Exercises of Algebraic Geometry I

Alex Massarenti

ALEX MASSARENTI, IMPA, ESTRADA DONA CASTORINA 110, 22460-320 RIO DE JANEIRO, BRAZIL
E-mail address: `massaren@impa.br`

Contents

Introduction

These notes collect a series of solved exercises for the course of *Algebraic Geometry I*, I gave at IMPA from August 4 to November 26, 2014. Most of them from the book *Algebraic Geometry* by *R. Hartshorne* [**Har**]. Many others from the notes by *Ph. Ellia* [**PhE**].

CHAPTER 1

Affine varieties

Exercise 1. [**Har**, Exercise 1.1]

(a) The coordinate ring of the curve $C = \{y - x^2 = 0\} \subset \mathbb{A}^2$ is given by
$$A(C) = k[x,y]/(y-x^2) \cong k[x,x^2] \cong k[x].$$
(b) $A(Z) = k[x,y]/(xy-1)$ is isomorphic to the localization of $k[x]$ at x. Let $f : A(Z) \to k[x]$ be a morphism of k-algebras. Since $x \in A(Z)$ is invertible $f(x) \in k$. Therefore, f can not be an isomorphism.

(c) Let $f(x,y)$ be an irreducible quadratic polynomial, and let $F(X,Y,Z)$ be the degree two homogeneous polynomial induced by f. Consider $F_{|\{Z=0\}} = F(X,Y,0)$. This is a degree two homogeneous polynomial on $\mathbb{P}^1$. Therefore, it has two roots counted with multiplicity. If it has a double root this means that the line $\{z=0\}$ is tangent to the conic $C = \{F = 0\}$ in a point $p \in C$. The conic C is isomorphic to $\mathbb{P}^1$. Therefore $C \setminus \{p\} \cong \mathbb{A}^1$, and we recover (a).
If $F(X,Y,0)$ has two distinct roots p,q, then $C \setminus \{p,q\} \cong \mathbb{P}^1 \setminus \{p,q\} \cong \mathbb{A}^1 \setminus \{q\}$ and we are in case (b).

Exercise 2. [**Har**, Exercise 1.3] Consider $Y = \{x^2 - yz = xz - x = 0\} \subset \mathbb{A}^3$. Then
$$Y = \{x^2 - y = z - 1 = 0\} \cup \{x = y = 0\} \cup \{x = z = 0\},$$
and Y is the union of two lines and a plane irreducible curve of degree two. In particular, the coordinate ring of each irreducible component is isomorphic to $k[t]$.

Exercise 3. [**Har**, Exercise 1.5] Let B be a finitely generated k-algebra. Then we may write $B = k[x_1,...,x_n]/I$ for some ideal $I = (f_1,...,f_r)$ in $k[x_1,...,x_n]$. Let $X = \{f_1 = ... = f_r = 0\} \subseteq \mathbb{A}^n$. Let $f \in I(X)$ then, by the Nullstellensatz we have $f^k \in I$ for some $k > 0$. Now, B does not have nilpotents, so $f \in I$. Clearly $I \subseteq I(X)$. This yields $I = I(X)$ and $B \cong A(X)$.
Conversely, assume to have $B = A(X)$ for some algebraic set $X \subset \mathbb{A}^n$. Let $I(X)$ be the ideal of X. Then $B \cong k[x_1,...,x_n]/I(X)$ is a finitely generated k-algebra. Let $f \in B$ be a nilpotent element. Then $f^k = 0$ for some k, that is $f^k \in I$. Since I is radical we get $f \in I$, that is $f = 0$ in B.

Exercise 4. [**Har**, Exercise 1.8] Let $Y \subset \mathbb{A}^n$ be an affine variety of dimension r. Let $H \subset \mathbb{A}^n$ be an hypersurface such that Y is not contained in H and $Y \cap H \neq \emptyset$. Since Y is not contained in H we have $I(H) \nsubseteq I(Y)$. Let f be the polynomial defining H. Then, the irreducible components of $Y \cap H$ corresponds to the minimal prime ideals of $A(Y)$

containing f. Note that $Y \nsubseteq H$ implies that f is not a zero-divisor in $A(Y)$. By the Hauptidealsatz any minimal prime ideal containing f has height one. Finally, by [**Har**, Theorem 1.8A] we get that the any irreducible component of $Y \cap H$ has dimension $\dim(Y) - 1$.

Exercise 5. [**Har**, Exercise 1.9] Let $\mathfrak{a} \subseteq k[x_1, ..., x_n]$ be an ideal that can be generated by r elements $f_1, ..., f_r$. Note that $\{f_i = 0\}$ defines an hypersurface for any $i = 1, ..., r$. We apply r times Exercise 1.8 and we distinguish two cases:

- at any step the variety $H_k = \{f_1 = ...f_k = 0\}$ is not contained in the hypersurface $\{f_{k+1} = 0\}$. Then at each step the dimension of the intersection drops by one. We get that the dimension of each irreducible component of Y is $n - r$,
- if H_k is contained in $\{f_{k+1} = 0\}$ for some k, then the intersection with $\{f_{k+1} = 0\}$ will not drop the dimension. Then each irreducible component of Y has dimension greater than $n - r$.

In any case we have that the dimension of each irreducible component of Y is greater or equal than $n - r$.

Exercise 6. [**Har**, Exercise 1.11] The curve Y is the image of the morphism

$$\begin{array}{rcc} \phi : \mathbb{A}^1 & \longrightarrow & \mathbb{A}^3 \\ t & \longmapsto & (t^3, t^4, t^5) \end{array}$$

Note that since $\mathbb{A}^1$ is irreducible Y is irreducible as well. Therefore $I = I(Y)$ is prime. Furthermore $\dim(Y) = \dim(A(Y)) = 1$ and by [**Har**, Theorem 1.8A] we get $\mathrm{height}(I(Y)) = 2$. Note that the three polynomials $z^2 - x^2y$, $xz - y^2$ and $yz - x^3$ are in $I(Y)$ and they are independent.
Let $J = (z^2 - x^2y, xz - y^2, yz - x^3) \subseteq I(Y)$. By [**Ku**, Page 138] we have that $I(Y) = J$ and that we need three elements to generate $I(Y)$.

Exercise 7. [**Har**, Exercise 1.12] Consider the polynomial

$$f = (x^2 - 1 + iy)(x^2 - 1 - iy) = x^4 - 2x^2 + y^2 + 1.$$

Since $\mathbb{R}[x, y] \subset \mathbb{C}[x, y]$ are unique factorization domains and f splits in $\mathbb{C}[x, y]$ as a product of two irreducible polynomials of degree two, we conclude that f is irreducible in $\mathbb{R}[x, y]$. On the other hand, $Z(f) = \{(1, 0), (-1, 0)\}$ is the union of two points. Therefore $f \in \mathbb{R}[x, y]$ is irreducible but $Z(f) \subset \mathbb{A}^2$ is reducible.

Exercise 8. [**PhE**] Let $M_n(k)$ be the set of $n \times n$ matrices with coefficients in k. Prove that

$$R_{n-1} = \{A \in M_n(k) \mid \mathrm{rank}(A) < n\}$$

is an algebraic subset of $M_n(k) \cong \mathbb{A}^{n^2}$.
Prove that if $A, B \in M_n(k)$ then AB and BA have the same characteristic polynomial.

The subset $R_{n-1} \subset \mathbb{A}^{n^2}$ is defined by the vanishing of finitely many polynomials. Therefore it is an algebraic subset.
Let us assume that B is invertible. Then $AB = B^{-1}(BA)B$, and

$$\begin{aligned} p_{AB}(\lambda) = \; & \det(\lambda I - B^{-1}(BA)B) = \det(\lambda B^{-1}IB - B^{-1}(BA)B) \\ & = \det(B^{-1})\det(\lambda I - BA)\det(B) = \det(\lambda I - BA) = p_{BA}(\lambda). \end{aligned}$$

Now, $\lambda \in \mathbb{A}^1$, and for any matrix A the polynomials $p_{AB}(\lambda)$ and $p_{BA}(\lambda)$ coincides on an open subset of $\mathbb{A}^{n^2} \times \mathbb{A}^1$. Let Z be the closed subset defined in $\mathbb{A}^{n^2}$ by $\det(B) = 0$, and let $W = Z \times \mathbb{A}^1$. Then $p_{AB}(\lambda)$ and $p_{BA}(\lambda)$ coincides on $\mathcal{U} = \mathbb{A}^{n^2} \times \mathbb{A}^1 \setminus W$. Since $p_{AB}(\lambda)$ and $p_{BA}(\lambda)$ are regular function on $\mathbb{A}^{n^2} \times \mathbb{A}^1$ we conclude that they coincide, that is $p_{AB} = p_{BA}$ for any pair of square matrices A, B.

Exercise 9. **[PhE]** Let us consider the morphism

$$\phi : \mathbb{A}^1 \to \mathbb{A}^3,\ t \mapsto (t, t^2, t^3)$$

and let $C = \phi(\mathbb{A}^1)$.

