Exercises of the Algebraic Geometry course held by Prof. Ugo Bruzzo

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These notes collect a series of solved exercises from the book *Algebraic Geometry* by *R. Hartshorne* [**Ha**]. Chapters and exercises are numbered as in the book.

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Exercise 1.8. *For any open subset $U \subseteq X$, show that the functor $\Gamma(U,-)$ from sheaves on X to abelian groups is a left exact functor, i.e. if*

$$0 \mapsto \mathcal{F}^{'} \stackrel{\phi}{\rightarrow} \mathcal{F} \stackrel{\psi}{\rightarrow} \mathcal{F}^{''}$$

is an exact sequence of sheaves, then

$$0 \mapsto \Gamma(U,\mathcal{F}^{'}) \stackrel{\phi(U)}{\rightarrow} \Gamma(U,\mathcal{F}) \stackrel{\psi(U)}{\rightarrow} \Gamma(U,\mathcal{F}^{''})$$

is an exact sequence of groups.

PROOF. Since ϕ is an injective morphism of sheaves, $\phi(U)$ is injective for any open subset $U \subseteq X$. So it is enough to prove that

$$Im(\phi(U)) = ker(\psi(U))$$

for any $U \subseteq X$. Since

$$0 \mapsto \mathcal{F}^{'} \stackrel{\phi}{\rightarrow} \mathcal{F} \stackrel{\psi}{\rightarrow} \mathcal{F}^{''}$$

is exact, the induced sequence on stalks

$$0 \mapsto \mathcal{F}^{'}_p \stackrel{\phi_p}{\rightarrow} \mathcal{F}_p \stackrel{\psi_p}{\rightarrow} \mathcal{F}^{''}_p$$

is exact for any $p \in X$. Let $s \in \Gamma(U,\mathcal{F}^{'})$ be a section of $\mathcal{F}^{'}$ on U, then $\psi_p((\phi_p)(s_p)) = 0$ for any $p \in U$, that is $\psi(\phi(s))_p = 0$ for any $p \in U$. So for any $p \in U$ there is an open neighborhood U_p of p in U such that $\psi(\phi(s))_{|U_p} = 0$. So $\psi(U)(\phi(U)(s)) = 0$ and $Im(\phi(U)) \subseteq ker(\psi(U))$.
Now take $v \in ker(\psi(U))$, then for any $p \in U$ there exits $s_p \in \mathcal{F}^{'}_p$ such that $\phi_p(s_p) = v_p$. Thus the are an open covering $\{U_i\}$ of U and sections $s_i \in \mathcal{F}^{'}(U_i)$ such that $\phi(s_i) = v_{|U_i}$. Now

$$\phi(s_{i|U_i \cap U_j}) = v_{|U_i \cap U_j} = \phi(s_{j|U_i \cap U_j}),$$

for any i, j, and since ϕ in injective we get

$$s_{i|U_i \cap U_j} = s_{j|U_i \cap U_j}$$

for any i, j. Since $\mathcal{F}^{'}$ is a sheaf there is a section $s \in \mathcal{F}^{'}(U)$ such that $s_{|U_i} = s_i$ for any i. Now, from $\phi(s_{|U_i}) = v_{|U_i}$ for any i we get $\phi(U)(s) = v$ and $ker(\psi(U)) \subseteq Im(\phi(U))$. ♠

Remark. The functor $\Gamma(U,-)$ need not to be exact, if $\psi : \mathcal{F} \rightarrow \mathcal{F}^{''}$ is surjective the maps on sections $\psi(U) : \mathcal{F}(U) \rightarrow \mathcal{F}^{''}(U)$ need not to be surjective. So it make sense to consider its right derived functors and to define the cohomology of a sheaf. If X is a topological space the category of sheaves of abelian groups on X has enough injectives, that is any sheaf $\mathcal{F}$ on X admits an injective resolution

$$0 \mapsto \mathcal{F} \rightarrow \mathcal{I}^0 \rightarrow \mathcal{I}^1 \rightarrow ...$$

of injective sheaves $\mathcal{I}^j$. The cohomology groups of $\mathcal{F}$ are defined as $H^i(X,\mathcal{F}) := h^i(\Gamma(X,\mathcal{I}^\bullet))$. Since any two resolution are homotopy equivalent the definition does not depend on the one we choose.

Exercise 1.17. *Skyscraper Sheaves. Let X be a topological space, let p be a point, and let A be an abelian group. Define a sheaf $i_p(A)$ as follows: $i_p(A)(U) = A$ if $p \in U$, 0 otherwise. Verify that the stalk of $i_p(A)$ is A at every point $q \in \overline{\{p\}}$, and 0 elsewhere. Show that this sheaf could also be described as $i_*(A)$, where A denotes the constant sheaf A on the closed subspace $\overline{\{p\}}$, and $i : \overline{\{p\}} \to X$ is the inclusion.*

