Cubic Surfaces in $\mathbb{P}^3$

Alex Massarenti

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Abstract

These notes give an elementary introduction to cubic surfaces in projective three–space. Starting from the most basic definitions, we classify reducible cubics, discuss cones and singular irreducible cubics, and then focus on smooth cubic surfaces and their lines. A central goal is to present a self-contained and classical proof that every cubic surface contains a line and that every smooth cubic surface contains exactly 27 lines. Throughout, the arguments are deliberately kept at the level of classical projective geometry: plane sections, tangency, projection, dimension counts, and Bézout-type reasoning. No use is made of blow-ups, divisors, linear systems on surfaces.

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Introduction

Cubic surfaces in $\mathbb{P}^3$ occupy a special place in classical algebraic geometry: they are among the simplest projective surfaces beyond quadrics, yet they already display a remarkably rich and rigid geometry. The interplay between plane sections, singularities, and the configuration of lines contained in the surface is highly structured, and it was a major testing ground for the methods of nineteenth-century projective geometry [CS49, Sch58, Cle71, Seg42]. The most famous manifestation of this rigidity is the *theorem of the* 27 *lines*: every smooth cubic surface in $\mathbb{P}^3$ contains exactly 27 lines [CS49, Sch58, Seg42]. Classical work also revealed the finer combinatorics of these lines (tritangent planes, the double-six, and related configurations), which later became tightly connected with root systems and Weyl groups, notably of type E_6 [Cob29, Loo81, Dol12].

Today, the theorem of the 27 lines is often presented through modern birational geometry: a smooth cubic surface is a del Pezzo surface of degree 3, hence (over an algebraically closed field) the blow-up of $\mathbb{P}^2$ in six points in general position; from this one reads off the Picard group, intersection form, and the configuration of (-1)-curves that correspond to the 27 lines. This perspective is conceptually clean and extremely powerful, and it is treated in many standard references [Man86, Bea96, Dol12, Dem80, Sha94]. In particular, the E_6-lattice structure on $\mathrm{Pic}(S)$ and the action of the Weyl group clarify the incidence combinatorics of the 27 lines and their moduli [Dem80, Loo81, DO94, Dol12]. However, it is not the only route to the result.

The purpose of these notes is to develop the basic theory of cubic surfaces *by elementary projective methods*, using only classical geometry and a small amount of basic algebraic geometry at the level of plane curves. Concretely, we work with explicit homogeneous equations, projective varieties defined by polynomials, and standard tools: plane sections, tangent planes, projection from a point, parameter counts, and Bézout-type arguments for plane curves. Even when we discuss rationality, we do so by writing down explicit birational maps via projection, rather than invoking general classification theorems. In particular, the proof of the theorem of the 27 lines is presented *without* appealing to the blow-up model, divisor theory on surfaces, or intersection theory on surfaces—although we keep in mind that our results are consistent with, and illuminated by, the modern del Pezzo viewpoint [Man86, Bea96, Dol12, Dem80]. (For background on the basic scheme-theoretic language of projective varieties and plane curves, we implicitly rely on standard sources such as [Har77, Sha94, Rei88].)

The guiding philosophy is that much of the geometry of cubic surfaces can be recovered from two simple observations:

- a plane section of a cubic surface is a plane cubic curve, hence subject to strong classical constraints (for instance, an irreducible plane cubic has at most one singular point);
- a line on a surface rigidly constrains how plane sections through that line can factor: a plane through a line on a cubic surface cuts the surface as the union of that line and a residual conic.

By repeatedly exploiting these facts, one can control singular loci, produce birational parametrizations, and ultimately organize the configuration of lines. This is very much in the spirit of the classical literature, where plane sections and projection arguments are the basic engines behind the structure theory [CS49, Sch58, Seg42].

We begin with the easiest case, namely reducible cubics, to fix ideas and notation. A reducible cubic surface is either the union of a plane and a quadric (smooth or a cone), or a union of three planes counted with multiplicity. This classification is immediate from factorization in the homogeneous coordinate ring and serves as a warm-up for later arguments in which factorization of plane sections plays a decisive role. Next we restrict to irreducible cubic surfaces and separate them into three broad families:

- *cubic cones* (surfaces obtained by joining a point to a plane cubic);
- *singular irreducible cubics that are not cones*;
- *smooth cubic surfaces*.

For cones, we give a characterization in terms of multiplicity at a point: a degree-d surface is a cone with vertex v if and only if v is a singular point of multiplicity d. In degree 3 this reduces the discussion to irreducible plane cubics, which are either smooth or have exactly one singular double point (node or cusp), a classical fact used throughout elementary treatments of plane curves [Rei88, Har77]. This leads to concrete normal forms for cones over smooth, nodal, and cuspidal plane cubics, and it clarifies the structure of their singular loci.

For singular irreducible cubic surfaces that are not cones, the main point is that they are rational, and one can see this directly by projection from a singular point. Here we avoid birational classification theory: the birational map is written down explicitly and its inverse is understood by elementary incidence considerations. We also treat separately the case where the singular locus is one-dimensional; a key elementary argument shows that if a cubic surface is singular along a curve, then that curve must be a line, necessarily appearing as a double line in plane sections through it. For broader discussions of singular cubic surfaces and their classical classification problems one may consult, for example, the classical sources and later accounts such as [Seg42, BW79].

The final part of the notes is devoted to smooth cubic surfaces and their lines. We first prove that *every* (smooth) cubic surface in $\mathbb{P}^3$ contains at least one line. Rather than using intersection theory or the del Pezzo classification, we use a classical incidence correspondence between lines in $\mathbb{P}^3$ (parametrized by the Grassmannian, classically embedded as the Klein quadric) and cubic surfaces (parametrized by $\mathbb{P}^{19}$). A dimension count shows that the incidence variety projects dominantly to $\mathbb{P}^{19}$, once one exhibits a single cubic surface containing only finitely many lines. This yields existence of lines on every cubic surface by a purely projective argument—again echoing the classical tradition [CS49, Sch58, Seg42].

With existence in hand, we then study the configuration of lines on a smooth cubic surface by analyzing plane sections through a fixed line $L \subset S$. Any plane containing L cuts S as

$$S \cap P = L \cup (\text{a conic}),$$

and the conic splits into two lines exactly when a certain determinant vanishes. The crucial observation is that this determinant is a polynomial of degree 5 in the parameter of the pencil of planes through L. Therefore there are exactly five planes through L for which the residual conic becomes reducible, producing ten further lines meeting L in five coplanar pairs. From this point on, the argument remains entirely combinatorial and projective: we prove the existence of two skew lines, analyze how many lines can pass through a given point, and control common transversals via elementary facts about quadrics in $\mathbb{P}^3$. These incidence patterns are the concrete geometric shadow of the "(-1)-curve" picture on the blow-up of $\mathbb{P}^2$ and its E_6-symmetry, as explained in modern accounts [Man86, Dol12, Dem80, Loo81].