- Prove that $C = Z(I)$, where $I = (y - x^2, z - x^3)$.
- Prove that $A(C) \cong k[x]$.
- Prove that C is smooth using the Jacobian criterion.
- Prove that C is not contained in any plane of $\mathbb{A}^3$, and that a general plane intersects C in three distinct points.
- Prove that any line of $\mathbb{A}^3$ intersects C in at most two distinct points.
- Prove that C is a complete intersection.

Both $y - x^2, z - x^3$ vanish on the points of the form (t, t^2, t^3). Any polynomial $f = f(x, y, z)$ can be written as

$$f(x, y, z) = f_1(y - x^2) + f_2(z - x^3) + r(x).$$

If $f(t, t^2, t^3) = r(t) = 0$ for any $t \in \mathbb{A}^1$ then $r \equiv 0$, and $f \in I$. This proves that $I(C) = (y - x^2, z - x^3)$. In particular, $C = Z(I)$. Note that this proves the last point as well.
The morphism ϕ is an isomorphism onto its image C. Therefore $C \cong \mathbb{A}^1$ and $A(C) \cong k[x]$.
The Jacobian matrix of C is given by

$$Jac(C) = \left(\begin{array}{ccc} -2x & 1 & 0 \\ -3x^2 & 0 & 1 \end{array} \right)$$

Therefore, $\text{rank}(Jac(C)) = 2$ for any $p \in C$, and C is smooth.
A plane Π is given by a linear equation of the form $\alpha x + \beta y + \gamma z + \delta = 0$. Therefore, its intersection with C is given by the solutions of $\alpha t + \beta t^2 + \gamma t^3 + \delta = 0$. Now, $C \subset \Pi$ is and only if $\alpha t + \beta t^2 + \gamma t^3 + \delta \equiv 0$, that is $\alpha = \beta = \gamma = \delta = 0$. We see also that for a general Π the equation $\alpha t + \beta t^2 + \gamma t^3 + \delta \equiv 0$ has three distinct solutions, that is Π intersects C in three distinct points.
Finally, assume that there is a plane Π intersecting C in four points. Then $\alpha t + \beta t^2 + \gamma t^3 + \delta \equiv 0$ and $C \subset \Pi$. This contradicts the fourth point.

Exercise 10. **[PhE]** Let $C \subset \mathbb{A}^3$ be a smooth, irreducible curve such that $\mathbb{I}(C) = (f, g)$. Prove that $T_xC = T_xF \cap T_xG$ for any $x \in C$, where F, G are the surfaces defined by f, g respectively. In particular F and G are smooth and transverse along C.

Without loss of generality we can assume that $x \in C$ is the origin. The Jacobian matrix of C is

$$Jac(C)(0) = \left(\begin{array}{ccc} \frac{\partial f}{\partial x}(0) & \frac{\partial f}{\partial y}(0) & \frac{\partial f}{\partial z}(0) \\ \frac{\partial g}{\partial x}(0) & \frac{\partial g}{\partial y}(0) & \frac{\partial g}{\partial z}(0) \end{array} \right)$$

Therefore, the tangent line T_0C is given by the intersection of the two planes

$$T_0F = \left\{ \frac{\partial f}{\partial x}(0)x + \frac{\partial f}{\partial y}(0)y + \frac{\partial f}{\partial z}(0)z = 0 \right\},$$
$$T_0G = \left\{ \frac{\partial g}{\partial x}(0)x + \frac{\partial g}{\partial y}(0)y + \frac{\partial g}{\partial z}(0)z = 0 \right\}.$$

Exercise 11. **[PhE]** Let $X \subset \mathbb{A}^n$ be a reducible hypersurface and let $X = X_1 \cup ... \cup X_r$ be its decomposition in irreducible components. Prove that if $x \in X_i \cap X_j$ then x is a singular point for X.

We may assume $X = X_1 \cup X_2$. If $X = Z(f)$, $X_1 = Z(g)$ and $X_2 = Z(h)$ we have $f = gh$. Therefore

$$\frac{\partial f}{\partial x_i} = \frac{\partial g}{\partial x_i}h + g\frac{\partial h}{\partial x_i}.$$

If $x \in X_1 \cap X_2$ then $g(x) = h(x) = 0$. Then $\frac{\partial f}{\partial x_i}(x) = 0$ for any $i = 1, ..., n$, and $x \in X_1 \cup X_2$ is singular.

CHAPTER 2

Projective varieties

Exercise 1. [**Har**, Exercise 2.1] Consider a homogeneous ideal $\mathfrak{a} \subseteq k[x_0, ..., x_n]$, and $f \in k[x_0, ..., x_n]$ a polynomial such that $\deg(f) > 0$ and $f(p) = 0$ for any $p \in Z(\mathfrak{a})$. We may interpret $p = [a_0 : ... : a_n] \in \mathbb{P}^n$ as the point $(a_0, ..., a_n) \in \mathbb{A}^{n+1}$, and the polynomials f as a polynomial on $\mathbb{A}^{n+1}$. By the Nullstellensatz we have $f^k \in \mathfrak{a}$ for some $k > 0$.

Exercise 2. [**Har**, Exercise 2.2] Let $C_a(Z(\mathfrak{a}))$ be the affine cone over $Z(\mathfrak{a})$. Then $I(C_a(Z(\mathfrak{a}))) = I(Z(\mathfrak{a}))$.
Now, $Z(\mathfrak{a}) = \emptyset$ if and only if $C_a(Z(\mathfrak{a})) \subseteq \{(0, ..., 0)\}$. By the Nullstellensatz we have $I(C_a(Z(\mathfrak{a}))) = r(\mathfrak{a})$. Now, we have two possibilities:

- $C_a(Z(\mathfrak{a})) = \emptyset$ if and only if $I(C_a(Z(\mathfrak{a}))) = r(\mathfrak{a}) = k[x_0, ..., x_n]$,
- $C_a(Z(\mathfrak{a})) = \{(0, ..., 0)\}$ if and only if $I(C_a(Z(\mathfrak{a}))) = r(\mathfrak{a}) = \bigoplus_{d>0} k[x_0, ..., x_n]_d$.

This proves $(i) \Leftrightarrow (ii)$. Now, let us prove $(ii) \Rightarrow (iii)$. If $r(\mathfrak{a}) = k[x_0, ..., x_n]$, then $1 \in r(\mathfrak{a})$. So $1 \in \mathfrak{a}$ and $\mathfrak{a} = k[x_0, ..., x_n]$. In particular $S_d \subseteq \mathfrak{a}$ for any $d > 0$. Now, assume $r(\mathfrak{a}) = \bigoplus_{d>0} S_d$. Then $x_i \in r(\mathfrak{a})$ for any $i = 0, ..., n$. Therefore, for any i there exists k_i such that $x_i^{k_i} \in \mathfrak{a}$. Let $m = max\{k_i\}$. Then $x_i^m \in \mathfrak{a}$ for any $i = 0, ..., n$. Now any monomial of degree $d = m(n+1)$ is divisible by x_i^m for some i. We conclude that $S_d \subseteq \mathfrak{a}$, where $d = m(n+1)$.
Finally we prove $(iii) \Rightarrow (i)$. If $S_d \subseteq \mathfrak{a}$ for some $d > 0$, in particular $x_i^d \in \mathfrak{a}$ for any $i = 0, ..., n$. Now, it is enough to observe that $Z(\mathfrak{a}) \subseteq Z((x_0^d, ..., x_n^d)) = \emptyset$.

Exercise 3. [**Har**, Exercise 2.9] Let $Y \subseteq \mathbb{A}^n$ be an affine variety. Consider the homeomorphism

$$\begin{array}{rcl} \phi_0 : U_0 = \mathbb{P}^n \setminus \{x_0 = 0\} & \longrightarrow & \mathbb{A}^n \\ [x_0 : ... : x_n] & \longmapsto & (\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}) \end{array}$$

Finally, let $\overline{Y}$ be the projective closure of Y.
Let $F \in I(\overline{Y})$, then $f(y_1, ..., y_n) = F(1, x_0, ..., x_n)$ where $y_i = \frac{x_i}{x_0}$ vanishes on $Y = \overline{Y} \cap U_0$. We get that $f \in I(Y)$ and $x_0^s \beta(f) = F$ for some s. Therefore, $F \in (\beta(I(Y)))$, where β is the homogeneization with respect to x_0.
Now let $F \in \beta(I(Y))$, then $F = g_1\beta(f_1) + ... + g_r\beta(f_r)$ for some $f_1, ..., f_r \in I(Y)$, that is $F = g_1 x_0^{s_1} f_1(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0}) + ... + g_r x_0^{s_r} f_r(\frac{x_1}{x_0}, ..., \frac{x_n}{x_0})$. Hence $F \in I(\overline{Y})$.
Let $Y \subset \mathbb{A}^3$ be the affine twisted cubic. Then $I(Y) = (x^3 - z, x^2 - y)$ while $I(\overline{Y}) = (xz - y^2, yw - z^2, xw - yz)$. Note that $I(\overline{Y})$ can not be generated by two elements because $\overline{Y} \subset \mathbb{P}^3$ is not a scheme-theoretic complete intersection.