PROOF. If $U \subseteq X$ is an open subset and $q \in U \cap \overline{\{p\}}$ then $p \in U$. In fact if $p \in U^c$ then $\overline{\{p\}} \subseteq U^c$ and $\overline{\{p\}} \cap U = \emptyset$.
Take a point $q \in \overline{\{p\}}$, then any open subset $U \subseteq X$ containing q has to contain p. So $i_p(U) = A$ for any open subset U containing q. The stalk $i_p(A)_q$ is the direct limit

$$i_p(A)_q = \varinjlim_{q \in U} i_p(U) = \varinjlim_{q\in U} A = A.$$

Now take $q \notin \overline{\{p\}}$, then there is a closed subset C containing p such that $q \notin C$, so $V = C^c$ is an open subset containing q such that $p \notin U$.
Since $i_p(V) = 0$ any section s of $i_p(A)$ on V is zero and considering the stalk on q we have $s_q = 0$ for any $s_q \in i_p(A)_q$.
The direct image sheaf is defined as $i_*(A)(U) := A(i^{-1}(U))$. If $p \in U$ then $i^{-1}(U) = \overline{\{p\}}$ and $i_*(A)(U) = A$. If $p \notin U$ then $i^{-1}(U) = \emptyset$ and $i_p(A)(\emptyset) = 0$. ♠

Remark. Let C be a smooth projective curve over a field k, and $p \in C$ be a point. The ideal sheaf $\mathcal{I}_p$ is the sheaf of regular function on X vanishing at p, it is the invertible sheaf $\mathcal{O}_C(-p)$. We have an exact sequence

$$0 \mapsto \mathcal{O}_C(-p) \to \mathcal{O}_C \to \mathcal{O}_p \mapsto 0.$$

The structure sheaf $\mathcal{O}_p$ of the point p is a skyscraper sheaf. Its stalk is isomorphic to the base field k on p and zero elsewhere.

Exercise 1.21. *Some Examples of Sheaves on Varieties. Let X be a variety over an algebraically closed field k. Let $\mathcal{O}_X$ be the sheaf of regular functions on X.*

(a) Let Y be a closed subset of X. For each open set $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the ideal in the ring $\mathcal{O}_X(U)$ consisting of those regular functions which vanish at all points of $Y \cap U$. Show that the presheaf $U \mapsto \mathcal{I}_Y(U)$ is a sheaf. It is called the sheaf of ideals $\mathcal{I}_Y$ of Y, and it is a subsheaf of the sheaf of rings $\mathcal{O}_X$.

PROOF. Let $U \subseteq X$ be an open subset, and let $\{U_i\}$ be an open cover of U. Consider a collection of regular functions $f_i \in \mathcal{I}_Y(U_i)$ such that $f_{i|U_i\cap U_j} = f_{j|U_i \cap U_j}$ for any i,j. In particular $f_i \in \mathcal{O}_X(U_i)$, and since $\mathcal{O}_X$ is a sheaf there exists a regular function $f \in \mathcal{O}_X(U)$ such that $f_{|U_i} = f_i$ for any i. Let $y \in Y \cap U$ be a point, then $y \in Y \cap U_i$ for some i. Since on U_i by construction $f = f_i$, we have $f(y) = f_i(y) = 0$. That is $f \in \mathcal{I}_Y(U)$. ♠

(b) If Y is a subvariety, then the quotient $\mathcal{O}_X/\mathcal{I}_Y$ is isomorphic to $i_*(\mathcal{O}_Y)$, where $i : Y \to X$ is the inclusion, and $\mathcal{O}_Y$ is the sheaf of regular functions on Y.

PROOF. Restriction of regular functions

$$\mathcal{O}_X(U) \to i_*(\mathcal{O}_Y)(U) = \mathcal{O}_Y(Y \cap U),\ f \mapsto f_{|Y\cap U},$$

gives a morphism of sheaves $\mathcal{O}_X \to i_*\mathcal{O}_Y$. By definition of $\mathcal{I}_Y$ the sequence

$$0 \mapsto \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y$$

is exact. Let $x \in X$ be a point. We distinguish two cases.

- If $x \notin Y$ then there is an open neighborhood U_x of x such that $U_x \cap Y = \emptyset$, and the stalk $i_*\mathcal{O}_{Y,x}$ is zero. So the morphism on the stalks
$$\mathcal{O}_{X,x} \to i_*\mathcal{O}_{Y,x}$$
is trivially surjective.
- If $x \in Y$ and $f_y \in i_*\mathcal{O}_{Y,x}$ there exists an open neighborhood U_x of x in X and a section $f \in i_*\mathcal{O}_Y(U_x)$ representing f_y. We can assume U_x to be affine, then the inclusion $U_x \cap Y \to U_x$ corresponds to a surjection between the coordinate rings $A(U_x) \to A(U_x \cap Y)$. So there exists a section $s \in \mathcal{O}_X(U_x)$ restricting to f and again the morphism on the stalks $\mathcal{O}_{X,x} \to i_*\mathcal{O}_{Y,x}$ is surjective.

We conclude that the sequence
$$0 \mapsto \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \mapsto 0$$
is exact, and $i_*\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{I}_Y$. ♠

(c) *Now let $X = \mathbb{P}^1$, and let Y be the union of two distinct points $p, q \in X$. Then there is an exact sequence of sheaves*
$$0 \mapsto \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_p \oplus i_*\mathcal{O}_q \mapsto 0.$$
Show however that the induced map on global sections in not surjective. This show that the global section functor $\Gamma(X, -)$ is not right exact.