Finally, assembling these constraints yields a complete count. Starting from two skew lines, one constructs 17 distinguished lines and then shows that every remaining line is uniquely determined as a transversal to an appropriate triple among five skew lines; a final counting argument gives 27 lines in total, recovering the classical theorem [CS49, Sch58, Seg42]. For readers interested in how these configurations connect to moduli, arithmetic quotients, and related geometric structures, there are further perspectives in the literature, for instance [Hun96, DO94].

The reader is assumed to be familiar with the basic language of projective space, homogeneous polynomials, and the elementary geometry of plane curves (in particular, Bézout's theorem and the classification of singularities of irreducible plane cubics); good entry points at this level include [Rei88, Har77, Sha94]. Beyond this, no specialized background is required: all constructions are explicit and the key steps reduce either to computations with equations or to standard incidence arguments.

Whenever a modern statement could be proved by invoking general theorems about surfaces, we instead provide a concrete argument tailored to cubics. This is not meant to replace modern methods, but to complement them: the classical approach makes the geometry visible, and it explains why the number 27 emerges from simple projective constraints rather than from heavy formalism [CS49, Sch58, Seg42, Dol12, Man86].

1 Cubic surfaces in $\mathbb{P}^3$

Let $\mathbb{P}^3(K)$ be the projective space over an algebraically closed field K. We consider projective algebraic sets of the form

$$S = V(F) \subset \mathbb{P}^3, \qquad F \in K[X, Y, Z, T] \text{ homogeneous of degree 3.}$$

Such projective varieties are the *cubic surfaces* in $\mathbb{P}^3$. We begin with the case in which F is reducible.

1.1 Reducible cubic surfaces

1. Suppose that F factors as $F = GH$, where G is homogeneous of degree 2 and H is homogeneous of degree 1. Then

$$S = V(G) \cup V(H),$$

so S is reducible: it is the union of an irreducible quadric $Q = V(G)$ and a plane $P = V(H)$ in $\mathbb{P}^3$. There are two possibilities for Q: it may be a quadric cone, or a smooth quadric.

2. Suppose that F factors as $F = GHL$ with G, H, L homogeneous of degree 1. Then S is a union of three planes, counted with multiplicity: three distinct planes, or two planes with one of them of multiplicity 2, or a single plane of multiplicity 3.

In conclusion, a reducible cubic surface in $\mathbb{P}^3$ is one of the following:

- The union of a plane P and a smooth quadric Q. In this case $P \cap Q$ is a conic, and it is singular for S.

- The union of a plane P and a quadric cone C. In this case $P \cap C$ is a conic, and it is singular for S.
- The union of three planes in $\mathbb{P}^3$, counted with multiplicity.

1.2 Irreducible cubic surfaces

From now on we assume that S is irreducible. We will use the standard fact that if $S \subset \mathbb{P}^3$ is an irreducible surface of degree d and H is a general plane, then $S \cap H$ is an irreducible plane curve of degree d.

Among irreducible cubic surfaces we distinguish three cases:

1. cubic cones;
2. singular cubic surfaces that are not cones;
3. smooth cubic surfaces.

Definition 1. A surface $S \subset \mathbb{P}^3$ is a *cone* with vertex $v \in \mathbb{P}^3$ and base curve C if

$$S = \bigcup_{p \in C} \langle v, p \rangle ,$$

i.e. S is the union of the lines joining v to points of C.

We may always assume that the base curve C is planar: if S is a cone over a curve in $\mathbb{P}^3$, then intersecting S with a general plane H not passing through v produces a plane curve that can be used as a base.

1.3 Cubic cones

Lemma 1. *A surface $S \subset \mathbb{P}^3$ of degree d is a cone with vertex v if and only if v is a singular point of multiplicity d on S. In that case, S is a cone over a plane curve of degree d.*

Proof. Assume that S is a cone with vertex v. A general plane H through v meets S in a plane curve C of degree d. Since S is ruled by lines through v, the curve C is a union of d lines through v (counted with multiplicity). Hence v is singular on S and has multiplicity d.

Conversely, assume that v is a singular point of multiplicity d on S. Let H be a general plane through v and set $C = S \cap H$. Then C is a plane curve of degree d, and v has multiplicity d on C, hence C is a union of d lines through v. Thus every general plane section through v is a union of d lines through v, and therefore S is a cone with vertex v. □

To describe cubic cones in $\mathbb{P}^3$ it suffices to look at irreducible plane cubic curves.

Lemma 2. *Let $C \subset \mathbb{P}^2$ be an irreducible plane cubic. Then C has at most one singular point, and any singular point is necessarily a double point.*

Proof. Let p be a singular point of C. Suppose there is another singular point $q \neq p$ on C. Then the line $R = \langle p, q \rangle$ meets C at least four times counting multiplicities, contradicting Bézout's theorem.

Now let $p \in C$ be singular. If the multiplicity $m_p(C) \geq 3$, then for any $q \in C$ with $q \neq p$, the line $R = \langle p, q \rangle$ meets C with multiplicity at least 3 at p and at least 1 at q, again giving at least 4 intersection points counted with multiplicity, a contradiction. Thus $m_p(C) = 2$. □

Irreducible plane cubics are either smooth or singular; any singular plane cubic has exactly one singular point, which is either a node or an ordinary cusp. Two singular plane cubics are projectively equivalent if and only if their singularities are of the same type. Accordingly, we distinguish cubic cones over a smooth cubic, over a nodal cubic, and over a cuspidal cubic.