Exercise 4. [**Har**, Exercise 2.10] Let $Y \subset \mathbb{P}^n$ be a non-empty algebraic set, and let $C(Y)$ be the affine cone over Y. Let $p = (a_0, ..., a_n) \in C(Y)$ be a point. Then p represents the point $[a_0, ..., a_n] \in Y$. In particular $f(p) = 0$ for any $f \in I(Y)$. On the other hand if $f \in I(Y)$

is homogeneous then f vanished on any line joining the origin of $\mathbb{A}^{n+1}$ and a point of Y, because f is homogeneous. Then $C(Y) = Z(I)$ is an algebraic set.
If $f \in I(C(Y))$ and $p \in Y$ then f vanishes on the line in $\mathbb{A}^{n+1}$ joining $(0, ..., 0)$ and p. Then $f \in I(Y)$. Conversely, if $g \in I(Y)$ we may write $g = g_i + ... + g_j$ where g_r is homogeneous of degree r. Since $I(Y)$ is homogeneous we have $g_r \in I(Y)$ for any r. Furthermore, Y non-empty implies that $I(Y)$ does not contain constants, that us $\deg(g_r) \geq 1$ for any r. This yields $g(0, ..., 0) = 0$. So $g \in I(C(X))$.
Now, $C(Y)$ is irreducible if and only if $I(C(Y))$ is prime, if and only if $I(Y) = I(C(Y))$ is prime, if and only if Y is irreducible.
Finally, we have

$$\dim(C(Y)) = \dim(A(C(Y))) = n + 1 - \mathrm{height}(I(C(Y))),$$

and

$$\dim(Y) = \dim(S(Y)) - 1 = n - \mathrm{height}(I(Y)).$$

Therefore, $\mathrm{height}(I(C(Y))) = \mathrm{height}(I(Y)) = n - \dim(Y)$, and

$$\dim(C(Y)) = n + 1 - (n - \dim(Y)) = \dim(Y) + 1.$$

Exercise 5. [**Har**, Exercise 2.11] If $I(Y) = (L_1, ..., L_k)$ where L_i is linear for any i, then $H_i = Z(L_i)$ are hyperplanes and $Y = \bigcap_{i=1}^{k} H_i$. Conversely, if $Y = \bigcap_{i=1}^{k} H_i$ up to an automorphism of $\mathbb{P}^n$ we may assume $H_i = Z(x_i)$. Then $I(Y) = I(\bigcap_{i=1}^{k} H_i) = (x_1, ..., x_k)$.
Let Y be a linear subspace of dimension r. Then Y is an intersection of hyperplanes. Intersecting with an hyperplane drops the dimension at most by one. Since $\dim(Y) = r$ then Y is the intersection of at least $n - r$ hyperplanes. We may assume that $n - r$ of them are $Z(x_i)$ for $i = 1, ..., n - r$. Then Y is the intersection of at least $n - r$ hyperplanes and $I(Y)$ is generated by $n - r$ linear polynomials.
Let Y, Z be linear varieties in $\mathbb{P}^n$ of dimension r, s. Then Y is intersection of $n - r$ hyperplanes and Z is intersection of $n - s$ hyperplanes. Therefore $I(Y \cap Z)$ is generated by at most $2n - r - s$ linear polynomials. Then $Y \cap Z$ is linear and $\dim(Y \cap Z) \geq n - (2n - r - s) = r + s - n$.

Exercise 6. [**Har**, Exercise 2.13] Let Y be the image of the Veronese embedding

$$\begin{array}{ccc} \nu : \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ [x_0 : x_1 : x_2] & \longmapsto & [x_0^2 : x_0x_1 : x_0x_2 : x_1^2 : x_1x_2 : x_2^2] \end{array}$$

Let $Z \subset Y$ be curve. Then $f^{-1}(Z) = C \subset \mathbb{P}^2$ is a plane curve. Therefore, $C = Z(f)$ where $f \in k[x_0, x_1, x_2]_d$ is a homogeneous polynomial. Then f^2 is a homogeneous polynomial of degree $2d$ on $\mathbb{P}^2$. Let $X_0, ..., X_5$ be homogeneous coordinates on $\mathbb{P}^5$. Then $f(x_0^2, ..., x_2^2) = F(X_0, ..., X_5)$ is a homogeneous polynomial of degree d on $\mathbb{P}^5$, and $Y \cap Z(F) = Z$.
For instance let Z be the image of the line $C = \{x_0 = 0\}$. Then $F = x_0^2 = X_0$, and $Z = Y \cap Z(X_0)$. Note that Z is a conic and since $\deg(Y) = 4$ we have $\deg(Y \cap Z(X_0)) = 4$. Then $Z = Y \cap Z(X_0)$ set-theoretically. Indeed, scheme-theoretically the intersection $Y \cap Z(X_0)$ is twice Z.

Exercise 7. [**Har**, Exercise 2.16] On the affine chart $w = 1$ we have the equations $x^2 - y, xy = z$, that is $y = x^2, z = x^3$. The points in the intersection $Q_1 \cap Q_2 \cap \{w \neq 0\}$ are of the form $(x, x^2, x^3, 1)$, and we get the twisted cubic. If $w = 0$ then $x = 0$, and we get the line $\{x = w = 0\}$.
Let C be the conic $\{x^2 - yz = 0\}$ and L the line $\{y = 0\}$. Then C and L intersects in the point $p = [0 : 0 : 1]$. Now, $I(\{p\}) = (x, y)$ but $x \notin I(C) + I(L)$. Therefore $I(C) + I(L) \neq I(\{p\})$. Note that L is tangent to C in p. Indeed $I(C \cap L) = (x^2, y)$.

Exercise 8. [**Har**, Exercise 2.17]

(a) Let $Y = Z(\mathfrak{a}) \subseteq \mathbb{P}^n$ be a variety. Assume $\mathfrak{a} = (f_1, ..., f_q)$. If $q = 1$ the Y is an hypersurface, and $dim(Y) = n - 1$. Assume that the statement is true for $q - 1$, and consider $\mathfrak{a} = (f_1, ..., f_{d-1}, f_q)$ with $f_q \notin (f_1, ..., f_{d-1})$. Let $X = Z((f_1, ..., f_{d-1}))$. By induction $\dim(X) \geq n - q + 1$. Furthermore, since $f_q \notin (f_1, ..., f_{d-1})$ intersecting X with the hypersurface $Z(f_q)$ drops the dimension by one. Then, $\dim(Y) = \dim(X \cap Z(f_q)) \geq n - q$.

(b) Let Y be a strict complete intersection. Then $I(Y) = (f_1, ..., f_{n-r})$ where $r = \dim(Y)$. Let $X_i = Z(f_i)$ be the hypersurface defined by f_i. Then $Y = \bigcap_{i=1}^{n-r} X_i$ is a set-theoretic complete intersection.

(c) Let Y be the twisted cubic in $\mathbb{P}^3$. Assume $I(Y) = (f, g)$. Then $Y = Z(f) \cap Z(g)$ scheme-theoretically. By Bézout's theorem we have $\deg(Z(f)) \cdot \deg(Z(g)) = deg(Y) = 3$. Therefore, either $\deg(Z(f)) = 1$ or $\deg(Z(g)) = 1$. In any case Y is contained in a plane. A contradiction.
Another way to see this fact is the following. In $I(Y)$ there are not linear polynomials because Y us not plane. On the other hand in $I(Y)$ there are the three independent quadratic polynomials $xz - y^2, yw - z^2, xw - yz$. Therefore $I(Y)$ can not be generated by two polynomials.
Now, consider the quadric surface Q given by

$$\det \begin{pmatrix} x & y \\ y & z \end{pmatrix} = 0$$

and the cubic surface S given by

$$\det \begin{pmatrix} x & y & z \\ y & z & w \\ z & w & x \end{pmatrix} = 0$$

On a general point $p = [u^3 : u^2v : uv^2 : v^3] \in Y$ we have $Jac(Q)(p) = (uv^2, -2u^2v, u^3, 0)$ and $Jac(S)(p) = (v^2(u^4 - v^4), -2uv(u^4 - v^4), u^2(u^4 - v^4), 0)$. Therefore, $\mathbb{T}_pQ = \mathbb{T}_pS$ for a general point $p \in Y$. This means that $Q \cap S = Y$ set-theoretically. However, scheme-theoretically Q and S cut Y twice.