PROOF. In this case $X = \mathbb{P}^1$ is a complete variety over an algebraically closed field k, then regular functions on X are constant, that is $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. On the other hand $i_*\mathcal{O}_p \oplus i_*\mathcal{O}_q$ is a skyscraper sheaf supported on $Y = \{p, q\}$, so $\Gamma(Y, i_*\mathcal{O}_p \oplus i_*\mathcal{O}_q) = k \oplus k$. Clearly a surjection $k \to k \oplus k$ does not exist. ♠

(d) *Again let $X = \mathbb{P}^1$, and let $\mathcal{O}$ be the sheaf of regular functions. Let $\mathcal{K}$ the constant sheaf on X associated to the function field K of X. Show that there is a natural injection $\mathcal{O} \to \mathcal{K}$. Show that the quotient sheaf $\mathcal{O}/\mathcal{K}$ is isomorphic to the direct sum of sheaves $\sum_{p \in X} i_p(I_p)$ where I_p is the group $K/\mathcal{O}_p$, and $i_p(I_p)$ denotes the skyscraper sheaf given by I_p at the point p.*

PROOF. A regular function on an open subset $U \subseteq X$ is a rational function $f : U \to k$ such that on an open covering $\{U_i\}$ of U the restricted functions $f_{|U_i}$ are regular on U_i. Then f defines a section in $\mathcal{K}(U)$. In this way we get an injective morphism $\mathcal{O} \hookrightarrow \mathcal{K}$.
For any rational function $f : U \to k$ we can consider its image in the quotient $K/\mathcal{O}_p$ for any $p \in U$. So we have a morphism $\mathcal{K} \to \sum_{p \in X} i_p(I_p)$ whose kernel clearly contains $\mathcal{O}$. To conclude we have to prove that the sequence
$$\mathcal{O} \to \mathcal{K} \to \sum_{p \in X} i_p(I_p)$$
is exact. Thus $\mathcal{K}/\mathcal{O} \cong \sum_{p \in X} i_p(I_p)$. On the stalk at $q \in X$ the sequence is $\mathcal{O}_q \to \mathcal{K}_q \to (\sum_{p \in X} i_p(I_p))_q$. Now it is enough to observe that $\mathcal{K}_q \cong K$, $(\sum_{p \in X} i_p(I_p))_q \cong K/\mathcal{O}_q$, and the sequence
$$0 \mapsto \mathcal{O}_q \to K \to K/\mathcal{O}_q \mapsto 0.$$
is exact. ♠

(e) *Finally show that in the case of (d) the sequence*
$$0 \mapsto \Gamma(X, \mathcal{O}_X) \to \Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O}_X) \mapsto 0$$
is exact.

PROOF. The functor Γ is left exact, so it is enough to prove that $\Gamma(X,\mathcal{K}) \to \Gamma(X,\mathcal{K}/\mathcal{O}_X)$ is surjective.
Since $\mathcal{K}/\mathcal{O}_X \cong \sum_{p\in X} i_p(I_p)$ we have to prove that given a rational function $f \in K$ and a point p there exists a rational function $g \in K$ such that $g \in \mathcal{O}_q$ for any $q \neq p$ and $g - f \in \mathcal{O}_p$.
We can write f as a ratio of polynomials

$$f = \frac{P(z)}{Q(z)} = \frac{\prod(z-a_i)}{\prod(z-b_j)} = z^{-h}\frac{\prod(z-a_i)}{\prod(z-c_j)},$$

and assume $p = 0 \in \mathbb{A}^1 \subset \mathbb{P}^1$. If $h \leq 0$ then f is regular in p, and $g = \textit{constant}$ will work. If $h > 0$ write $\prod(z-a_i) = \sum_i \alpha_i z^i$, $\prod(z-c_j) = \sum_i \beta_i z^i$ and choose

$$g = \frac{\sum_{i=0}^{h} \gamma_i}{z^h},$$

where the γ_i are defined recursively as $\gamma_0 = \frac{\alpha_0}{\beta_0}$, $\gamma_i = \frac{\alpha_i - \sum_{j=0}^{i-1} c_j\beta_{i-j}}{\beta_0}$ for $i > 0$. With these choices $g \in \mathcal{O}_q$ for any $q \neq p$ and $g - f \in \mathcal{O}_p$. ♠

Remark. With a bit more technology (e) can be solved easily as follows.
The exact sequence of sheaves

$$0 \mapsto \mathcal{O}_X \to \mathcal{K} \to \mathcal{K}/\mathcal{O}_X \mapsto 0$$

yields the following exact sequence in cohomology

$$0 \mapsto \Gamma(X,\mathcal{O}_X) \to \Gamma(X,\mathcal{K}) \to \Gamma(X,\mathcal{K}/\mathcal{O}_X) \to H^1(X,\mathcal{O}_X) \to ...$$

Now, it is enough to observe that, by Serre duality, on $\mathbb{P}^1$ we have

$$H^1(X,\mathcal{O}_X) \cong H^0(X,\omega_X) = H^0(X,\mathcal{O}_X(-2)) = 0,$$

being $\omega_X \cong \mathcal{O}_{\mathbb{P}^1}(-2)$ the canonical sheaf of $\mathbb{P}^1$.

