing that hypersurfaces and complete intersections of fixed degree become unirational when the ambient dimension is sufficiently large. The guiding idea is that, in high dimension, low degree varieties contain many linear spaces or many highly tangent lines. These can be used to build dominant rational parametrizations.

The first results go back to Morin for hypersurfaces and Predonzan for complete intersections [Mor42, Pre49]. In modern language, they say that, if the multidegree is fixed and the ambient dimension is large enough, then the general complete intersection is unirational.

Theorem 10.1 (Morin–Predonzan). *Fix positive integers $d_1,\dots,d_r$. Then there exists an integer $N(d_1,\dots,d_r)$ such that a general complete intersection $X_{d_1,\dots,d_r}\subset\mathbb{P}^N$ of multidegree $(d_1,\dots,d_r)$ is unirational for every $N\geq N(d_1,\dots,d_r)$* [Mor42, Pre49].

The construction is geometric. One first finds a sufficiently large linear space contained in X. Projection from this linear space turns X into a fibration whose general fiber is again a complete intersection, but of smaller multidegree. One then argues inductively. Thus the problem is reduced to the existence of enough linear spaces on a general complete intersection.

Paranjape and Srinivas gave a modern exposition of this argument for general complete intersections of small multidegree [PS92]. Ramero later obtained effective estimates for the required bound $N(d_1,\dots,d_r)$ [Ram90]. Shimada generalized the Morin–Predonzan theorem further, allowing more flexible incidence constructions [Shi95].

For hypersurfaces, stronger results are known. Harris, Mazur and Pandharipande proved that, in characteristic zero, for every d there exists an integer $N'(d)$ such that every smooth hypersurface $X_d\subset\mathbb{P}^N$ is unirational for $N\geq N'(d)$ [HMP98]. Thus, in the hypersurface case, one can replace "general" by "smooth", provided the dimension is sufficiently large.

A more recent result of Beheshti and Riedl gives an explicit bound [BR21]. They prove that an arbitrary smooth hypersurface $X_d\subset\mathbb{P}^n_{\mathbb{C}}$ is unirational as soon as $n\geq 2^{d!}$. The important point is that this is not just a theorem for the general hypersurface. It holds for every smooth hypersurface in this range. Their proof is based on the geometry of linear subspaces on hypersurfaces. They prove the de Jong–Debarre conjecture for lines in the range $n\geq 2d-4$, and also prove expected-dimension and irreducibility results for spaces of higher-dimensional linear subspaces. These linear spaces then provide enough room to run an inductive unirationality construction.

Even more recently, Cheng introduced a different construction using highly tangent lines [Che25]. For a general hypersurface $X_d\subset\mathbb{P}^n$ of degree $d\geq 6$, he proves unirationality as soon as $n\geq 2^{(d-1)2^{d-5}}$. The construction considers lines having contact of order at least $d-1$ with X at a point. Such a line has one residual intersection point with X, and the residual point construction gives a rational map to X. The fibers of the relevant incidence correspondence are complete intersections of smaller degrees, so the proof again proceeds by induction on the degree.

Thus the known results have slightly different strengths. Morin–Predonzan and Paranjape–Srinivas prove unirationality for general complete intersections. Harris–Mazur–Pandharipande and Beheshti–Riedl prove unirationality for every smooth hypersurface in sufficiently large dimension. Cheng gives a much sharper bound than the classical ones, but for general hypersurfaces.

There is also a related, but different, circle of results on rational connectedness and rational simple connectedness. A smooth complete intersection $X_{d_1,\dots,d_r}\subset\mathbb{P}^N$ is Fano if $\sum_i d_i\leq N$, hence rationally connected in characteristic zero. A stronger property, rational simple connectedness, is known for low degree complete intersections. This is not the same as unirationality, but it has strong consequences over function fields.