Exercise 9. [**PhE**] Let $R, L \subset \mathbb{P}^3$ be two skew lines. Let $p \in \mathbb{P}^3$ be a points such that $p \notin R \cap L$. Prove that there exists a unique line L_p such that $p \in L_p$, $L_p \cap R \neq \emptyset$ and $L_p \cap L \neq \emptyset$.
Now, let $L_1, L_2, L_3 \subset \mathbb{P}^3$ be three, pairwise skew, lines. Then for any point $p \in L_1$ there exists a unique line L_p such that $p \in L_p$, $L_p \cap L_2 \neq \emptyset$ and $L_p \cap L_3 \neq \emptyset$. Prove that if $p \neq q$

then $L_p \cap L_q = \emptyset$. Let

$$Q = \bigcup_{p \in L_1} L_p.$$

Compute the dimension and the degree of Q.

Consider the plane $H = \langle p, R \rangle$. Since $R \cap L = \emptyset$ we have that L is not contained in H. Therefore, $H \cap L = \{q\}$. The line $L_p = \langle p, q \rangle$ intersects R as well because L_p and R are both contained in H. Assume there is another line R_p with this property and consider the plane $\Pi = \left\langle L_p, R_p \right\rangle$. Then $L, R \subset \Pi$, and $L \cap R \neq \emptyset$. A contradiction.
The dimension of Q is two because for any point $p \in L_1$ we have a line L_p in Q. Now, quadric surfaces in $\mathbb{P}^3$ are parametrized by $\mathbb{P}^9 = \mathbb{P}(k[x_0, x_1, x_2, x_3]_2)$. A line L_i is contained in Q if and only if L_i intersects Q in at least three points. Therefore, to force L_1, L_2, L_3 to be contained in a quadric surface we get nine linear equations in the homogeneous coordinates of $\mathbb{P}^9$. We conclude that there is a quadric $\overline{Q} \subset \mathbb{P}^3$ containing L_1, L_2, L_3. Note that Q can not be neither a double plane not the union of two planes because we have three skew lines in Q. For the same reason $\overline{Q}$ can not be a quadric cone. Indeed all the lines contained in a quadric cone pass through the vertex. Therefore $\overline{Q}$ is a smooth quadric. Assume there is another quadric $\overline{Q}_2$ containing L_1, L_2, L_3. Then any line T in $\overline{Q}$ intersecting L_1, L_2, L_3 intersects $\overline{Q}_2$ in at least three points. Therefore $T \subset \overline{Q}_2$. This means that $\overline{Q} \cap \overline{Q}_2$ contains a surface. However, $\overline{Q}$ and $\overline{Q}_2$ are irreducible. Then $\overline{Q} = \overline{Q}_2$. We conclude that given three skew lines $L_1, L_2, L_3 \subset \mathbb{P}^3$ there exists a unique quadric surface $\overline{Q} \subset \mathbb{P}^3$ containing L_1, L_2, L_3. Furthermore, $\overline{Q}$ is smooth, and in particular irreducible.
Now, any line L_p intersects $\overline{Q}$ in at least three points. Then $L_p \subset \overline{Q}$ for any $p \in L_1$. This means that the surface Q is contained in $\overline{Q}$. Since $\overline{Q}$ is irreducible we conclude that $Q = \overline{Q}$. Finally, $\deg(Q) = \deg(\overline{Q}) = 2$.

Exercise 10. **[PhE]** Let $x = [1 : 0 : ... : 0] \in \mathbb{P}^n$ and $H = \{x_0 = 0\}$. The projection from x on the hyperplane H is defined as

$$\begin{array}{rcl} \pi_x : \mathbb{P}^n & \dashrightarrow & H \cong \mathbb{P}^{n-1} \\ y & \longmapsto & \langle x, y \rangle \cap H \end{array}$$

Prove that, if $n \geq 2$, it is not possible to extend π_x on the whole of $\mathbb{P}^n$. Now, consider the case $n = 2$ and the conic $C = \{x_2^2 - x_0x_1\}$. Prove that the restriction

$$\pi_{x|C} : C \dashrightarrow \mathbb{P}^1$$

can be extended on the whole of C.

We may try to extend π_x defining $\pi_x(x) = z \in H$ for some $z \in H$. However, if $n \geq 2$ the extend map can not be continuous in x.
Since $x = [1 : 0 : 0] \in C$ is a smooth point there is a natural way to extend $\pi_{x|C} : C \dashrightarrow \mathbb{P}^1$, that is considering $T_xC = \{x_1 = 0\}$. We may define

$$\pi_x(x) = T_xC \cap \{x_0 = 0\} = \{[0 : 0 : 1]\}.$$

Exercise 11. **[PhE]** Let $S_d = k[x_0, ..., x_n]_d$ the k-vector space of degree d homogeneous polynomials in $n+1$ variables. Prove that

$$\dim(S_d) = \binom{d+n}{n}.$$

Let us look at the case $d = 1$. Then $\dim(S_1) = n+1$. On the other hand, if $n = 0$ we have $\dim(S_d) = 1$ for any d. Therefore we may proceed by double induction on n and d. Note that we have

$$k[x_0, ..., x_n]_d = k[x_0, ..., x_{n-1}]_d \oplus k[x_0, ..., x_n]_{d-1}.$$

By induction hypothesis $\dim(k[x_0, ..., x_{n-1}]_d) = \binom{d+n-1}{n-1}$ and $\dim(k[x_0, ..., x_n]_{d-1}) = \binom{d-1+n}{n}$. Finally,

$$\dim(S_d) = \binom{d+n-1}{n-1} + \binom{d-1+n}{n} = \binom{d+n}{n}.$$

Exercise 12. **[PhE]** Let $C \subset \mathbb{P}^2$ be a smooth curve of degree $d \geq 2$. We define a morphism

$$\begin{array}{rcl} f : C & \longrightarrow & \mathbb{P}^{2*} \\ p & \longmapsto & T_pC \end{array}$$

Prove that the image $C^* = f(C) \subset \mathbb{P}^{2*}$ is a curve. The curve C^* is called the dual curve of C.
Let $C \subset \mathbb{P}^2$ be the conic given by $x_1^2 - x_0x_2 = 0$. Prove that C is smooth and determine C^*.

Since $d \geq 2$ and we are in characteristic zero the morphism f is not constant. Therefore its image has dimension one. Since f is projective $f(C)$ is closed.
The tangent line at $C = Z(g)$ in a point $p \in C$ corresponds to point of $\mathbb{P}^2$ whose homogeneous coordinates are the partial derivatives of g evaluated in p. That is

$$\begin{array}{rcl} f : C = Z(g) & \longrightarrow & \mathbb{P}^{2*} \\ p & \longmapsto & \left[\frac{\partial g}{\partial x}(p) : \frac{\partial g}{\partial y}(p) : \frac{\partial g}{\partial z}(p)\right] \end{array}$$

Let d be the degree of C, and let L be a line in $\mathbb{P}^{2*}$. The pulling-back the equation of L via f we get a polynomial of degree $d-1$ on $\mathbb{P}^2$. By Bézout theorem we have $\deg(C^*) = d(d-1)$ In our case $d = 2$, so C^* is a conic. Indeed the morphism is

$$\begin{array}{rcl} f : C = Z(g) & \longrightarrow & \mathbb{P}^{2*} \\ [x_0 : x_1 : x_2] & \longmapsto & [-x_2 : 2x_1 : -x_0] \end{array}$$

If z_0, z_1, z_2 are the homogeneous coordinates on $\mathbb{P}^{2*}$ note that $z_1^2 = 4x_1^2 = 4x_0x_1 = 4z_0z_2$. Therefore $C^* \subset \mathbb{P}^{2*}$ is the smooth conic defined by $\{z_1^2 - 4z_0z_2 = 0\}$. Note that the matrix of C^* is the inverse of the matrix of C.

Exercise 13. Let $X \subset \mathbb{P}^n$ be an irreducible hypersurface of degree d having a singular point $x \in X$ of multiplicity $d-1$. Prove that X is rational.

Consider the projection

$$\begin{array}{rcl} \pi_x : X & \dashrightarrow & H \cong \mathbb{P}^{n-1} \\ y & \longmapsto & \langle x, y \rangle \cap H \end{array}$$

where H is a general hyperplane. Note that for $y \in X$ general the line $\langle x, y\rangle$ is not contained in X. Otherwise, X would be a cone with vertex x, and x would be of multiplicity d for X.
Since $\deg(X) = d$ and x has multiplicity $d-1$, by Bézout's theorem the general line $\langle x, y\rangle$ intersects X only in x with multiplicity $d-1$ and in y with multiplicity one. This means that π_x is birational. So X is rational.