2 - Schemes

Exercise 2.5. *Describe* Spec$(\mathbb{Z})$*, and show that it is a final object for the category of schemes, i.e. each scheme X admits a unique morphism to* Spec$(\mathbb{Z})$*.*

PROOF. The ideals of $\mathbb{Z}$ are of the form $I = (n)$, and prime ideals are of the form $P = (p)$ such that $p \in \mathbb{Z}$ is a prime integer. Furthermore there is a generic point corresponding to the ideal (0).
The closed subsets of Spec$(\mathbb{Z})$ are of the form

$$D(n) = \{(p) \mid p|n\}.$$

The functors Spec and Γ are adjoint. If X is a scheme and A is a ring, there is a bijective correspondence between morphisms $X \to \mathrm{Spec}(A)$ and morphisms of rings $A \to \Gamma(X,\mathcal{O}_X)$. Recall that we consider morphisms of rings with identity, so there is an unique morphism $\mathbb{Z} \to \Gamma(X,\mathcal{O}_X)$. ♠

Exercise 2.7. *Let X be a scheme. For any $x \in X$, let $\mathcal{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the residue field of x on X to be the field $k(x) = \mathcal{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of* Spec(K) *to X it is equivalent to giving a point $x \in X$ and an inclusion map $k(x) \to K$.*

PROOF. Assume to have a morphism $(f, f^\sharp) : \mathrm{Spec}(K) \to X$. Immediately we get a point $x = f(\mathrm{Spec}(K)) \in X$. Let Spec$(A)$ be an open affine subset of X containing x, then $x \in \mathrm{Spec}(A)$ corresponds to a prime ideal $\mathfrak{p}$. The morphism $\mathrm{Spec}(K) \to \mathrm{Spec}(A)$ induces a morphism of rings $\alpha : A \to K$ whose kernel is $\mathfrak{p}$, finally α induces an inclusion $A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p} = k(x) \hookrightarrow K$.

Fix a point $x \in X$ and assume to have a morphism $k(x) \to K$. The topological space $\operatorname{Spec}(K)$ has a unique point, given a point $x \in X$ we get a continuous map

$$f_x : \operatorname{Spec}(K) \to X, \operatorname{Spec}(K) \mapsto x.$$

The sheaf $f_{x*}(\mathcal{O}_{\operatorname{Spec}(K)})$ is a skyscraper sheaf whose stalk in $x \in X$ is isomorphic to K. So to give a morphism of sheaves

$$f_x^\sharp : \mathcal{O}_X \to f_{x*}(\mathcal{O}_{\operatorname{Spec}(K)})$$

is equivalent to giving for any open subset $U \subseteq X$ containing x a natural morphism $\mathcal{O}_X(U) \to K$. We take this morphism to be the composition

$$\mathcal{O}_x(U) \to \mathcal{O}_{X,x} \to k(x) \to K.$$

♠

Exercise 2.8. *Let X be a scheme. For any point $x \in X$, we define the Zariski tangent space T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\epsilon]/\epsilon^2$ be the ring of dual numbers over k. Show that to give a k-morphism of* $\operatorname{Spec} k[\epsilon]/\epsilon^2$ *to X is equivalent to giving a point $x \in X$, rational over k (i.e. such that $k(x) = k$), and an element of T_x.*

Proof. Let $D = \operatorname{Spec} k[\epsilon]/\epsilon^2$. Suppose to have a morphism

$$(f, f^\sharp) : D \to X.$$

We take $x \in X$ to be the image via f of the unique point in D. The morphism induces an inclusion $k(x) \to k$. On the other hand, since f is a k-morphism, the inclusion has to be compatible with the structure morphisms over k. So we have a chain of inclusions $k \subseteq k(x) \subseteq k$, and $k(x) = k$. The induced morphism on the stalks $f_x^\sharp : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2$, maps the maximal ideal $\mathfrak{m}_x$ to the maximal ideal (ϵ), so $f_x^\sharp(\mathfrak{m}_x^2) \subseteq (\epsilon^2)$. Then we get a k-morphism

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to k[\epsilon]/\epsilon^2 \to k.$$

Now, suppose to have a point $x \in X$ with $k(x) = k$ and a k-linear morphism $L : \mathfrak{m}_x/\mathfrak{m}_x^2 \to k$. We consider the continuous map $f : D \to X$ mapping D to $x \in X$. Consider now the morphisms $ev : \mathcal{O}_{X,x} \to k(x) = k$, and

$$\phi : \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2,\ g \mapsto ev(g) + L(g - ev(g))\epsilon.$$

Note that $g - ev(g) \in \mathfrak{m}_x$, and that ϕ is a k-linear ring homomorphism. Now for any open subset $U \subseteq X$ containing x we consider the composition

$$\mathcal{O}_X(U) \to \mathcal{O}_{X,x} \to k[\epsilon]/\epsilon^2.$$

This gives a morphism of sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_D$. ♠

Remark. The meaning of the previous exercise is that to give a morphism of schemes $\operatorname{Spec}(k[\epsilon]/\epsilon^2) \to X$ is equivalent to giving a k-rational point $x \in X$ and a tangent direction to X at x. Suppose it is given a moduli problem for a certain class of schemes, and let $\pi : X \to D$ by a family of these scheme parametrized by D. Let X_0 be the central fiber of this family. Suppose the moduli problem is represented by a scheme M and let $x_0 \in M$ be the point corresponding to X_0. Then to give morphism from D to M sending D to x_0 is equivalent to give a tangent direction of M at x_0. Naively speaking that's why the scheme D is closely related to the concept of infinitesimal deformation and tangent space to a moduli space.

