A NOTE ON THE YONEDA LEMMA

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ABSTRACT. We state, without proofs, the Yoneda lemma, and the 2-Yoneda lemma. Then we prove that a functor is representable if and only if it admits a universal object using the Yoneda lemma, and that a fibered category is representable if and only if it is fibered in groupoids using the 2-Yoneda lemma.

CONTENTS

1. THE YONEDA LEMMA

Let $\mathcal{C}$ be a category. The functors from $\mathcal{C}^{op}$ to (Set) can be thought as a category $Hom(\mathcal{C}^{op},(Set))$, in which the arrows are the natural transformations. To any object $X \in \mathcal{C}$ we can associate a functor $h_X : \mathcal{C}^{op} \rightarrow (Set)$, which maps an object $Y \in \mathcal{C}$ in $Hom_{\mathcal{C}}(Y,X)$, and a morphism $f : Y \rightarrow Z$ in the map

$$h_X f : Hom_{\mathcal{C}}(Z,X) \rightarrow Hom_{\mathcal{C}}(Y,X),\ \alpha \mapsto \alpha \circ f.$$

Now let now $\rho : X \rightarrow Y$ be an arrow in $\mathcal{C}$. For any object $Z \in \mathcal{C}$ we get a function $h_f Z : h_X Z \rightarrow h_Y Z$, $\alpha \mapsto \rho \circ \alpha$. This defines a morphism $h_f : h_X \rightarrow h_Y$, i.e. for any arrow $\beta : Z \rightarrow W$ in $\mathcal{C}$ the following diagram commute

$$\begin{array}{ccc} h_X Z & \xrightarrow{h_f Z} & h_Y Z \\ {\scriptstyle h_X \beta}\downarrow & & \downarrow{\scriptstyle h_Y \beta} \\ h_X W & \xrightarrow{h_f W} & h_Y W \end{array}$$

In this way we have defined a functor

$$\mathcal{H} : \mathcal{C}^{op} \rightarrow Hom(\mathcal{C}^{op},(Set)),\ X \rightarrow h_X,\ f \rightarrow h_f.$$

Lemma 1.1. *(Weak Yoneda Lemma) The functor $\mathcal{H}$ is fully faithful, that is the function*

$$Hom(X,Y) \rightarrow Hom(h_X,h_Y),\ f \mapsto h_f$$

is bijective.

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The functor $\mathcal{H}$ is not an equivalence of categories because it is not essentially surjective. This means that there are functors $\mathcal{C}^{op} \to (Set)$ not isomorphic to a functor of the form h_X. The lemma says that the category $\mathcal{C}$ can be embedded in the category $Hom(\mathcal{C}^{op}, (Set))$.

Definition 1.2. A functor $F : \mathcal{C}^{op} \to (Set)$ is said to be representable if it is isomorphic to a functor of the form h_X for some $X \in \mathcal{C}^{op}$.

If we restrict to the full subcategory of $Hom(\mathcal{C}^{op}, (Set))$ of representable functor then $\mathcal{H}$ is an equivalence. Note that if $F \cong h_X$ and $F \cong h_Y$ then the isomorphism between h_X and h_Y comes form an unique isomorphism form X and Y, by the weak Yoneda lemma. This means that two objects representing the same functor are canonically isomorphic.
Our aim is to reformulate the definition of representable functor using a more general version of Yoneda's lemma. Let $X \in \mathcal{C}$ be an object and let $F : \mathcal{C}^{op} \to (Set)$ be a functor. Given a natural transformation $\alpha : h_X \to F$ we can consider the map $\alpha_X : h_X(X) = Hom(X, X) \to F(X)$, and the element $\xi = \alpha_X(Id_X)$. We get a function