1.3.1 Cones over a smooth plane cubic

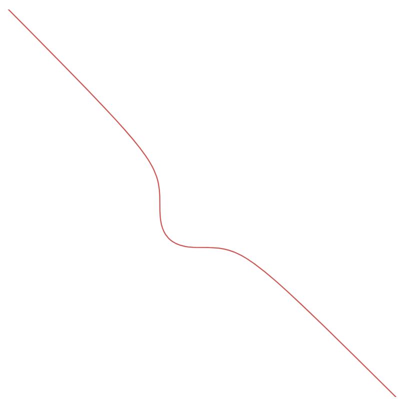
Consider the cubic surface $S \subset \mathbb{P}^3$ defined by

$$F(X, Y, Z, T) = X^3 + Y^3 + Z^3 = 0.$$

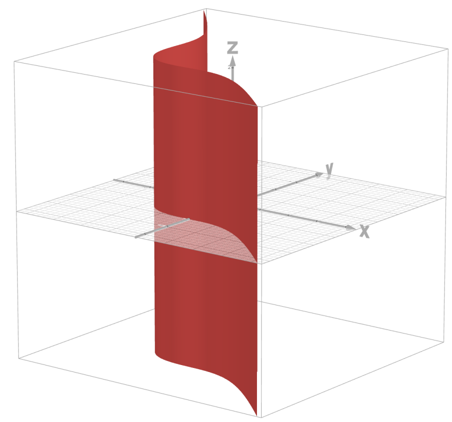
Then

$$(\partial_X F, \partial_Y F, \partial_Z F, \partial_T F) = (3X^2, 3Y^2, 3Z^2, 0),$$

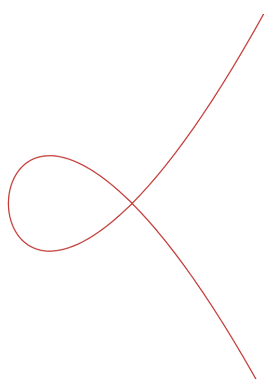
so all first derivatives vanish at the point $v = [0 : 0 : 0 : 1]$, and hence $\mathrm{Sing}(S) = \{v\}$. Intersecting with the plane $H = \{T = 0\}$ gives a smooth plane cubic, so S is a cubic cone with vertex v over a smooth plane cubic. Moreover, all second derivatives vanish at v, while some third derivative does not, showing that v has multiplicity 3 on S, in agreement with Lemma 1.



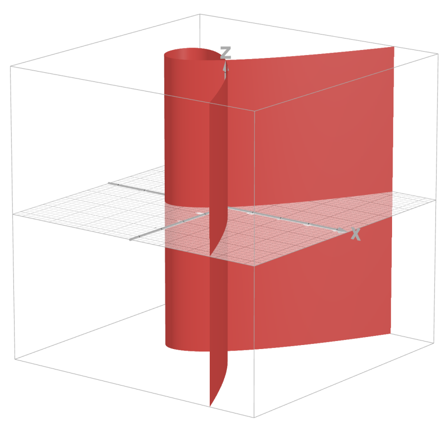
Smooth cubic.



Cone over a smooth cubic.



Nodal cubic.



Cone over a nodal cubic.

1.3.2 Cones over a nodal plane cubic

Consider the cubic surface $S \subset \mathbb{P}^3$ defined by

$$F(X, Y, Z, T) = Y^2 Z - X^2 Z - X^3 = 0.$$

We have

$$(\partial_X F, \partial_Y F, \partial_Z F, \partial_T F) = (-2XZ - 3X^2,\ 2YZ,\ Y^2 - X^2,\ 0).$$

A direct computation gives

$$\mathrm{Sing}(S) = R \cup \{v\}, \qquad R = \{X = 0,\ Y = 0\}, \qquad v = [0 : 0 : 0 : 1].$$

Intersecting with the plane $H = \{T = 0\}$ yields the plane cubic

$$C: \quad G(X, Y, Z) = Y^2Z - X^2Z - X^3 = 0.$$

In the affine chart $Z = 1$ this becomes

$$C^*: \quad y^2 - x^2 - x^3 = 0,$$

and $(0, 0)$ is a double point. The tangent cone at $(0, 0)$ consists of the two lines $y - x = 0$ and $y + x = 0$, so C has a node at $p = [0 : 0 : 1]$, with principal tangents $Y - X = 0$ and $Y + X = 0$. Therefore S is a cubic cone with vertex v over the nodal plane cubic C. Notice that the singular line R is precisely the line through p and v.

1.3.3 Cones over a cuspidal plane cubic

Consider the cubic surface $S \subset \mathbb{P}^3$ defined by

$$F(X, Y, Z, T) = Y^2Z - X^3 = 0.$$

Then

$$(\partial_X F, \partial_Y F, \partial_Z F, \partial_T F) = (-3X^2,\ 2YZ,\ Y^2,\ 0),$$

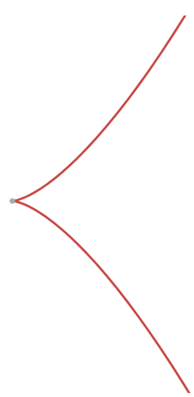
and again

$$\mathrm{Sing}(S) = R \cup \{v\}, \qquad R = \{X = 0,\ Y = 0\}, \qquad v = [0 : 0 : 0 : 1].$$

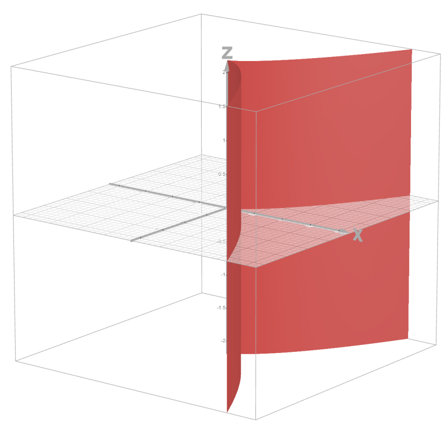
Intersecting with $H = \{T = 0\}$ gives the plane cubic

$$C: \quad Y^2Z - X^3 = 0,$$

which is cuspidal at $p = [0 : 0 : 1]$. Hence S is a cubic cone with vertex v over a cuspidal plane cubic.



Cuspidal cubic.



Cone over a cuspidal cubic.

Lemma 3. *Let S be a cubic cone with vertex v, and let H be a plane not passing through v. If the plane cubic $C = S \cap H$ is singular, then S is rational.*

Proof. Since C is singular, it has a double point $p \in C$. The line $R = \langle p, v\rangle$ is a double ruling line on S. Let $V = S \setminus R$, which is a Zariski open subset of S, and let P be a plane not passing through p. For any $q \in V$, the line $\langle p, q\rangle$ does not meet V again (otherwise it would be contained in S), so projection from p to P is well-defined on V and induces an isomorphism between an open subset of S and an open subset of $\mathbb{P}^2$. Thus S is rational. □

1.4 Singular irreducible cubic surfaces that are not cones

For any algebraic variety X, the singular locus $\mathrm{Sing}(X)$ is a proper closed subset of X. For a singular cubic surface S we distinguish two cases:

1. $\dim \mathrm{Sing}(S) = 1$, i.e. S is singular along a curve;
2. $\dim \mathrm{Sing}(S) = 0$, i.e. S has only isolated singular points.