Exercise 2.16. *Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.*

(a) If $U = \text{Spec}(B)$ is an open affine subscheme of X, and if $\overline{f} \in B = \Gamma(U, \mathcal{O}_{X|U})$ is the restriction of f, show that $U \cap X_f = D(\overline{f})$. Conclude that X_f is an open subset of X.

PROOF. Let $x \in U = \text{Spec}(B)$ be the point associated to the prime ideal $\mathfrak{p} \subset B$. The maximal ideal of $\mathcal{O}_x$ is $\mathfrak{p}B_{\mathfrak{p}}$. We have

$$x \in U \cap X_f \iff \overline{f} \notin \mathfrak{p} \iff x \in D(\overline{f}).$$

Since for any affine open subset $U \subseteq X$ the intersection $U \cap X_f$ is open we conclude that X_f is open. ♠

Exercise 2.18. *In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.*

(a) Let A be a ring $X = \text{Spec}(A)$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

PROOF. We have, $f^n = 0$ for some non negative integer $n \iff f \in \mathcal{N} = \bigcap_{\mathfrak{p} \subset A} \mathfrak{p} \iff f \in \mathfrak{p}$ for any prime ideal $\mathfrak{p} \subset A \iff \mathfrak{p} \notin D(f)$ for any prime ideal $\mathfrak{p} \subset A$. ♠

(b) Let $\phi : A \to B$ be a homomorphism of rings, and let $f : Y = \text{Spec}(B) \to X = \text{Spec}(A)$ be the induced morphism of affine schemes. Show that ϕ is injective if and only if the map of sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore in that case f is dominant, i.e. $f(Y)$ is dense in X.

PROOF. Take a point $\mathfrak{p} \in \text{Spec}(A)$, the stalk $(f_*\mathcal{O}_{\text{Spec}(B)})_{\mathfrak{p}}$ is $B \otimes_A A_{\mathfrak{p}}$. If ϕ is injective clearly the induced morphism $(\mathcal{O}_{\text{Spec}(A)})_{\mathfrak{p}} \to (f_*\mathcal{O}_{\text{Spec}(B)})_{\mathfrak{p}}$ is injective for any $\mathfrak{p} \in \text{Spec}(A)$, that is $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective.
Conversely if $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective, then the induced morphism on the global sections $\Gamma(X, \mathcal{O}_X) \to \Gamma(Y, f_*\mathcal{O}_Y)$ is injective, but this morphism is exactly $\phi : A \to B$.
Let U be the complement of $\overline{f(Y)}$ in X. The open subset U is covered by subsets of the form $D(f)$, with $f \in \phi^{-1}(\mathfrak{p})$ for any $\mathfrak{p} \in \text{Spec}(B)$. Then $\phi(f) \in \mathfrak{p}$ for any $\mathfrak{p} \in \text{Spec}(B)$, so $\phi(f)$ is nilpotent. Since ϕ is injective f is nilpotent and $D(f) = \emptyset$. ♠

(c) With the same notation, show that if ϕ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjective.

PROOF. The morphism ϕ induces a bijection between the prime ideals of $B \cong A/Ker(\phi)$ and the prime ideals of A containing $Ker(\phi)$. The preimage of the open subset $D(f) \subseteq \text{Spec}(A)$ is $D(f + Ker(\phi)) \subseteq \text{Spec}(A/Ker(\phi))$, so principal open subsets of $\text{Spec}(A/Ker(\phi))$ are open in the image with respect the induced topology. So $f : Y \to X$ is continuous and open, therefore it is an homeomorphism onto its image. Finally if ϕ is surjective, then the induce morphism on the stalks $A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}$ is surjective. ♠

3 - First Properties of Schemes

Exercise 3.10.

(a) If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $sp(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

PROOF. Let $y \in Y$ be a point, and let $k(y)$ be the residue field of y. We have an induced morphism $\operatorname{Spec} k(y) \to Y$. The fibre of the morphism f over $y \in Y$ is the scheme

$$X_y = X \times_Y \operatorname{Spec} k(y).$$

The continuous map induce by the morphism of scheme $\operatorname{Spec} k(y) \to Y$ simply maps $g : \operatorname{Spec}(k(y)) \mapsto y$. The underlying topological space of X_y is

$$sp(X_y) = \{(x, \xi) \mid f(x) = g(\xi) = y\} = f^{-1}(y)$$

clearly the projection $sp(X_y) \to f^{-1}(y),\ (x, \xi) \mapsto x$ is an homeomorphism. ♠

(b) Let $X = \operatorname{Spec} k[s,t]/(s - t^2)$, let $Y = \operatorname{Spec} k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \mapsto s$. If $y \in Y$ is a point $a \in k$ with $a \neq 0$, show that the fiber X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$ show that the fiber X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k algebraically closed).

PROOF. The fibre is the spectrum of the tensor product

$$k[s,t]/(s - t^2) \otimes_{k[s]} k[s]/(s - a) \cong k[s,t]/(s - t^2, s - a).$$

The fibre is the zero dimensional subscheme cut out on the parabola $s = t^2$ by the line $s = a$. If $a \neq 0$ this intersection consists of two distinct points with residue field k. On the other hand, if a $a = 0$, the line $s = 0$ is tangent to the parabola at the origin. So the ring of the fibre is $k[t]/(t^2)$, that is X_0 is a nonreduced double point whose residue field is an extension of degree 2 of the residue field of the generic point $\eta \in Y$. ♠

Exercise 3.11.