$$Hom(h_X, F) \to F(X), \alpha \mapsto \alpha_X(Id_X) = \xi.$$

Conversely fixed an element $\xi \in F(X)$, we define a natural transformation $\alpha : h_X \to F$ as follows. Let $Y \in \mathcal{C}$ be an object, and let $f \in h_X(Y) = Hom(Y, X)$ be an arrow, the arrow f induces a function $F(f) : F(X) \to F(Y)$. We define $\alpha_Y : h_X(Y) \to F(Y)$ mapping $f \mapsto F(f)(\xi)$. We have defined a function

$$F(X) \to Hom(h_X, F).$$

Lemma 1.3. *(Yoneda Lemma) The two function above define a bijective correspondence*

$$Hom(h_X, F) \leftrightarrows F(X)$$

Note that for $F = h_Y$ we get a bijective correspondence $Hom(h_X, h_Y) \leftrightarrows h_Y(X) = Hom(X, Y)$, that is the weak Yoneda lemma.

Definition 1.4. Let $F : \mathcal{C}^{op} \to (Set)$ be a functor. An *universal object* for F is a pair (X, ξ) where $X \in \mathcal{C}$ and $\xi \in F(X)$ such that for each $Y \in \mathcal{C}$ and each $\beta \in F(Y)$, there is a unique arrow $f : Y \to X$ such that $\beta = F(f)(\xi)$.

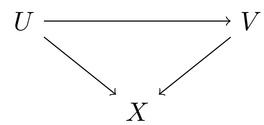
Recall that for any $\xi \in F(X)$ we defined a morphism $\alpha : h_X \to F$, in other words (X, ξ) is an universal object if and only if the morphism α defined by $\xi \in F(X)$ is an isomorphism. By Yoneda lemma any natural transformation $h_X \to F$ is induced by a $\xi \in F(X)$. So we get the following proposition.

Proposition 1.5. *A functor $F : \mathcal{C}^{op} \to (Set)$ is representable if and only if it has an universal object.*

Clearly if (X, ξ) is an universal object fro F then X represents F. The Yoneda lemma establish an equivalence of categories between $\mathcal{C}$ and the full subcategory of $Hom(\mathcal{C}^{op}, (Set))$ defined by representable functors. In particular if $\mathcal{C} = (Sch/k)$ is the category of schemes over the field k, we get that a scheme X is determined by its functor of points $h_X : (Sch/k)^{op} \to (Set)$.

2. The 2-Yoneda Lemma

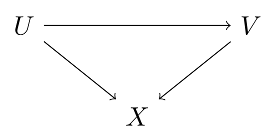
Let $\mathcal{C}$ be a category and let $X \in \mathcal{C}$ be an object. We con construct the *Comma category* $(\mathcal{C}/X)$ as follows. The objects of $(\mathcal{C}/X)$ are morphism $U \to X$ in $\mathcal{C}$, and the arrows between two objects $U \to X$ and $V \to X$ are the arrows $U \to V$ in $\mathcal{C}$ such that the following diagram commutes



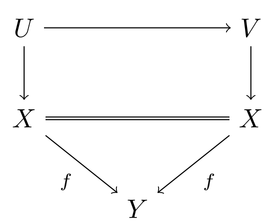
The fibered category $(\mathcal{C}/X) \to \mathcal{C}$, $(U \to X) \to U$ is the category fibered in sets associated to the functor $h_X : \mathcal{C}^{op} \to (Set)$. Recall that we have an embedding $\mathcal{C} \to Hom(\mathcal{C}^{op}, (Set))$, $X \mapsto h_X$, and by composing we get an embedding

$$\mathcal{C} \to Hom(\mathcal{C}^{op}, (Set)) \to (FC/\mathcal{C}),\ X \mapsto h_X \mapsto ((\mathcal{C}/X) \to \mathcal{C}),$$

of $\mathcal{C}$ in $(FC/\mathcal{C})$, the 2-category of fibered categories over $\mathcal{C}$. Note that a morphism $f : X \to Y$ in $\mathcal{C}$ goes to the morphism $(\mathcal{C}/f) : (\mathcal{C}/X) \to (\mathcal{C}/Y)$ of fibered categories defined as follows. Let $U \to X$ be an object in $(\mathcal{C}/X)$ then $(\mathcal{C}/f)(U \to X) = (U \to X \to Y)$ i.e. a morphism goes in its composition with f. The functor $(\mathcal{C}/f)$ sends an arrow