Proposition 1. *An irreducible singular cubic surface $S \subset \mathbb{P}^3$ that is not a cone is always rational.*

Proof. Let $p \in \mathrm{Sing}(S)$. Since S is not a cone, the point p must have multiplicity 2 on S (otherwise it would be a vertex of a cone by Lemma 1). Set $V = S \setminus \mathrm{Sing}(S)$, a nonempty open subset of S. Choose a plane H disjoint from $\mathrm{Sing}(S)$; in particular H does not pass through p. For any $q \in V$, the line $R = \langle p, q\rangle$ meets S only at p and q; otherwise R would be contained in S, contradicting irreducibility and the choice of q. Hence projection from p to H is well-defined on V and induces a birational map from S to $H \simeq \mathbb{P}^2$. Therefore S is rational. □

1.4.1 The case $\dim \mathrm{Sing}(S) = 1$

We now show that if a cubic surface is singular along a curve, then its singular locus is necessarily a line.

Proposition 2. *Let $S \subset \mathbb{P}^3$ be a cubic surface singular along a curve. Then that curve is a line R. Moreover, R appears with multiplicity* 2 *in the scheme-theoretic intersection with a general plane through R (so R is a* double line *on S), and* $\mathrm{Sing}(S) = R$.

Proof. Let $C \subset S$ be a curve contained in $\mathrm{Sing}(S)$, and let $\deg(C) = d$. A plane H meets C in d points (counted with multiplicity); all these points are singular points of the plane cubic $G = H \cap S$. But an irreducible plane cubic has at most one singular point (Lemma 2), hence $d = 1$ and C is a line R.

The line R cannot occur with multiplicity 3 on S: otherwise, for a plane $H = \langle p, R\rangle$ with $p \in S$ and $p \notin R$, we would have $H \subset S$, contradicting irreducibility. Thus R is not a triple line.

Finally, suppose there exists a singular point $p \in \mathrm{Sing}(S)$ with $p \notin R$. Consider the plane $P = \langle p, R\rangle$. The plane cubic $P \cap S$ has degree 3 and contains R with multiplicity at least 2, hence

$$P \cap S = 2R \cup L$$

for some line L passing through p. But then $P \cap S$ would have to be singular at p, which forces the component L to be singular at p, impossible since a line is smooth. Therefore no such p exists and $\mathrm{Sing}(S) = R$, with R a double line on S. □

Example (a ruled cubic surface of the first kind). The cubic surface

$$S: \quad F(X, Y, Z, T) = XY^2 - Z^2T = 0$$

has a double line (and is called a *ruled cubic surface of the first kind*). Indeed

$$(\partial_X F, \partial_Y F, \partial_Z F, \partial_T F) = (Y^2,\ 2XY,\ -2ZT,\ -Z^2),$$

so $\mathrm{Sing}(S) = R$ where $R = \{Y = 0,\ Z = 0\}$.

Let $p = [a : 0 : 0 : b] \in R$. One checks that $\partial^2_{YY} F = 2X$ and $\partial^2_{ZZ} F = -2T$; at a point of R these second derivatives never vanish simultaneously, hence every point of R is a double point of S.

Let H be a plane containing R. Then $H \cap S$ is a plane cubic containing R with multiplicity 2, hence it is of the form $2R \cup L$ for some line $L \subset S$, necessarily distinct from R. Now fix $p \in S \setminus R$ and consider the plane $P = \langle p, R\rangle$. By the previous discussion, $P \cap S$ contains R with multiplicity 2 and thus determines another line $L_p \subset S$ passing through p and meeting R. Moreover, this line is unique: if there were another line T_p through p meeting R and contained in S, then both L_p and T_p would lie in P, and $P \cap S$ would have degree at least 4, forcing $P \subset S$, a contradiction.

The line $D = \{X = 0,\ T = 0\}$ is contained in S and does not meet R. For $p \in D$, let T_pS be the tangent plane to S at p. Then $T_pS \cap S$ is a plane cubic containing D. Since R and D are skew, the plane T_pS does not contain R, hence

$$T_pS \cap S = D \cup C$$

where C is a conic. The curve $T_pS \cap S$ is singular at p, so C is singular and therefore splits as a union of two lines. Thus

$$T_pS \cap S = D \cup R_1 \cup R_2,$$

and by what we have proved above, one of the lines R_1, R_2 is of the form L_p.

Let $q_1 = [0 : 0 : 0 : 1]$ and $q_2 = [1 : 0 : 0 : 0]$. If $q \in R$ is different from q_1 and q_2, then there are two distinct lines L, L' through q, contained in S, and meeting D. If $q = q_1$ or $q = q_2$, then the line through q meeting D is unique. Indeed, if $q = [a : 0 : 0 : b] \in R$, then planes through q have equation $bX - aT = 0$; substituting into the equation of S shows:

- If $a \neq 0$, then $T = (b/a)X$ and the equation becomes $X(aY^2 - bZ^2) = 0$. If $b = 0$ we get one line; if $b \neq 0$ we get two distinct lines.
- If $b \neq 0$, then $X = (a/b)T$ and the equation becomes $T(aY^2 - bZ^2) = 0$. If $a = 0$ we get one line; if $a \neq 0$ we get two distinct lines.

Hence the two lines coincide if and only if $q = q_1$ or $q = q_2$.

We have exhibited the lines R, D, and the family of lines L_p. One can show that these are *all* the lines contained in S. Indeed, let L be a line not of the form L_p, so L does not meet R. Assume $L \subset S$ and $L \neq D$. Then either L meets D, or L and D are skew. In both cases one obtains a contradiction by considering tangent plane sections and the fact that S is irreducible (the details follow exactly the argument in the original notes).

1.4.2 The case $\dim \mathrm{Sing}(S) = 0$

Lemma 4. *A cubic surface with only isolated singularities has at most four singular points.*

Proof. Assume for contradiction that $\mathrm{Sing}(S) = \{p_1, p_2, p_3, p_4, p_5\}$ consists of five distinct points. Each p_i has multiplicity at least 2 on S. Let $R = \langle p_1, p_2\rangle$; then $R \subset S$. Let H be a plane through p_3, p_4, p_5. The line R meets H at a point p. The plane cubic $H \cap S$ has (at least) three singular points, hence it must be a union of three lines $L \cup T \cup F$. Then $p \in (R \cap L) \cup (R \cap T) \cup (R \cap F)$, forcing p to be the common intersection point of L, T, F, which contradicts the fact that the plane cubic $L \cup T \cup F$ has three singular points. □

Example (four singular points). The cubic surface

$$S: \quad YZT + XZT + XYT + XYZ = 0$$

has exactly four singular points. Indeed, its partial derivatives are

$$(\partial_X F, \partial_Y F, \partial_Z F, \partial_T F) = (ZT + YT + YZ,\ ZT + XT + XZ,\ YT + XT + XY,\ YZ + XZ + XY),$$

which vanish simultaneously exactly at

$$[1 : 0 : 0 : 0], \quad [0 : 1 : 0 : 0], \quad [0 : 0 : 1 : 0], \quad [0 : 0 : 0 : 1].$$

1.5 Smooth cubic surfaces

For instance, the Fermat cubic

$$S: \quad X^3 + Y^3 + Z^3 + T^3 = 0$$

is smooth because

$$(\partial_X F, \partial_Y F, \partial_Z F, \partial_T F) = (3X^2, 3Y^2, 3Z^2, 3T^2)$$

does not vanish identically at any point of $\mathbb{P}^3$.