(a) Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

PROOF. Since both X and X' can be covered by open affine subsets, we can assume $X = \operatorname{Spec}(A)$ and $X' = \operatorname{Spec}(B)$ to be affine. Since f is a closed immersion Y is an affine subscheme of X, and we can write $Y = \operatorname{Spec}(A/I)$. Let $\phi : A \to A/I$ be the morphism induced by f. Consider $f' : \operatorname{Spec}(B \otimes_A (A/I)) \to \operatorname{Spec}(B)$. Now $B \otimes_A (A/I) \cong B/\langle \phi(I) \rangle$, so f' is a closed immersion. ♠

4 - Separated and Proper Morphisms

Exercise 4.2. *Let S be a scheme, let X be a reduced scheme over S, and let Y be a separated scheme over S. Let f and g be two S-morphisms of X to Y which agree on an open dense subset of X. Show that $f = g$. Give examples to show that this result fails if either (a) X is nonreduced, or (b) Y is nonseparated.*

PROOF. Let $\alpha : X \to Y \times_S Y,\ x \mapsto (f(x), g(x))$ be the diagonal morphism of f, g. Let $U \subseteq X$ be the open subset on which $f_{|U} = g_{|U}$. Consider the fiber product $W = X \times_{Y \times_S Y} Y$ and the following diagram

$$\begin{array}{ccc} U & = & U \\ \downarrow & & \downarrow \\ W & \xrightarrow{\overline{\Delta}} & X \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ Y & \xrightarrow{\Delta} & Y \times_S Y \end{array}$$

Note that since f and g agree on U we have $\alpha(U) \subseteq \Delta(Y)$, so $U \subseteq W$ is an open subset also in the fiber product.
The morphism $\Delta : Y \to Y \times_S Y$ is a closed immersion, since Y is separated over S, furthermore closed immersion are stable under base change, so $\overline{\Delta} : W \to X$ is a closed immersion as well. The injection $U \hookrightarrow X$ factors through W which is a closed subset of X, and since U is dense we get $sp(W) = sp(X)$. Let $V = \mathrm{Spec}(A) \subseteq X$ be an open affine subset. The morphism $\overline{\Delta}_{|V} : W \cap V \to V$ is a closed immersion. So $W \cap V$ is a closed subscheme homeomorphic to V and determined by an ideal I of A. The morphism $\mathrm{Spec}(A/I) \to \mathrm{Spec}(A)$ is injective, so $I \subseteq \mathcal{N}$. On the other hand $I = \mathcal{N} = 0$ since A is reduced. So $W \cap V = V$ for any $V \subseteq X$ open affine, and this implies $W = X$ as schemes.
Now it is enough to recall that

$$W = \{(x,y) \mid \alpha(x) = (f(x), g(x)) = \Delta(y) = (y,y)\} = \{x \mid f(x) = g(y)\}.$$

Since $W = X$ we get $f = g$.

(a) Consider $X = Y = \mathrm{Spec}(k[x,y]/(x^2, xy))$ the affine line with the origin counted twice. The scheme X is nonreduced, it has nilpotents at the origin. The identity $Id : X \to Y$, $x \mapsto x$, $y \mapsto y$, and the morphism $f : X \to Y$, $x \mapsto 0$, $y \mapsto y$ mapping X on the affine line, agree on a dense open subset of X but they differs at the origin. Indeed f kill the nilpotent elements.

(b) Take $X = \mathbb{A}^1$ and $Y = \mathbb{A}^1_{0_1,0_2}$ be the affine line with two origins. Consider $f_1 : X \to Y$, mapping $x \mapsto x$ for any $x \neq 0$ and $0 \mapsto 0_1$ and $f_2 : X \to Y$, mapping $x \mapsto x$ for any $x \neq 0$ and $0 \mapsto 0_2$. Clearly f_1, f_2 coincides on $X \setminus \{0\}$ but $f_1 \neq f_2$.

♠

Exercise 4.3. *Let X be a separated scheme over an affine scheme S. Let U, V be open affine subsets of X. Then $U \cap V$ is also affine. Give an example to show that this fails if X is not separated.*

PROOF. Consider the fiber product

$$\begin{array}{ccc} W = U \cap V & \xrightarrow{\alpha} & U \times_S V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\Delta} & X \times_S X \end{array}$$

Now, X is separated over S so Δ is a closed immersion. Closed immersions are stable under base change, then α is a closed immersion as well. Furthermore $U \times_S V$ is affine, being U, V, S affine, and $U \cap V$ is a closed subscheme of the affine scheme $U \times_S V$. Then $U \cap W$ has to be affine.
Take $X = \mathbb{A}^2_{0_1,0_2}$ the affine plane with doubled origin, U a copy of the affine plane, and V the other copy of $\mathbb{A}^2$. Then $U \cap V = \mathbb{A}^2 \setminus \{0\}$, and this is not affine. Suppose $\mathbb{A}^2 \setminus \{0\}$ to be affine, then $\mathbb{A}^2 \setminus \{0\} \cong \mathrm{Spec}(A)$. In this case A is the ring of regular functions $A \cong \Gamma(\mathbb{A}^2 \setminus \{0\}, \mathcal{O}_{\mathbb{A}^2 \setminus \{0\}})$. On the other hand the origin has codimension two in $\mathbb{A}^2$ and $\mathbb{A}^2$ is a normal scheme. So any regular function on $\mathbb{A}^2 \setminus \{0\}$ extends to a regular function on $\mathbb{A}^2$, that is $A \cong \Gamma(\mathbb{A}^2, \mathcal{O}_{\mathbb{A}^2})$. So $\mathbb{A}^2 \setminus \{0\} = \mathbb{A}^2$, a contradiction. ♠