Exercise 14. Let x, y, z be homogeneous coordinates on $\mathbb{P}^2$. Consider the conic $C = \{y^2 - xz = 0\}$, and the point $p = [0:1:0] \notin C$. Let α, β, γ the dual coordinates on $\mathbb{P}^{2*}$. Compute the dual conic $C^* \subset \mathbb{P}^{2*}$ of C, and the line $L_p \subset \mathbb{P}^{2*}$ dual to the point p. Prove that the tangents lines to C through p corresponds to points in $L_p \cap C^*$. Finally, compute explicitly the tangent lines to C through p.
Now, let x, y, z, w be homogeneous coordinates on $\mathbb{P}^3$. Compute the equations of the line L through $p = [1:0:0:0]$ and $q = [1:1:1:1]$. Write down the equation of a smooth quadric surface $Q \subset \mathbb{P}^3$ such that $L \subset Q$.

Let $F = y^2 - xz$, and consider the morphism

$$\begin{array}{cccc} f: C & \longrightarrow & \mathbb{P}^{2*} & \\ {[x:y:z]} & \longmapsto & \left[\frac{\partial F}{\partial x}(x,y,z) : \frac{\partial F}{\partial y}(x,y,z) : \frac{\partial F}{\partial z}(x,y,z)\right] & = [-z:2y:-x] \end{array}$$

The $C^* = f(C)$. If α, β, γ are homogeneous coordinates on $\mathbb{P}^{2*}$ note that $\beta^2 = 4y^2 = 4xz = 4\alpha\gamma$. Therefore $C^* \subset \mathbb{P}^{2*}$ is the smooth conic defined by $\{\beta^2 - 4\alpha\gamma = 0\}$.
The dual of $p = [0:1:0]$ is the space of linear forms $\{L = \alpha x + \beta y + \gamma z\}$ vanishing at p. This forces $\beta = 0$. Therefore, $L_p = \{\beta = 0\} \subset \mathbb{P}^{2*}$. The map f associates to a point $q \in C$ the tangent line $\mathbb{T}_q C$. Hence, by duality the tangent lines of C through p corresponds to the points of intersection between C^* and L_p.
We have $C^* \cap L_p = \{[0:0:1], [1:0:0]\}$. Therefore, the two tangent lines are the dual lines of these two points, that is $R_1 = \{z = 0\}$ and $R_2 = \{x = 0\}$.
Let us consider the matrix

$$M = \left(\begin{array}{cccc} x & y & z & w \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

The line L is the locus of $\mathbb{P}^3$ where M has rank two. Note that there are two 3×3 minors of M giving $\{y - z = 0\}$, and $\{w - z = 0\}$. Then $L = \{y - z = w - z = 0\}$. Consider the quadric Q given by

$$Q = \{F = x(y-z) + y(w-z) = xy - xz + yw - yz = 0\}.$$

Clearly $L \subset Q$. Furthermore,

$$\frac{\partial F}{\partial x} = y - z,\ \frac{\partial F}{\partial y} = x + w,\ \frac{\partial F}{\partial z} = -x - y,\ \frac{\partial F}{\partial w} = y.$$

Now, $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = \frac{\partial F}{\partial w} = 0$ forces $x = y = z = w = 0$. Then Q is smooth.

Exercise 15. Let $\nu : \mathbb{P}^2 \to \mathbb{P}^5$ be the degree two Veronese embedding, and let $V \subset \mathbb{P}^5$ be its image. Compute the degree of $V \subset \mathbb{P}^5$.
Let $C \subset \mathbb{P}^2$ be a curve of degree d. Compute the degree of $\nu(C) \subset V \subset \mathbb{P}^5$. Prove that V does not contain a line.
Now, interpret $\mathbb{P}^5$ as the projective space parametrizing conics in $\mathbb{P}^2$.
Explain why $V \subset \mathbb{P}^5$ is the locus parametrizing rank one conics, that is double lines.
Consider the matrix representation of a general conic in $\mathbb{P}^2$. Let X be the locus in $\mathbb{P}^5$ parametrizing rank two conics, that is union of two lines. Prove that $X \subset \mathbb{P}^5$ is an hypersurface of degree three.
Deduce that there exists a Zariski open subset $\mathcal{U} \subset \mathbb{P}^5$ parametrizing rank three conics, that is smooth conics, and that the general conic can not be written as a sum of powers of two squares of linear forms.

The Veronese embedding is defined as

$$\begin{array}{ccc} \nu : \mathbb{P}^2 & \longrightarrow & \mathbb{P}^5 \\ [a:b:c] & \longmapsto & [a^2 : ab : ac : b^2 : bc : c^2] \end{array}$$

Since $\dim(V) = 2$ to compute $\deg(V)$ we have to intersect with a general linear subspace of dimension three H. Let us write $H = H_1 \cup H_2$ where the H_i's are hyperplanes. Then $\nu^{-1}(V \cap H_i) = C_i$ are two conics in $\mathbb{P}^2$. Since, ν is an isomorphism we get

$$\deg(V) = \#(H \cap V) = \#(C_1 \cap C_2) = 4.$$

Let $C \subset \mathbb{P}^2$ be a curve of degree d. Since ν is an isomorphism the image $\Gamma = \nu(C)$ is a curve isomorphic to C. Let $H \subset \mathbb{P}^5$ be a general hyperplane. Then $\nu^{-1}(V \cap H) = C_1$ is a conic, and

$$\deg(\Gamma) = \#(H \cap \Gamma) = \#(H \cap V \cap \Gamma) = \#(C_1 \cap C) = 2d.$$

The lowest degree of a curve in $\mathbb{P}^2$ is one. Then, the lowest degree of a curve in V is two. In particular V does not contain any line.
Let $L = ax + by + cz$ be a linear form on $\mathbb{P}^2$. Then

$$L^2 = a^2x^2 + 2abxy + 2acxz + b^2y^2 + 2bcyz + c^2z^2.$$

Note that modulo re-scaling the coefficients of the mixed terms these are exactly the coordinates of ν. Therefore, V parametrizes double lines. Let $Z_0, ..., Z_5$ be homogeneous coordinates on $\mathbb{P}^5$. The we may write a plane conic as

$$C = \{Z_0x^2 + 2Z_1xy + 2Z_2xz + Z_3y^2 + 2Z_4yz + Z_5z^2 = 0\}.$$

The matrix of C is

$$M = \begin{pmatrix} Z_0 & Z_1 & Z_2 \\ Z_1 & Z_3 & Z_4 \\ Z_2 & Z_4 & Z_5 \end{pmatrix}$$

Hence, the locus X parametrizing rank two conics in defined by $X = \{\det(M) = 0\}$. So, X is an hypersurface of degree three. Any point in the open subset $\mathcal{U} = \mathbb{P}^5 \setminus X$ represents a smooth conics. Assume that the general conic can be written as sum of two square of linear forms $F = L_1^2 + L_2^2$. Then $C = Z(F)$ would be singular in $\{L_1 = L_2 = 0\}$. A contradiction, because we know that the general conic is smooth.

Exercise 16. Let $X \subset \mathbb{P}^n$ be an irreducible, reduced and non-degenerate variety. Prove that

$$\deg(X) \geq \operatorname{codim}(X) + 1.$$

Provide an example where the equality is achieved. We say that an irreducible, reduced and non-degenerate variety $X \subset \mathbb{P}^n$ is a *variety of minimal degree* if $\deg(X) = \operatorname{codim}(X) + 1$. Provide an example of a variety of minimal degree which is not an hypersurface. Prove that a cone over a variety of minimal degree is of minimal degree.

If $\operatorname{codim}(X) = 1$, being X non-degenerate, we have $\deg(X) \geq 2 = \operatorname{codim}(X) + 1$. We proceed by induction on $\operatorname{codim}(X)$. Let $x \in X$ be a general point, and

$$\pi_x : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$

be the projection from x. The variety $Y = \overline{\pi_x(X)} \subset \mathbb{P}^{n-1}$ has degree $\deg(Y) = \deg(X) - 1$, and codimension $\operatorname{codim}(Y) = \operatorname{codim}(X) - 1$. By induction hypothesis we have $\deg(Y) \geq \operatorname{codim}(Y) + 1$, which implies $\deg(X) \geq \operatorname{codim}(X) + 1$. The simplest example of a variety of minimal degree is a quadric hypersurface in $\mathbb{P}^n$. Let $V_2^2 \subset \mathbb{P}^5$ be the Veronese surface. Then $\operatorname{codim}(V_2^2) = 3$, and $\deg(V_2^2) = 4$. Therefore, the Veronese surface is of minimal degree.
Now, let $X \subset \mathbb{P}^n$ be a variety of minimal degree, and let $C_p(X) \subset \mathbb{P}^{n+1}$ be the cone with vertex $p \in \mathbb{P}^{n+1}$ over X. Then, $\deg(C_p(X)) = \deg(X)$, and $\dim(C_p(X)) = \dim(X) + 1$, that is $\operatorname{codim}(C_p(X)) = n + 1 - \dim(C_p(X)) = \operatorname{codim}(X)$. Finally,

$$\deg(C_p(X)) = \deg(X) = \operatorname{codim}(X) + 1 = \operatorname{codim}(C_p(X)) + 1,$$

and $C_p(X)$ is of minimal degree.