in $(\mathcal{C}/X)$, in the commutative diagram



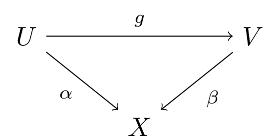
Lemma 2.1. (<u>Weak 2-Yoneda Lemma</u>) *The function that sends $f : X \to Y$ in the morphism $(\mathcal{C}/f) : (\mathcal{C}/X) \to (\mathcal{C}/Y)$ states a bijection between $Hom_{\mathcal{C}}(X, Y)$ and $Hom((\mathcal{C}/X), (\mathcal{C}/Y))$.*

Definition 2.2. A fibered category over $\mathcal{C}$ is representable if it is equivalent to a category of the form $(\mathcal{C}/X)$.

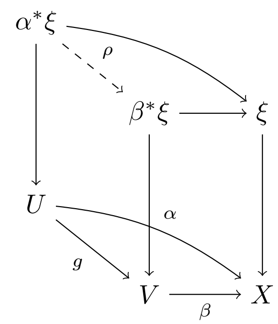
Let F be a category fibered over $\mathcal{C}^{op}$, and let X be an object of $\mathcal{C}^{op}$. Let $f \in Hom_{\mathcal{C}}((\mathcal{C}/X), F)$ be a morphism of fibered categories. Then we have $f_X : (\mathcal{C}/X)(X) \to F(X)$ and we can consider $f(Id_X) \in F(X)$. Furthermore if $\alpha : f \to g$ is a base preserving natural transformation between $f, g : (\mathcal{C}/X) raF$, we get an arrow $\alpha_{Id_X} : f(Id_X) \to g(Id_X)$. This defines a functor

$$Hom_{\mathcal{C}}((\mathcal{C}/X), F) \to F(X).$$

Conversely, if $\xi \in F(X)$ is an object we can construct a functor $f_\xi : (\mathcal{C}/X) \to F$ as follows. For any $\psi : U \to X$ in $(\mathcal{C}/X)$ we define $f_\xi(\psi) = \psi^*\xi \in F(U)$. If



is an arrow in $(\mathcal{C}/X)$ we associate the unique arrow $\rho : \alpha^*\xi \to \beta^*\xi$ in F such that the diagram



commutes.

Lemma 2.3. *(2-Yoneda Lemma) The functors above define an equivalence of categories*

$$Hom_{\mathcal{C}}((\mathcal{C}/X), F) \cong F(X).$$

Then a morphism $(\mathcal{C}/X) \to F$ corresponds to an object $\xi \in F(X)$, which in turn defines the functor $f_\xi : (\mathcal{C}/X) \to F$ defined above, this is isomorphic to the original functor f. Now the fibered category F is representable if and only if $f_\xi : (\mathcal{C}/X) \to F$ is an equivalence for some $X \in \mathcal{C}$ and $\xi \in F(X)$. Recall that f_ξ is an equivalence if and only if for each object $Y \in \mathcal{C}^{op}$ the functor

$$f_\xi(Y) : (\mathcal{C}/X)(Y) = Hom_{\mathcal{C}}(Y, X) \to F(Y),\ g \mapsto g^*\xi,$$

is an equivalence of categories. Since $Hom_C(Y, X)$ is a set this means that $F(Y)$ is a groupoid. Thanks to the 2-Yoneda lemma we get another characterization for representable fibered categories.

Proposition 2.4. *A fibered category F over $\mathcal{C}$ is representable if and only if it is fibered in groupoids, and there exist an object $Y \in \mathcal{C}$, and an object $\xi \in F(Y)$, such that for any object ν of F there exists a unique arrow $\nu \to \xi$ in F.*

References

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