For weighted complete intersections, a result in this direction is due to Minoccheri [Min20]. Let $X_{d_1,\dots,d_c}\subset\mathbb{P}(e_0,\dots,e_n)$ be a smooth weighted complete intersection over $\mathbb{C}$. Under suitable low degree hypotheses, including $c\leq n-3$, $e_0=e_1=e_2=e_3=1$, and numerical inequalities comparing the d_i with the weights e_j, Minoccheri proves that X is rationally simply connected. In particular, over the function field of a complex curve, such varieties satisfy weak approximation and have strong consequences for R-equivalence and zero-cycles.

At present, however, this weighted result should not be confused with a weighted Morin–Predonzan theorem. It gives rational simple connectedness, not unirationality. A general unirationality theorem for low degree weighted complete intersections, with explicit bounds in terms of the weights and degrees, seems much less developed. We can summarize the situation as follows:

Varieties	Hypothesis	Conclusion
General hypersurfaces	$N \gg d$	Unirational [Mor42]
General complete intersections	$N \gg (d_1, \dots, d_r)$	Unirational [Pre49, PS92]
Smooth hypersurfaces	$N \gg d$, char $k = 0$	Unirational [HMP98]
Smooth hypersurfaces over $\mathbb{C}$	$n \geq 2^{d!}$	Unirational [BR21]
General hypersurfaces	$d \geq 6,\ n \geq 2^{(d-1)2^{d-5}}$	Unirational, Cheng [Che25]
Smooth weighted complete intersections	Minoccheri's low degree hypotheses	Rationally simply connected [Min20].

The general philosophy is clear: if the degrees are fixed and the dimension grows, complete intersections become increasingly flexible. What is still missing, especially in the weighted setting, is a general unirationality theorem with effective bounds comparable to the classical projective-space case.

11. Open problems

We have seen that even-dimensional cubic hypersurfaces can be rational. For instance, the Fermat cubic $X^{2n} \subset \mathbb{P}^{2n+1}$ is rational in characteristic different from 3. On the other hand, smooth cubic curves have genus one, and smooth cubic threefolds are not rational by Clemens–Griffiths [CG72]. This suggests the following natural question.

Question 11.1. Does there exist a smooth rational odd-dimensional cubic hypersurface $X^{2n+1} \subset \mathbb{P}^{2n+2}$ for some $n \geq 2$?

The same question can be asked for hypersurfaces of higher degree. General hypersurfaces of fixed degree become unirational when the dimension is sufficiently large. This leads to a rationality analogue.

Question 11.2. Does there exist a smooth rational complex hypersurface $X_d \subset \mathbb{P}^{n+1}$ of degree d for some $n \gg d$?

The relation between unirationality and rational connectedness is still poorly understood in dimension three. The minimal model program reduces many questions to Fano threefolds, del Pezzo fibrations, and conic bundles. For Fano threefolds of Picard rank one, the following cases remain especially important.

Question 11.3. Are the following threefolds unirational?

- A smooth quartic threefold $X_4 \subset \mathbb{P}^4$.
- A double cover of a smooth quadric threefold branched along a divisor of degree 8.
- A degree 6 hypersurface in $\mathbb{P}(1, 1, 1, 2, 3)$.

For surfaces the analogous problem is also open in low degree over arbitrary fields.

Question 11.4. Let S be a del Pezzo surface over a field k, with $S(k) \neq \emptyset$ and $K_S^2 \in \{1, 2\}$. Is S unirational over k?

Question 11.4 has a positive answer when S admits a conic bundle structure, by Kollár–Mella [KM17]. Degree 2 is also known over finite fields, by Salgado–Testa–Várilly-Alvarado and Festi–van Luijk [STVA14, FvL16]. Over arbitrary fields, however, the problem remains open. The three-dimensional analogue for conic bundles is the following.

Question 11.5. Is every complex conic bundle threefold $X \to \mathbb{P}^2$ unirational, independently of the degree of its discriminant?