Exercise 17. Let $a_1 \leq a_2 \leq ... \leq a_k$ be natural numbers, and let $n = \sum_{i=1}^{k} a_i + k - 1$. Fix $H_i \cong \mathbb{P}^{a_i} \subset \mathbb{P}^n$ complementary linear subspaces, and $C_i \subset H_i$ a rational normal curve of degree a_i for any i. Finally, we choose isomorphisms $\phi_i : C_1 \to C_i$ for $i = 2, ..., k$ and consider the rational normal scroll of dimension k

$$S_{a_1,...,a_k} = \bigcup_{p \in C_1} \langle p, \phi_2(p), ..., \phi_k(p) \rangle .$$

Compute the degree of $S_{a_1,...,a_k}$.

We proceed by induction on k. If $k = 1$ then S_{a_1} is just a rational normal curve of degree a_1. We want to prove that in general

$$\deg(S_{a_1,...,a_k}) = a_1 + ... + a_k.$$

Let us consider a general hyperplane H containing H_1,...,$H_{a_{k-1}}$, and let $S_{a_1,...,a_{k-1}}$ the corresponding rational normal scroll of dimension $k - 1$. Note that H intersects C_{a_k} is a_k points $p_1, ..., p_{a_k}$. These points determines a_k k-planes $\Lambda_1, ..., \Lambda_{a_k}$ of the scroll $S_{a_1,...,a_k}$, and we have

$$S_{a_1,...,a_k} \cap H = S_{a_1,...,a_{k-1}} \cup \Lambda_1 \cup ... \cup \Lambda_{a_k}.$$

We conclude that $\deg(S_{a_1,...,a_k}) = \deg(S_{a_1,...,a_{k-1}}) + \deg(\Lambda_1) + ... + \deg(\Lambda_{a_k}) = \sum_{i=1}^{k} a_i$.

Exercise 17. Let $X \subset \mathbb{P}^n$ be a variety set-theoretically defined by polynomials $F_1, ..., F_m$ of degree $d_i = \deg(F_i)$. Prove that if $d_1 + ... + d_m \leq n - 1$ then through any point $x \in X$ there is a line contained in X.

We may assume $x = [1 : 0 : ... : 0]$. The lines through x are parametrized by the hyperplane $\{x_0 = 0\}$. The line spanned by x and $[0 : x_1, ..., x_n]$ is the set of the points $[u : vx_1 : ... : vx_n]$ for $[u : v] \in \mathbb{P}^1$.
Now, $F_i(u : vx_1 : ... : vx_n)$ is a polynomial of degree d_i on $\mathbb{P}^1$ whose coefficients depend on $x_1, ..., x_n$. Note that these coefficient are d_i and not $d_i + 1$ because $x \in X$ forces the coefficient of u^{d_i} to be zero. Therefore, $F_i(u : vx_1 : ... : vx_n) \equiv 0$ on $\mathbb{P}^1$ yields $d_1 + ... + d_m$ equations on $\mathbb{P}^{n-1}$. Finally, if $d_1 + ... + d_m \leq n - 1$ these system of equations has a solution, that is there is a line through x contained in X.

CHAPTER 3

Morphisms

Exercise 1. [**Har**, Exercise 3.1]

(a) Let C be a conic in $\mathbb{A}^2$. By [**Har**, Exercise 1.1] the conic C is isomorphic either to the curve $\{y - x^2 = 0\}$ or to the curve $\{xy = 1\}$. The first curve is isomorphic to $\mathbb{A}^1$ while the second is isomorphic to $\mathbb{A}^1 \setminus \{0\}$.

(b) A proper open subsets of $\mathbb{A}^1$ is of the form $U = \mathbb{A}^1 \setminus Z$ where Z is a finite set of points $Z = \{x_1, ..., x_k\}$. Note that, since $Z \subset \mathbb{A}^1$ is an hypersurface U is affine. In particular, by [**Har**, Lemma 4.2] U is isomorphic to an hypersurface in $\mathbb{A}^2$. In the coordinate ring $A(Y)$ of U the polynomial $x - x_1$ is a unit. Therefore, any morphism $A(U) \rightarrow k[x]$ has to send $x - x_1$ in an element of k. Note that also x_1 has to be mapped to an element of k. Therefore x is mapped to k as well, and the morphism can not be surjective.

(c) Let $C \subset \mathbb{P}^2$ an irreducible conic. Then, modulo a change of variables the equation of C can be written as $\{xz - y^2 = 0\}$. Then C is the image of the embedding

$$\begin{array}{rcc} \nu : \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ [x_0 : x_1] & \longmapsto & [x_0^2 : x_0x_1 : x_1^2] \end{array}$$

and $C \cong \mathbb{P}^1$.

(d) Assume there is an homeomorphism $f : \mathbb{A}^2 \rightarrow \mathbb{P}^2$. Let L, R be two lines in $\mathbb{A}^2$ such that $L \cap R = \emptyset$. Then $f(L), f(R)$ are two curves in $\mathbb{P}^2$. So $f(L) \cap f(R) \neq \emptyset$ and f can not be injective. A contradiction.

(e) Let X, Y be an affine and a projective variety respectively. Assume $X \cong Y$. Then the ring of regular functions of X is isomorphic to the ring of regular functions on Y. Now, since Y is projective the regular functions on Y are constant. So the the regular functions on X are constant as well. Since X is affine it has to be a point.

Exercise 2. [**Har**, Exercise 3.3] Let $\phi : X \rightarrow Y$ be a morphism, and $p \in X$ be point. We define a morphism of local rings

$$\begin{array}{rcc} \phi_p^* : \mathcal{O}_{Y,\phi(p)} & \longrightarrow & \mathcal{O}_{X,p} \\ (U, f) & \longmapsto & (\phi^{-1}(U), f \circ \phi) \end{array}$$

If ϕ is an isomorphism, let ψ be its inverse. For any point $p \in X$ the morphism

$$\begin{array}{rcc} \psi_{\phi(p)}^* : \mathcal{O}_{X,p} & \longrightarrow & \mathcal{O}_{Y,\phi(p)} \\ (V, g) & \longmapsto & (\psi^{-1}(V), g \circ \psi) \end{array}$$

is the inverse of ϕ_p^*.

Now, assume that ϕ_p^* is an isomorphism for any p, and that ϕ is an homeomorphism. Then $\psi = \phi^{-1}$ is an homeomorphism as well. Now, let $V \subseteq Y$ be an open subset. Let $\phi(p) \in V$

be a point. Since ϕ_p^* is an isomorphism we have that for any regular function f on V the pull-back $f \circ \phi$ is regular on $\phi^{-1}(V)$. Therefore, ϕ is a morphism. In the same way ψ us a morphism. Then ϕ is an isomorphism.
Assume that for some $p \in X$ the morphism ϕ_p^* is not injective. This means that there for some $(U, f) \in \mathcal{O}_{Y,\phi(p)}$ with $f \neq 0$ we have $(\phi^{-1}(U), f \circ \phi) \in \mathcal{O}_{X,p}$ with $f \circ \phi = 0$. Now, since $f \neq 0$ we may consider the subvariety $Z(f) \subset Y$. Finally $f \circ \phi = 0$ yields $\phi(X) \subseteq Z(f)$. Therefore $\phi(X)$ is not dense in Y.

Exercise 3. **[Har**, Exercise 3.5] Let $H \subset \mathbb{P}^n$ be an hypersurface of degree d. Consider the Veronese embedding of degree d

$$\begin{array}{ccc} \nu : \mathbb{P}^n & \longrightarrow & \mathbb{P}^N \\ [x_0 : ... : x_n] & \longmapsto & [x_0^d : ... : x_n^d] \end{array}$$

and let $V \subset \mathbb{P}^N$ be its image. Note that $\nu(H)$ corresponds to an hyperplane section $V \cap \Pi$ of V. Since $\mathbb{P}^N \setminus \Pi \cong A^N$. The variety $V \setminus (V \cap \Pi)$ is affine. Therefore $\mathbb{P}^n \setminus H = \nu^{-1}(V \setminus (V \cap \Pi))$ is affine as well.

Exercise 4. **[Har**, Exercise 3.9] The image of

$$\begin{array}{ccc} \nu : \mathbb{P}^1 & \longrightarrow & \mathbb{P}^2 \\ [x_0 : x_1] & \longmapsto & [x_0^2 : x_0 x_1 : x_1^2] \end{array}$$

is the conic $C = \{xz - y^2 = 0\} \subset \mathbb{P}^2$, and $C \cong \mathbb{P}^1$. On the other hand in $S(C) = k[x, y, z]/(xz - y^2)$ we have three elements of degree one while in $S(\mathbb{P}^1) = k[x_0, x_1]$ we have just two elements of degree one.