Exercise 4.4. *Let $f : X \to Y$ be a morphism of separated schemes of finite type over a noetherian scheme S. Let Z be a closed subscheme of X which is proper over S. Show that $f(Z)$ is closed in Y, and that $f(Z)$ with its image subscheme structure is proper over S.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{f_{|Z}} & Y \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\beta} & \\ & S & \end{array}$$

Now, Z is proper over S, that is α is a proper morphism. So $\beta \circ f_{|Z} = \alpha$ is proper, furthermore β is separated. We conclude that $f_{|Z}$ is proper by Corollary 4.8. Since any proper morphism is closed $f(Z)$ is closed in Y.
To prove that $f(Z)$ is closed over S we will show that $f(Z) \to S$ is of finite type, separated, and universally closed.
Since $f(Z)$ is a closed subscheme of Y, and Y is of finite type over S we have that $f(Z)$ is of finite type over S.
Since $f(Z) \hookrightarrow Y$ is a closed immersion, and closed immersions are stable under base change, $f(Z) \times_S f(Z) \to Y \times_S Y$ is a closed immersion. Then $f(Z) \to S$ is separated.
Let V be a scheme over S. Consider the following situation

$$\begin{array}{ccc} V \times_S Z & \longrightarrow & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ V \times_S f(Z) & \longrightarrow & f(Z) \\ {\scriptstyle \gamma}\downarrow & & \downarrow{\scriptstyle \beta} \\ V & \longrightarrow & S \end{array}$$

Let $R \subseteq V \times_S f(Z)$ be a closed subset. Then $g^{-1}(R)$ is closed in $V \times_S Z$. Furthermore $(\gamma \circ g)(g^{-1}(R))$ is closed in V being $Z \to S$ universally closed. Finally, being g surjective, we get $(\gamma \circ g \circ g^{-1})(R) = \gamma(R)$. Then $\gamma(R)$ is closed in V. ♠

Exercise 4.9. *Show that the composition of projective morphisms is projective.*

PROOF. Consider the composition of two projective morphisms

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \pi_1} & {\scriptstyle j}\downarrow & \nearrow{\scriptstyle \pi_2} & \\ \mathbb{P}^n_Y & & \mathbb{P}^m_Z & & \end{array}$$

where i, j are closed immersions, and π_1, π_2 are projections. Let $Seg : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ be the Segre embedding. Considering the following diagram

$$\begin{array}{ccccccc} X & \xrightarrow{i} & \mathbb{P}^n \times Y & \xrightarrow{Id \times j} & \mathbb{P}^n \times \mathbb{P}^m \times Z & & \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \pi_1} & & \downarrow & {\scriptstyle Seg}\searrow & \\ & & Y & \xrightarrow{j} & \mathbb{P}^m \times Z & & \mathbb{P}^{nm+n+m} \times Z \\ & & & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \pi_2} & \swarrow{\scriptstyle \pi} & \\ & & & & Z & & \end{array}$$

it is clear that $g \circ f$ factors as $\pi \circ (Seg \circ (Id \times j) \circ i)$, and since Seg is a closed immersion, we conclude that $g \circ f$ is projective. ♠

Remark. If $X \to \mathrm{Spec}(k)$ is a scheme over a field k, the structure morphism is projective if it factors through a closed immersion $X \to \mathbb{P}^n_k$. We recover the usual notion of projective scheme. As the projective of a scheme over a field is related to the existence of an ample line bundle on X, the projectivity of a morphism $f : X \to Y$ is related to the existence of a relatively ample line bundle on X, that is a line bundle on X whose restriction to the fibers of f is ample.

5 - Sheaves of Modules

Let X be a scheme, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. For any open subset $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and we can consider the sheaves associated to the presheaves

$$U \mapsto T^r\mathcal{F}(U),\ S^r\mathcal{F}(U),\ \bigwedge^r \mathcal{F}(U).$$

These sheaves are respectively the *tensor algebra*, the *symmetric algebra*, and the *exterior algebra* of $\mathcal{F}$. If $\mathcal{F}$ is locally free of rank n then $T^r\mathcal{F},\ S^r\mathcal{F},\ \bigwedge^r \mathcal{F}$ are locally free of rank n^r, $\binom{n+r-1}{n-1}$ and $\binom{n}{r}$ respectively. Consider an exact sequence

$$0 \mapsto \mathcal{F}^{'} \to \mathcal{F} \to \mathcal{F}^{''} \mapsto 0$$

of $\mathcal{O}_X$-modules. For any r there is a finite filtration

$$\bigwedge^r \mathcal{F} = F^0 \supseteq F^1 \supseteq ... \supseteq F^r \supseteq F^{r+1} = 0$$

with quotients

$$F^p/F^{p+1} \cong \bigwedge^p \mathcal{F}^{'} \otimes \bigwedge^{r-p} \mathcal{F}^{''}$$

for each p. In particular

$$\bigwedge^n \mathcal{F} \cong \bigwedge^{n^{'}} \mathcal{F}^{'} \otimes \bigwedge^{n^{''}} \mathcal{F}^{''}.$$

This formula is very useful in a number of contests. As instance we can derive the *adjunction formula*.