For smooth cubic surfaces one can prove:

Lemma 5. *Every smooth cubic surface contains a line, and every smooth cubic surface contains exactly* 27 *lines.*

We will now justify these statements.

Lemma 6. *A cubic surface in* $\mathbb{P}^3$ *is rational if and only if it is* not *a cone over a smooth plane cubic.*

1.6 Every cubic surface contains at least one line

We now prove that every cubic surface in $\mathbb{P}^3$ contains at least one line.

1.6.1 Parameter spaces

In $\mathbb{P}^n$, giving a hypersurface of degree d is equivalent (up to nonzero scalar) to giving a homogeneous polynomial of degree d in $n+1$ variables, i.e. a point of

$$\mathbb{P}(S_n^d),$$

where S_n^d denotes the vector space of homogeneous degree-d polynomials in $n+1$ variables. We have

$$\dim(S_n^d) = \binom{n+d}{d} = \frac{(n+d)!}{d!\,n!}, \qquad \dim(\mathbb{P}(S_n^d)) = \binom{n+d}{d} - 1.$$

Thus hypersurfaces of degree d in $\mathbb{P}^n$ are parametrized by $\mathbb{P}^N$ with

$$N = \binom{n+d}{d} - 1.$$

For $n = d = 3$ we obtain $N = \binom{6}{3} - 1 = 19$, so cubic surfaces in $\mathbb{P}^3$ are parametrized by $\mathbb{P}^{19}$.

The Klein quadric $K \subset \mathbb{P}^5$ parametrizes the lines in $\mathbb{P}^3$; it is a smooth quadric of dimension 4.

The incidence variety

Let $X_d \subset \mathbb{P}^N \times K$ be the set of pairs (p,q) such that the line L_q corresponding to $q \in K$ is contained in the hypersurface S_p corresponding to $p \in \mathbb{P}^N$. (We assume X_d is a projective variety.) Consider the projection morphisms

$$F : \mathbb{P}^N \times K \to \mathbb{P}^N, \quad (p,q) \mapsto p, \qquad G : \mathbb{P}^N \times K \to K, \quad (p,q) \mapsto q,$$

and restrict them to X_d.

The map $G|_{X_d}$ is surjective: for every line $L \subset \mathbb{P}^3$ there exists a degree-d hypersurface containing it (for instance, a union of d planes in the pencil through L).

Fix $q \in K$ and compute $\dim(G^{-1}(q))$. Up to projective change of coordinates, we may assume

$$L_q : \quad T_0 = T_1 = 0$$

in homogeneous coordinates $[T_0 : T_1 : T_2 : T_3]$ on $\mathbb{P}^3$. A hypersurface of degree d contains L_q if and only if its defining form $P \in S_3^d$ can be written as

$$P = T_0 P_0 + T_1 P_1, \qquad P_0, P_1 \in S_3^{d-1}.$$

Let

$$V = \{P \in S_3^d : P = T_0 P_0 + T_1 P_1 \text{ with } P_0, P_1 \in S_3^{d-1}\}.$$

Then $G^{-1}(q) \simeq \mathbb{P}(V)$.

Consider the linear map

$$H : S_3^{d-1} \times S_3^{d-1} \to S_3^d, \qquad H(P_0, P_1) = T_0 P_0 + T_1 P_1.$$

It is linear and $\mathrm{Im}(H) = V$. Hence

$$\dim(V) = \dim(S_3^{d-1} \times S_3^{d-1}) - \dim(\mathrm{Ker}(H)).$$

If $(P, Q) \in \mathrm{Ker}(H)$ then $T_0P + T_1Q = 0$, so $T_0 \mid Q$ and $T_1 \mid P$. Write $Q = T_0Q'$ and $P = T_1P'$ with $P', Q' \in S_3^{d-2}$. Then $T_0T_1P' + T_1T_0Q' = 0$, hence $Q' = -P'$. Thus

$$\mathrm{Ker}(H) = \{(T_1P', -T_0P') : P' \in S_3^{d-2}\} \simeq S_3^{d-2},$$

so $\dim(\mathrm{Ker}(H)) = \binom{d+1}{3}$. Therefore

$$\dim(V) = 2\binom{d+2}{3} - \binom{d+1}{3} = \frac{d(d+1)(d+5)}{6}.$$

Consequently,

$$\dim \mathbb{P}(V) = \frac{d(d+1)(d+5)}{6} - 1 \quad \text{and} \quad \dim(G^{-1}(q)) = \dim \mathbb{P}(V) = N - (d+1).$$

We will use the following standard dimension statements.

Theorem 1. *Let $f : X \to Y$ be a surjective regular map between irreducible varieties. If $\dim(X) = n$ and $\dim(Y) = m$, then $m \leq n$ and:*

- *for every $y \in Y$ and every irreducible component X' of $f^{-1}(y)$ one has $\dim(X') \geq n - m$;*
- *there exists a nonempty open subset $U \subset Y$ such that $\dim(f^{-1}(y)) = n - m$ for all $y \in U$.*

Theorem 2. *Let $f : X \to Y$ be a surjective regular map between projective varieties, with Y irreducible. If every fiber $f^{-1}(y)$ is irreducible and has the same dimension, then X is irreducible.*

By Theorem 2 we may assume X_d is irreducible. Since $\dim(K) = 4$, Theorem 1 gives

$$\dim(X_d) = \dim(G^{-1}(q)) + \dim(K) = \big(N - (d+1)\big) + 4 = N + 3 - d.$$

Now consider the map $F|_{X_d} : X_d \to \mathbb{P}^N$. Its image $F(X_d)$ is closed in $\mathbb{P}^N$ because X_d is projective. Moreover $\dim(F(X_d)) \leq \dim(X_d)$.

If $d > 3$ then $\dim(X_d) = N+3-d < N$, hence $\dim(F(X_d)) < N$ and therefore $F(X_d) \neq \mathbb{P}^N$. Equivalently, there exist degree-d hypersurfaces in $\mathbb{P}^3$ containing no line, and in fact they form a nonempty open subset of $\mathbb{P}^N$.