Exercise 5. **[Har**, Exercise 3.13] Consider the ideal

$$\mathfrak{m} = \{(U, f) \mid f_{|U \cap Y} = 0\}.$$

The quotient $\mathcal{O}_{X,Y}/\mathfrak{m}$ consists of invertible rational functions on Y, that is $K(Y)$. Furthermore, any element in $\mathcal{O}_{X,Y} \setminus \mathfrak{m}$ is invertible. Therefore, $\mathfrak{m}$ is the unique maximal ideal of $\mathcal{O}_{X,Y}$.
Note that $\mathfrak{m}$ is the ideal of functions vanishing on Y. Therefore $\mathrm{height}(\mathfrak{m}) = \mathrm{codim}_X(Y) = \dim(X) - \dim(Y)$. Furthermore, $K(Y)$ has dimension zero being a field. We conclude that

$$\dim(\mathcal{O}_{X,Y}) = \mathrm{height}(\mathfrak{m}) = \dim(X) - \dim(Y).$$

Exercise 6. **[Har**, Exercise 3.14] We may assume $p = [1 : 0... : 0]$ and H to be the hyperplane $\{x_0 = 0\}$. Therefore the projection ϕ is given by

$$\begin{array}{ccc} \phi : \mathbb{P}^n \setminus \{p\} & \longrightarrow & H \cong \mathbb{P}^{n-1} \\ [x_0 : ... : x_n] & \longmapsto & [x_1 : ... : x_n] \end{array}$$

So, it is a morphism.
The projection of $\mathbb{P}^3$ from the point $p = [0 : 0 : 1 : 0]$ to the plane $\{z = 0\}$ is given by

$$\begin{array}{ccc} \phi : \mathbb{P}^3 \setminus \{p\} & \longrightarrow & H \cong \mathbb{P}^2 \\ [x : y : z : w] & \longmapsto & [x : y : w] \end{array}$$

Therefore $\phi(t^3, t^2u, tu^2, u^3) = (t^3, t^2u, u^3)$. Hence, the image $\Gamma = \phi(C)$ of the twisted cubic C is the cuspidal cubic $\Gamma = \{y^3 - x^2w = 0\}$. Let $q \in H$ be the singular point of Γ. Note that the line $\langle p, q \rangle$ is tangent to C.

Exercise 7. [**Har**, Exercise 3.19] Let

$$\begin{array}{ccc} \phi : \mathbb{A}^n & \longrightarrow & \mathbb{A}^n \\ (x_1, ..., x_n) & \longmapsto & (f_1, ..., f_n) \end{array}$$

be an automorphism. Since ϕ is surjective any polynomials $f_i = f_i(x_1, ..., x_n)$ is a linear non-constant polynomials. Therefore $J(f_1, ..., f_n)$ is a non-zero constant.
The converse is still a hard open problem in algebraic geometry known as *Jacobian conjecture*. It was posed in 1939 by Eduard Ott-Heinrich Keller.

Exercise 8. [**Har**, Exercise 3.21]

(a) Note that $(\mathbb{A}^1, +)$ is a group. The inverse is just $x \mapsto -x$. Therefore, $\mathbb{G}_a$ is a group variety.
(b) $(\mathbb{A}^1 \setminus \{0\}, \cdot)$ is a group, and the inverse is $x \mapsto \frac{1}{x}$. Therefore, $\mathbb{G}_m$ is a group variety.
(c) If X is a variety, and G is a group variety, we define a group structure on $\mathrm{Hom}(X, G)$ by $\alpha(f, g)(x) = f(x) \cdot g(x)$.
(d) We may identify $\mathbb{G}_a \cong \mathbb{A}^1 \cong k$. Therefore, $\mathcal{O}(X) \cong \mathrm{Hom}(X, \mathbb{G}_a)$.
(e) An element of $\mathrm{Hom}(X, \mathbb{G}_m)$ is a regular function on X which is never zero. Therefore, it is invertible and its inverse is a regular never vanishing function on X. This means that $\mathrm{Hom}(X, \mathbb{G}_m) \cong \mathcal{O}(X)^*$.

CHAPTER 4

Rational Maps

Exercise 1. [**Har**, Exercise 4.6] The standard Cremona transformation of $\mathbb{P}^2$ is the rational map

$$\begin{array}{rcl}\phi : \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^2 \\ [x_0, x_1, x_2] & \longmapsto & [x_1x_2, x_0x_2, x_0x_1]\end{array}$$

(a) Note that

$$\phi^2(x_0, x_1, x_2) = [x_0^2x_1x_2 : x_0x_1^2x_2 : x_0x_1x_2^2] = x_0x_1x_2[x_0 : x_1 : x_2] = [x_0 : x_1 : x_2].$$

Therefore $\phi^{-1} = \phi$, and ϕ is birational.

(b) ϕ is an isomorphism on the open subset $\mathcal{U} = \mathbb{P}^2 \setminus \{x_0x_1x_2 = 0\}$, that is on the complement of the three lines spanned by the fundamental points $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1]$.

(c) ϕ and ϕ^{-1} are defined on $\mathbb{P}^2 \setminus \{p_1 = [1 : 0 : 0], p_2 = [0 : 1 : 0], p_3 = [0 : 0 : 1]\}$. The map ϕ is an isomorphism on $\mathcal{U}$ and contracts the line $L_{i,j} = \langle p_i, p_j \rangle$ to the point p_k with $k \neq i, j$, for any $i, j = 1, 2, 3$.

Exercise 2. [**Har**, Exercise 4.9] Let H be a linear subspace of dimension $n - r - 1$ such that $X \cap H = \emptyset$. The projection from H

$$\pi_H : X \to \mathbb{P}^r$$

is surjective. Therefore we get an inclusion $K(\mathbb{P}^r) \hookrightarrow K(X)$. Now,

$$\operatorname{trdeg}_k K(\mathbb{P}^r) = \operatorname{trdeg}_k K(X) = \dim(X) = r.$$

Then $K(X)$ is a finite algebraic extension of $K(\mathbb{P}^r)$. Assume $H = \{x_0 = ... = x_r = 0\}$. Then $K(X)$ is generated over $K(\mathbb{P}^r)$ by $\frac{x_{r+1}}{x_0}, ..., \frac{x_n}{x_0}$, and by the theorem of the primitive element $K(X)$ is generated over $K(\mathbb{P}^r)$ is generated by an element of the form $\sum_{i=r+1}^{n} \lambda_i \frac{x_i}{x_0}$. Consider the linear subspace $E = H \cap \{\sum_{i=r+1}^{n} \lambda_i \frac{x_i}{x_0} = 0\}$. Then, the map π_H factorizes as the projection π_E from L composed with the projection π_p where $p = \pi_E(H)$. Note that $Y = \pi_E(X)$ is an hypersurface. This gives

$$K(\mathbb{P}^r) \hookrightarrow K(Y) \hookrightarrow K(X)$$

where $K(X) = K(\mathbb{P}^r)(\frac{x_{r+1}}{x_0}, ..., \frac{x_n}{x_0})$, and $K(Y) = K(\mathbb{P}^r)(\sum_{i=r+1}^{n} \lambda_i \frac{x_i}{x_0})$. We conclude that the degree of the extension $K(Y) \hookrightarrow K(X)$. Therefore π_E is generically injective. Since π_E is dominant, it follows that it is birational.

Geometrically, we can argue as follows. Fix a general linear subspace H of dimension $n - r - 2$. In particular, $X \cap H = \emptyset$. For any $x \in X$ consider the linear space $H_x = \langle H, x \rangle$.

Note that $\dim(H_x) = n - r - 1$. Now, consider a general linear subspace Π of dimension $r+1$, and define the projection from H as:

$$\begin{array}{rcl} \pi_H : X & \longrightarrow & \Pi \cong \mathbb{P}^{r+1} \\ x & \longmapsto & H_x \cap \Pi \end{array}$$

Since $\dim(H_x) + \dim(X) - n < 0$ we have $H_x \cap X = \{x\}$ for a general $x \in X$. Therefore π_H is generically injective, and X is birational to its image $Y = \pi_H(X)$ which is an hypersurface in $\mathbb{P}^{r+1}$.

Exercise 3. Let $p \in \mathbb{P}^n$, with $n \geq 3$, be a point, and let

$$\pi_p : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$$

be the projection from p. Consider a linear subspace $H \subset \mathbb{P}^{n-1}$ of dimension k. Prove that $\Pi = \overline{\pi_p^{-1}(H)}$ is a linear subspace of $\mathbb{P}^n$ of dimension $k+1$ and passing through p.
Now, Let $C \subset \mathbb{P}^n$ be an irreducible, smooth, non-degenerate curve of degree d. Compute the degree of $\Gamma = \overline{\pi_p(C)}$ in the two cases $p \in C$, $p \notin C$.
Now, consider the twisted cubic $C = \nu(\mathbb{P}^1) \subset \mathbb{P}^3$, where $\nu : \mathbb{P}^1 \to \mathbb{P}^3$ is the degree three Veronese embedding. Let x, y, z, w be homogeneous coordinates on $\mathbb{P}^3$. Consider $p = [1:0:0:0] \in C$. Describe the curve $\Gamma = \overline{\pi_p(C)} \subset \mathbb{P}^2$.