Exercise - Adjunction Formula. Let Y be a smooth subvariety of a smooth variety X. Consider the exact sequence

$$0 \mapsto T_Y \to T_X \to N_{Y/X} \mapsto 0.$$

Taking exterior powers we get

$$\bigwedge^n T_X \cong \bigwedge^m T_Y \otimes \bigwedge^{n-m} N_{X/Y} \otimes \mathcal{O}_X.$$

Then, on the canonical sheaves, we have

$$\omega_Y \cong \omega_X \otimes \bigwedge^{n-m} \check{N}_{X/Y} \otimes \mathcal{O}_X,$$

where $n = \dim(X)$ and $m = \dim(Y)$. In particular for a degree d hypersurface $Y \subseteq \mathbb{P}^n$ the canonical sheaf is given by $\omega_Y \cong \mathcal{O}_X(d-n-1)$, being $\omega_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1)$.

Exercise 5.1-(b), and a consequence. *Let* $(X, \mathcal{O}_X)$ *be a locally ringed space, and let* $\mathcal{E}$ *be a locally free* $\mathcal{O}_X$*-module of finite rank. For any* $\mathcal{O}_X$*-module* $\mathcal{F}$*,*

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \cong \check{\mathcal{E}} \otimes_{\mathcal{O}_X} \mathcal{F}.$$

PROOF. For any $U \subseteq X$ open subset we have to define a morphism

$$(\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes \mathcal{F})(U) \to (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}))(U),$$

that is a morphism

$$Hom(\mathcal{E}_{|U}, \mathcal{O}_{X|U}) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \to Hom(\mathcal{E}_{|U}, \mathcal{F}_{|U}).$$

Note that a section $s \in \mathcal{F}(U)$ yields a morphism

$$\phi_s : \mathcal{O}_{X|U} \to \mathcal{F}_{|U},\ f \mapsto sf.$$

We define our morphism as follows

$$Hom(\mathcal{E}_{|U}, \mathcal{O}_{X|U}) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \to Hom(\mathcal{E}_{|U}, \mathcal{F}_{|U}),\ (\psi, s) \mapsto \phi_s \circ \psi.$$

If $\mathcal{E}$ is locally free, then the stalk $\mathcal{E}_x$ is locally free, and the above morphism if clearly an isomorphism at the level of stalks.
In particular if $\mathcal{E} = \mathcal{F} = \mathcal{L}$ is a line bundle on X, then

$$\check{\mathcal{L}} \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) \cong \mathcal{O}_X.$$

The line bundle on a locally ringed space $(X, \mathcal{O}_X)$ form a group called the *Picard group* of X and denoted by $\mathrm{Pic}(X)$. ♠

Exercise 5.18. *Let* $(X, \mathcal{O}_X)$ *be a scheme. There is a one-to-one correspondence between isomorphism classes of locally free sheaves of rank r on X and isomorphism classes of rank r vector bundle on X.*

PROOF. Let $\pi : E \to X$ be a vector bundle on X. For any $U \subseteq X$ open subset, consider the $\mathcal{O}_X(U)$-module

$$\Gamma(U, E) := \{\sigma : U \to E \mid \pi \circ \sigma = Id_U\}.$$

The correspondence $U \mapsto \Gamma(U, E)$ defined a presheaf $\mathcal{E}$ on X, and since E is a vector bundle $\mathcal{E}$ is indeed a sheaf.
Let $\{U_i\}$ be an open cover of X on which E trivializes. Then $E_{|U_i} \cong U_i \times k^r$, on the other hand the sheaf of sections of $U_i \times k^r$ is isomorphic to $\mathcal{O}^r_{X|U_i}$. We conclude that $\mathcal{E}$ is locally free of rank r.
Now, let $\mathcal{F}$ be a locally free sheaf of rank r. There exists on open covering $\{U_i\}$ of X, and isomorphisms $\phi_i : \mathcal{O}^r_{X|U_i} \to \mathcal{F}_{|U_i}$. These induces on $U_{i,j}$ isomorphisms

$$\phi_{i,j} : \mathcal{O}^r_{X|U_{i,j}} \to \mathcal{O}^r_{X|U_{i,j}}$$

represented by an $r \times r$ matrix with entries in $\mathcal{O}_X(U_{i,j})$. Since $\phi_{i,j}$ is invertible, the matrix A_x is invertible for any $x \in U_{i,j}$. Then we get a morphism

$$g_{i,j} : U_{i,j} \to GL(k^r).$$

The morphisms $\{g_{i,j}\}$ satisfies cocycle conditions, so we can construct the vector bundle F associated to the collection $\{g_{i,j}\}$.
To conclude it is enough to observe that the assignments $E \rightsquigarrow \mathcal{E}$ and $\mathcal{F} \rightsquigarrow F$ are inverse to each other. ♠

Bibliography

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