Theorem 3. *For every $d > 3$ there exists a degree-d surface in $\mathbb{P}^3$ that contains no line. Moreover, the set of degree-d surfaces containing no line is a nonempty open subset of $\mathbb{P}^N$; in other words, a general surface of degree $d > 3$ contains no lines.*

The cubic case

For cubic surfaces ($d = 3$), cones obviously contain infinitely many lines. We first exhibit a cubic surface containing only finitely many lines.

Consider the affine cubic surface in $\mathbb{A}^3$ given by $xyz = 1$. A line in $\mathbb{A}^3$ is the intersection of two affine planes:

$$R = \{a_0x + a_1y + a_2z + a_3 = 0, \quad b_0x + b_1y + b_2z + b_3 = 0\}.$$

Solving (generically) for x and y as polynomials $Q(z)$ and $P(z)$ in z and substituting into $xyz = 1$ yields

$$Q(z)P(z)z = 1 \quad \text{for all } z \in K,$$

which is impossible since $Q(z)P(z)z - 1$ is a nonconstant polynomial over an algebraically closed field. Hence the affine surface $xyz = 1$ contains no lines.

Let $S^* \subset \mathbb{P}^3$ be its projective closure, defined by

$$XYZ = T^3.$$

The surface S^* meets the plane at infinity $\{T = 0\}$ along the plane cubic $XYZ = 0$, which is the union of the three coordinate lines

$$R' = \{X = 0,\ T = 0\}, \qquad R'' = \{Y = 0,\ T = 0\}, \qquad R''' = \{Z = 0,\ T = 0\}.$$

Since the affine part contains no lines, these three are the only lines contained in S^*. Therefore there exists a point $s \in \mathbb{P}^{19}$ corresponding to a cubic surface such that $F^{-1}(s)$ is nonempty and $\dim(F^{-1}(s)) = 0$.

Now $F : X_3 \to F(X_3)$ is surjective. For any s in the image, Theorem 1 implies

$$\dim(F^{-1}(s)) \geq \dim(X_3) - \dim(F(X_3)).$$

Since $\dim(F^{-1}(s)) = 0$ for our special s, we get $\dim(X_3) \leq \dim(F(X_3))$. But $\dim(X_3) = 19$ and $F(X_3) \subset \mathbb{P}^{19}$ is a closed irreducible subset of the same dimension, hence $F(X_3) = \mathbb{P}^{19}$. Thus every cubic surface contains at least one line. Moreover, by Theorem 1, there exists a nonempty open subset $U \subset \mathbb{P}^{19}$ such that $\dim(F^{-1}(s)) = 0$ for all $s \in U$, i.e. a general cubic surface contains finitely many (but nonzero) lines.

Theorem 4. *Every cubic surface in $\mathbb{P}^3$ contains at least one line. Moreover, there exists a nonempty open subset $U \subset \mathbb{P}^{19}$ such that every cubic surface corresponding to a point of U contains a finite, nonzero number of lines.*

2 Lines on a smooth cubic surface and the number 27

Let $S \subset \mathbb{P}^3$ be a smooth cubic surface.

Proposition 3. *For every point $p \in S$, there are at most three lines contained in S and passing through p. Moreover, if $P \subset \mathbb{P}^3$ is any plane, then $P \cap S$ is either an irreducible plane cubic, or a conic plus a line, or three distinct lines.*

Proof. Let $L \subset S$ be a line through p. Then the tangent plane T_pL is contained in the tangent plane T_pS. Hence every line through p and contained in S lies in the plane curve $T_pS \cap S$, which is a plane cubic; therefore there are at most three such lines.

For the second statement, assume $P = \{T = 0\}$ and let $L = \{Z = 0,\ T = 0\} \subset P$. We claim that $P \cap S$ cannot contain a multiple line. If L had multiplicity 2 in $P \cap S$, then we could write the defining equation of S as

$$f(X, Y, Z, T) = Z^2 F(X, Y, Z, T) + T\, G(X, Y, Z, T),$$

with F homogeneous of degree 1 and G homogeneous of degree 2. Computing partial derivatives gives

$$\begin{aligned}(\partial_X f, \partial_Y f, \partial_Z f, \partial_T f) = (&Z^2 \partial_X F + T\partial_X G,\ Z^2 \partial_Y F + T\partial_Y G,\\ &2ZF + Z^2\partial_Z F + T\partial_Z G,\ Z^2 \partial_T F + G + T\partial_T G).\end{aligned}$$

On L we have $Z = T = 0$, and from $\partial_T f$ we obtain $G(X, Y, 0, 0) = 0$. But G is a homogeneous quadratic, so it has two roots (counted with multiplicity) on the line L, producing a singular point of S, contradicting smoothness. Therefore no plane section contains a multiple line, and the only possible reducible plane cubics are a line plus a (possibly reducible) conic, or three distinct lines. □

By Theorem 4, the surface S contains a line L. After a change of coordinates, we may assume

$$L = \{X = 0,\ Y = 0\}.$$

Consider the pencil of planes through L; a general plane in the pencil has equation $aX + Y = 0$. Since $L \subset S$, the cubic equation $F(X, Y, Z, T) = 0$ of S can be written in the form

$$F = A(X,Y)Z^2 + 2B(X,Y)ZT + C(X,Y)T^2 + 2D(X,Y)Z + 2E(X,Y)T + F_3(X,Y),$$

where A, B, C are homogeneous of degree 1 in X, Y, D, E are homogeneous of degree 2, and F_3 is homogeneous of degree 3.

Restricting to the plane $P_a = \{Y = -aX\}$, and factoring out the line L, one finds that $P_a \cap S$ is the union of L and a conic K_a. The conic K_a splits as a union of two distinct lines if and only if its defining 3×3 matrix is singular. In our notation the matrix is

$$M(K_a) = \begin{pmatrix} A & C & D \\ C & B & E \\ D & E & F \end{pmatrix},$$

and the splitting condition is

$$\det(M(K_a)) = A(BF - E^2) - C(CF - ED) + D(CE - BD) = 0.$$

The determinant is a polynomial of degree 5 in the parameter a, hence it has five roots counted with multiplicity.

Lemma 7. *Let $\sum_{i,j=0}^{n} a_{ij}(t)X_iX_j$ be a one-parameter family of quadrics, and let $D(t) = \det(a_{ij}(t))$. Then the number of singular quadrics in the family (counted with multiplicity) equals $\deg(D(t))$.*

Therefore, given a line $L \subset S$, there exist exactly five planes P_a through L such that

$$S \cap P_a = L \cup L_a \cup L'_a,$$

with L_a and L'_a distinct lines. This produces ten further lines on S meeting L.

Moreover, if we consider the pencil of conics determined by two conics of the above form, the presence of a double line would force all conics in the pencil to be double lines, which is impossible here because we have only five singular conics in the pencil.