This question should be compared with rationality. A conic bundle over $\mathbb{P}^2$ with discriminant of degree at most 4 is rational. When the discriminant has degree 5, rationality depends on the parity of the theta-characteristic associated with the double cover of the discriminant. For larger discriminant degree, rationality is usually expected to fail, but unirationality may still hold.

Remark 11.6. Questions 11.3 and 11.5 are related. A quartic threefold can be studied through families of conics. Let $X_4 \subset \mathbb{P}^4$ be a quartic threefold which is not a cone and has at most double points. Let $\Gamma \subset X_4$ be a rational curve and let $x \in \Gamma$ be general. The projective tangent cone $TC_x X_4$ is a quadric cone. If $H \subset \mathbb{P}^4$ is a general hyperplane, then $C_x = TC_x X_4 \cap H$ is a conic.

Consider the incidence variety $\mathcal{S} = \{(x, y) \mid x \in \Gamma,\ y \in C_x\} \subset \Gamma \times H$. The projection $\mathcal{S} \to \Gamma$ is a conic fibration over a rational curve, hence $\mathcal{S}$ is rational. For $y \in C_x$, let $L_y \subset TC_x X_4$ be the line through x and y. Since L_y has contact of order at least 3 with X_4 at x, it meets X_4 in one residual point. This gives a rational map $\mathcal{S} \dashrightarrow X_4$, whose image is a rational surface $S \subset X_4$.

Now repeat the construction with S in place of Γ. Define $\mathcal{Y} = \{(x, y) \mid x \in S,\ y \in C_x\} \subset S \times H$. The projection $\mathcal{Y} \to S$ is a conic fibration. The residual intersection construction gives a rational map $g \colon \mathcal{Y} \dashrightarrow X_4$.

The expectation is that, for a sufficiently general choice of the initial rational curve, this map should be dominant. If $\mathcal{Y}$ is unirational, then X_4 is unirational.

Thus Question 11.5 gives a possible route to Question 11.3. If one could prove sufficiently general unirationality results for conic bundle threefolds over rational surfaces, then one could hope to apply them to conic fibrations naturally associated with quartic threefolds and with other Fano threefolds.

More generally, an analogous construction applies to quartic hypersurfaces $X_4 \subset \mathbb{P}^{n+1}$. Tangent cones at points produce quadric hypersurfaces, and residual intersection gives rational maps from fibrations in quadrics to X_4. Understanding the unirationality of these auxiliary quadric fibrations is therefore closely related to the unirationality of quartic hypersurfaces.

Question 11.7. Does there exist a complex rationally connected variety of dimension at least 3 which is not unirational?

This is perhaps the most basic conceptual question behind the comparison between unirationality and rational connectedness. We know that X unirational implies X rationally connected. For curves and surfaces over $\mathbb{C}$, the converse holds: rational connectedness is equivalent to rationality, and hence to unirationality. Starting from dimension 3, however, no example is known which separates rational connectedness from unirationality.

The difficulty is that rational connectedness is a very flexible geometric condition: it only asks that two general points can be joined by a rational curve. Unirationality is much stronger: it asks for a single dominant rational parametrization from projective space. Thus unirationality requires a global parametrization, while rational connectedness only requires many local rational curves.

At present, there is no known birational invariant that detects the difference between these two conditions over $\mathbb{C}$. Classical invariants such as holomorphic forms, plurigenera, and Kodaira dimension vanish or behave trivially on rationally connected varieties, just as they do on unirational varieties. More refined invariants, such as intermediate Jacobians, Brauer groups, unramified cohomology, or decompositions of the diagonal, are powerful for distinguishing rationality or stable rationality from unirationality, but they do not currently provide a general obstruction to unirationality inside the class of rationally connected varieties.

Thus Question 11.7 asks whether rational connectedness is merely the correct flexible replacement for rationality, or whether it is genuinely larger than unirationality in the complex projective category. A positive answer would exhibit a new kind of rationally connected variety, one covered by rational curves in abundance but not dominated by projective space.

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