In particular we obtain:

Proposition 4. *Every line L on a smooth cubic surface S meets exactly 10 other lines on S. These 10 lines can be grouped into 5 pairs (T, T') such that the two lines in each pair meet each other.*

Lemma 8. *There exist two skew (disjoint) lines L and R on a smooth cubic surface S.*

Proof. Choose two distinct planes P_1 and P_2 through a fixed line L such that

$$P_1 \cap S = L \cup L_1 \cup L'_1, \qquad P_2 \cap S = L \cup L_2 \cup L'_2.$$

We claim that L_1 and L_2 are skew. If they met at a point q, then $q \in P_1 \cap P_2$. Since P_1 and P_2 share the line L, this would force $P_1 = P_2$, a contradiction. □

Theorem 5. *Every smooth cubic surface $S \subset \mathbb{P}^3$ is rational.*

Proof. Let L and R be two skew lines on S (Lemma 8). There are only finitely many lines on S; let $U \subset S$ be the open subset obtained by removing all the lines contained in S. For $p \in U$, there exists a unique line $T_p \subset \mathbb{P}^3$ through p that meets both L and R. Fix a plane H not containing L or R. The line T_p meets H in a point $f(p)$, giving a map $f : U \to H \simeq \mathbb{P}^2$. This map is birational (indeed an isomorphism on suitable open subsets), so S is rational. □

If R is a line on S skew to L, then R meets exactly one line in each of the five pairs (T, T') associated to L in Proposition 4: indeed R meets the plane P_a with $P_a \cap S = T \cup T' \cup L$, and if R met both T and T' then R would meet P_a in two points, forcing $R \subset P_a$, contradicting that L and R are skew.

Lemma 9. *Let L_1, L_2, L_3, L_4 be four pairwise disjoint lines in $\mathbb{P}^3$. Then either the four lines lie on a smooth quadric and have infinitely many common transversals, or they do not lie on a smooth quadric and have one or two common transversals.*

Proof. Start with three skew lines L_1, L_2, L_3 and fix a point $p \in L_1$. There is a unique line L_p through p meeting both L_2 and L_3. If $p' \in L_1$ with $p' \neq p$, then $L_{p'}$ does not meet L_p (otherwise L_1 and L_2 would be coplanar). Hence the union of the lines L_p as p varies in L_1 is a smooth quadric Q containing L_1, L_2, L_3. On Q there are two rulings by skew lines. If $L_4 \subset Q$, then L_4 lies in the same ruling as L_1, L_2, L_3, and every line in the other ruling meets all L_i, giving infinitely many common transversals. If $L_4 \not\subset Q$, then L_4 meets Q in one or two points, and the lines of the other ruling through those points are the only common transversals, giving one or two of them. □

Theorem 6. *A smooth cubic surface contains exactly* 27 *lines.*

Proof. Let L and M be two skew lines on S. The line L meets exactly 10 lines on S, arranged in 5 coplanar pairs (L_i, L_i') for $i = 1, \ldots, 5$ (Proposition 4). The line M meets exactly one of L_i and L_i' in each pair; assume M meets L_i for each i. Similarly, M determines 5 coplanar pairs (L_i, L_i''). So far we have listed the lines

$$L,\ M,\ L_i,\ L_i',\ L_i'' \qquad (i = 1, \ldots, 5),$$

for a total of $1 + 1 + 5 + 5 + 5 = 17$ lines.

We claim:

1) Any line $D \subset S$ different from the above 17 meets exactly three of the five lines L_i.

2) Conversely, for every triple of distinct indices $(i, j, k) \subset \{1, 2, 3, 4, 5\}$ there exists a unique line $L_{ijk} \subset S$, different from L and M, meeting L_i, L_j, L_k.

(1) Four skew lines on S cannot lie on a smooth quadric: otherwise all lines of the opposite ruling would meet S in four points and hence would be contained in S, forcing a quadric component of S and contradicting irreducibility. If D met more than three of the L_i, then D would be a common transversal to at least four skew lines among them, but by Lemma 9 there are at most two such transversals; this forces $D = L$ or $D = M$, a contradiction. If D met fewer than three of the L_i, then it would meet more than three of the L_i', and the same argument applies.

(2) Take, for instance, the triple L_1, L_2, L_3. There is a unique smooth quadric Q containing three skew lines in $\mathbb{P}^3$. Then $Q \cap S$ is a curve of degree 6, so

$$Q \cap S = L_1 \cup L_2 \cup L_3 \cup C$$

with $\deg(C) = 3$. Since L and M meet Q in three points each, they are contained in Q, hence $C = L \cup M \cup L_{123}$ for a line $L_{123} \subset Q$. The line L_{123} cannot belong to the same ruling as L_1, L_2, L_3 (otherwise we would have four skew lines on S contained in a smooth quadric), so it lies in the opposite ruling and therefore meets each of L_1, L_2, L_3. This gives existence. Uniqueness follows from Lemma 9: common transversals are at most two, and the two already accounted for are L and M.

Finally, the number of lines of the form L_{ijk} is the number of ways to choose three lines among the five L_i, namely

$$\binom{5}{3} = \frac{5!}{3!\,2!} = 10.$$

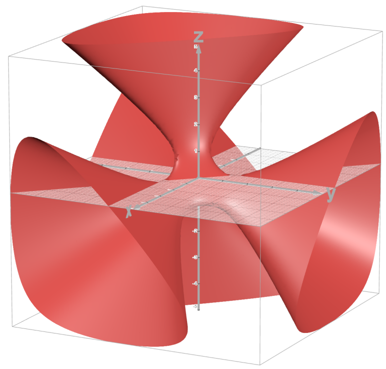
Counting: we have L and M (2 lines), the 10 lines coplanar with L, the 10 lines coplanar with M (five of which are the same L_i already counted), and the 10 lines L_{ijk}. Thus $1 + 1 + 10 + 10 - 5 + 10 = 27$. □

3 The Clebsch cubic surface

The picture below shows the real locus of the *Clebsch diagonal cubic*, a smooth cubic surface with the largest possible symmetry group among smooth cubic surfaces (its projective automorphism group is isomorphic to $\mathfrak{S}_5$).

A convenient way to write it is to realize it as a cubic surface in a hyperplane $H \simeq \mathbb{P}^3$ of $\mathbb{P}^4$:

$$S_{\mathrm{Cl}} = \Big\{ [x_0 : x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^4 \ \Big|\ x_0 + x_1 + x_2 + x_3 + x_4 = 0,\ \ x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0 \Big\}.$$



Clebsch cubic surface.

Identifying the hyperplane $H : \sum_{i=0}^{4} x_i = 0$ with $\mathbb{P}^3$ (for instance by eliminating $x_4 = -(x_0 + x_1 + x_2 + x_3)$), one obtains an explicit cubic equation in $\mathbb{P}^3$:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 - \left(x_0 + x_1 + x_2 + x_3\right)^3 = 0.$$

A classical and striking feature of S_{Cl} is that its 27 are real.

The 27 lines on S_{Cl}. From now on we work inside the hyperplane $H : \; x_0 + \cdots + x_4 = 0 \simeq \mathbb{P}^3$. Thus, a line in H is cut out by two independent linear equations in the variables $x_0, \ldots, x_4$.

(A) Fifteen lines. They are obtained by choosing two disjoint pairs among $\{0, 1, 2, 3, 4\}$; on H they are given by the two equations $x_i + x_j = 0$ and $x_k + x_\ell = 0$ (the remaining coordinate is then forced to be 0 by $\sum x_i = 0$):

$$\begin{aligned}
L_1 : &\quad x_1 + x_2 = 0, \quad x_3 + x_4 = 0,\\
L_2 : &\quad x_1 + x_3 = 0, \quad x_2 + x_4 = 0,\\
L_3 : &\quad x_1 + x_4 = 0, \quad x_2 + x_3 = 0,\\
L_4 : &\quad x_0 + x_2 = 0, \quad x_3 + x_4 = 0,\\
L_5 : &\quad x_0 + x_3 = 0, \quad x_2 + x_4 = 0,\\
L_6 : &\quad x_0 + x_4 = 0, \quad x_2 + x_3 = 0,\\
L_7 : &\quad x_0 + x_1 = 0, \quad x_3 + x_4 = 0,\\
L_8 : &\quad x_0 + x_3 = 0, \quad x_1 + x_4 = 0,\\
L_9 : &\quad x_0 + x_4 = 0, \quad x_1 + x_3 = 0,\\
L_{10} : &\quad x_0 + x_1 = 0, \quad x_2 + x_4 = 0,\\
L_{11} : &\quad x_0 + x_2 = 0, \quad x_1 + x_4 = 0,\\
L_{12} : &\quad x_0 + x_4 = 0, \quad x_1 + x_2 = 0,\\
L_{13} : &\quad x_0 + x_1 = 0, \quad x_2 + x_3 = 0,\\
L_{14} : &\quad x_0 + x_2 = 0, \quad x_1 + x_3 = 0,\\
L_{15} : &\quad x_0 + x_3 = 0, \quad x_1 + x_2 = 0.
\end{aligned}$$

(B) Twelve further lines (defined over $\mathbb{Q}(\sqrt{5})$). Let $\zeta = e^{2\pi i/5}$ be a primitive fifth root of unity. For each of the following 12 cyclic orders of $\{0, 1, 2, 3, 4\}$ (taken up to reversal), define a line by the two linear equations

$$x_{i_0} + \zeta^2 x_{i_1} + \zeta^4 x_{i_2} + \zeta\, x_{i_3} + \zeta^3 x_{i_4} = 0, \qquad x_{i_0} + \zeta^3 x_{i_1} + \zeta\, x_{i_2} + \zeta^4 x_{i_3} + \zeta^2 x_{i_4} = 0.$$

(The two equations are complex conjugates; taking real and imaginary parts yields two real linear equations, exhibiting these lines as real.) Concretely, we obtain the remaining 12 lines:

$$
\begin{array}{lll}
L_{16}: & x_0+\zeta^2x_1+\zeta^4x_2+\zeta\,x_3+\zeta^3x_4=0, & x_0+\zeta^3x_1+\zeta\,x_2+\zeta^4x_3+\zeta^2x_4=0,\\
L_{17}: & x_0+\zeta^2x_1+\zeta^4x_2+\zeta\,x_4+\zeta^3x_3=0, & x_0+\zeta^3x_1+\zeta\,x_2+\zeta^4x_4+\zeta^2x_3=0,\\
L_{18}: & x_0+\zeta^2x_1+\zeta^4x_3+\zeta\,x_2+\zeta^3x_4=0, & x_0+\zeta^3x_1+\zeta\,x_3+\zeta^4x_2+\zeta^2x_4=0,\\
L_{19}: & x_0+\zeta^2x_1+\zeta^4x_3+\zeta\,x_4+\zeta^3x_2=0, & x_0+\zeta^3x_1+\zeta\,x_3+\zeta^4x_4+\zeta^2x_2=0,\\
L_{20}: & x_0+\zeta^2x_1+\zeta^4x_4+\zeta\,x_2+\zeta^3x_3=0, & x_0+\zeta^3x_1+\zeta\,x_4+\zeta^4x_2+\zeta^2x_3=0,\\
L_{21}: & x_0+\zeta^2x_1+\zeta^4x_4+\zeta\,x_3+\zeta^3x_2=0, & x_0+\zeta^3x_1+\zeta\,x_4+\zeta^4x_3+\zeta^2x_2=0,\\
L_{22}: & x_0+\zeta^2x_2+\zeta^4x_1+\zeta\,x_3+\zeta^3x_4=0, & x_0+\zeta^3x_2+\zeta\,x_1+\zeta^4x_3+\zeta^2x_4=0,\\
L_{23}: & x_0+\zeta^2x_2+\zeta^4x_1+\zeta\,x_4+\zeta^3x_3=0, & x_0+\zeta^3x_2+\zeta\,x_1+\zeta^4x_4+\zeta^2x_3=0,\\
L_{24}: & x_0+\zeta^2x_2+\zeta^4x_3+\zeta\,x_1+\zeta^3x_4=0, & x_0+\zeta^3x_2+\zeta\,x_3+\zeta^4x_1+\zeta^2x_4=0,\\
L_{25}: & x_0+\zeta^2x_2+\zeta^4x_4+\zeta\,x_1+\zeta^3x_3=0, & x_0+\zeta^3x_2+\zeta\,x_4+\zeta^4x_1+\zeta^2x_3=0,\\
L_{26}: & x_0+\zeta^2x_3+\zeta^4x_1+\zeta\,x_2+\zeta^3x_4=0, & x_0+\zeta^3x_3+\zeta\,x_1+\zeta^4x_2+\zeta^2x_4=0,\\
L_{27}: & x_0+\zeta^2x_3+\zeta^4x_2+\zeta\,x_1+\zeta^3x_4=0, & x_0+\zeta^3x_3+\zeta\,x_2+\zeta^4x_1+\zeta^2x_4=0.
\end{array}
$$

To work in $\mathbb{P}^3$ with coordinates $[x_0 : x_1 : x_2 : x_3]$ (eliminating $x_4 = -(x_0 + x_1 + x_2 + x_3)$), one can substitute this expression for x_4 into the above linear equations to obtain each L_i as the intersection of two planes in $\mathbb{P}^3$.

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