(a) Now, let $p = [1:0:0:1] \in \mathbb{P}^3$. Write down explicitly the projection $\pi_p : \mathbb{P}^3 \to \mathbb{P}^2 \cong \{x = 0\}$. Prove that $\overline{\pi_p(C)} = \Gamma$ is the curve given by $\{y^3 - z^3 + yzw = 0\}$. Prove that $\mathrm{Sing}(\Gamma) = \{q = [0:0:1]\}$, $\mathrm{mult}_q \Gamma = 2$, and that Γ has two distinct principal tangents in q. Consider the line $L = \{y = z = 0\}$. Note that $p \in L$. Compute the intersection $L \cap C$.

(b) Let $p = [1:1:0:0] \in \mathbb{P}^3$. Write down explicitly the projection $\pi_p : \mathbb{P}^3 \to \mathbb{P}^2 \cong \{x = 0\}$. Prove that $\overline{\pi_p(C)} = \Gamma$ is the curve given by $\{z^3 - z^2 w + yw^2 = 0\}$. Prove that $\mathrm{Sing}(\Gamma) = \{q = [1:0:0]\}$, $\mathrm{mult}_q \Gamma = 2$, and that Γ has a double principal tangent in q. Consider the line $L = \{z = w = 0\}$. Note that $p \in L$. Compute the intersection $L \cap C$.

Finally, Give a geometric interpretation of both (a) and (b).

Let $H \subset \mathbb{P}^{n-1}$ be a linear subspace of dimension k. Then, $\Pi = \overline{\pi_p^{-1}(H)}$ is the cone over H with vertex p. Hence Π is a linear subspace of $\mathbb{P}^n$ of dimension $k+1$ and passing through p.
Let $C \subset \mathbb{P}^n$ be an irreducible, smooth, non-degenerate curve of degree d, and consider $\Gamma = \overline{\pi}_p(C)$.
Let $H \subset \mathbb{P}^{n-1}$ be a general hyperplane. Then $\Pi = \overline{\pi_p^{-1}(H)}$ is a general hyperplane in $\mathbb{P}^n$ passing though p. We have $\#(\Pi \cap C) = \#\{q_1, ..., q_d\} = d$.

- If $p \notin C$ then $\deg(\Gamma) = \#(H \cap \Gamma) = \#\{\pi_p(q_1), ..., \pi_p(q_d)\} = d$.
- If $p \in C$ then $p = q_i$ for some i. Without loss of generality we may assume $p = q_1$. In this case we have $\deg(\Gamma) = \#(H \cap \Gamma) = \#\{\pi_p(q_2), ..., \pi_p(q_d)\} = d - 1$.

The degree three Veronese embedding is defined as

$$\begin{array}{rcl} \nu : \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3 \\ [a:b] & \longmapsto & [a^3 : a^2 b : ab^2 : b^3] \end{array}$$

The projection from $p = [1:0:0:0]$ is the rational map given by

$$\begin{array}{rcl} \pi_p : \mathbb{P}^3 & \longrightarrow & \mathbb{P}^2 = \{x=0\} \\ [x:y:z:w] & \longmapsto & [y:z:w] \end{array}$$

Let $C = \nu(\mathbb{P}^1)$ be the twisted cubic. Then, $\pi_p(C) = \pi_p \circ \nu(\mathbb{P}^1)$, and

$$\pi_p([a^3 : a^2b : ab^2 : b^3]) = [a^2b : ab^2 : b^3] = [a^2, ab, b^2].$$

We conclude that $\pi_p(C)$ is the conic $\{z^2 - yw = 0\}$. Note that $p \in C$.

(a) The general line in $\mathbb{P}^3$ through $p = [1:0:0:1]$ is of the form

$$L_p = \{yc - zb = yd - wb + xb - ya = 0\}$$

with $[a:b:c:d] \in \mathbb{P}^3$. The intersection $L_p \cap \{x=0\}$ is the point $[y : \frac{c}{b}y : \frac{d-a}{b}y]$. Therefore, the projection π_p is the map

$$\begin{array}{rcl} \pi_p : \mathbb{P}^3 & \longrightarrow & \mathbb{P}^2 = \{x=0\} \\ [a:b:c:d] & \longmapsto & [b:c:d-a] \end{array}$$

In this case

$$\pi_p([s^3 : s^2t : st^2 : t^3]) = [s^2t : st^2 : t^3 - s^3].$$

Now, in the homogeneous coordinates y, z, w on $\mathbb{P}^2$ we see that $y^3 - z^3 + yzw = s^6t^3 - s^3t^6 + s^3t^3(t^3 - s^3) = 0$. Therefore, $\Gamma = \{F = y^3 - z^3 + yzw = 0\}$. The partial derivatives of F are

$$\frac{\partial F}{\partial y} = 3y^2 + zw,\ \frac{\partial F}{\partial z} = -3z^2 + yw,\ \frac{\partial F}{\partial w} = yz,$$

and we see that $\mathrm{Sing}(\Gamma) = \{q = [0:0:1]\}$. Furthermore, $\frac{\partial^2 F}{\partial z \partial y} = w$, and $\frac{\partial^2 F}{\partial z \partial y}(q) \neq 0$. So $\mathrm{mult}_q \Gamma = 2$.
Let us consider the de-homogenization of F with respect to w, that is $f = y^3 - z^3 + yz$. We see that the affine curve $\Gamma_w = \{y^3 - z^3 + yz = 0\}$ as two distinct tangent direction, namely $\{y=0\}$, $\{z=0\}$ at the origin. Then Γ has two distinct tangent direction given by $\{y=0\}$ and $\{z=0\}$ in q.
Finally the intersection $L \cap C$ consists of the two points $p_1 = [1:0:0:0]$, $p_2 = [0:0:0:1]$.

(b) In this case the general line in $\mathbb{P}^3$ through $p = [1:1:0:0]$ is of the form

$$L_p = \{xc - za - yc + zb = zd - wc = 0\}$$

with $[a:b:c:d] \in \mathbb{P}^3$. The intersection $L_p \cap \{x=0\}$ is the point $[\frac{b-a}{c}z : z : \frac{d}{c}z]$. Therefore, the projection π_p is the map

$$\begin{array}{rcl} \pi_p : \mathbb{P}^3 & \longrightarrow & \mathbb{P}^2 = \{x=0\} \\ [a:b:c:d] & \longmapsto & [b-a:c:d] \end{array}$$

and

$$\pi_p([s^3 : s^2t : st^2 : t^3]) = [s^2t - s^3 : st^2 : t^3].$$

Now, in the homogeneous coordinates y, z, w on $\mathbb{P}^2$ we see that $z^3 - z^2w + yw^2 = s^3t^6 - s^2t^7 + (s^2t - s^3)t^6 = 0$. Therefore, $\Gamma = \{F = z^3 - z^2w + yw^2 = 0\}$. The partial derivatives of F are

$$\frac{\partial F}{\partial y} = w^2,\ \frac{\partial F}{\partial z} = 3z^2 - 2zw,\ \frac{\partial F}{\partial w} = -z^2 + 2yw,$$

and $\mathrm{Sing}(\Gamma) = \{q = [1 : 0 : 0]\}$. Furthermore, $\frac{\partial^2 F}{\partial w^2} = 2y$, and $\frac{\partial^2 F}{\partial w^2}(q) \neq 0$. So $\mathrm{mult}_q \Gamma = 2$.

The de-homogenization of F with respect to y, that is $f = z^3 - z^2w + w^2$. The affine curve $\Gamma_w = \{z^3 - z^2w + w^2 = 0\}$ as one double principal tangent at the origin, namely $\{w = 0\}$. Then Γ has one double principal tangent given by $\{w = 0\}$ in q. Finally, the intersection $L \cap C$ consists of the point $p_1 = [1 : 0 : 0 : 0]$ with multiplicity two.

In (a) the line L is secant to C. Indeed $L \cap C$ consists of the two points $p_1 = [1 : 0 : 0 : 0]$, $p_2 = [0 : 0 : 0 : 1]$. Since we are projecting from a point $p \in L$ this secant line gets contracted by the projection. The tangent lines $\mathbb{T}_{p_1}C$, $\mathbb{T}_{p_2}C$ are mapped by the projection in the two principal tangents of Γ in its singular point q. Note that the singularity in q is a node coming from the identification of p_1, and p_2 after the projection.

In (b) the line L is the tangent line $\mathbb{T}_{p_1}C$. Since, $p \in L$ we are contracting the tangent direction (you may think at this as a degeneration of (a) for $p_1 \mapsto p_2$). We see that contracting the tangent line we produce a curve with a cusp.

CHAPTER 5

Non-singular Varieties

Exercise 1. [**Har**, Exercise 5.3] We have $\mu_p(Y) = 1$ if and only if in the decomposition

$$f = f_1 + ... + f_d$$

there is a term of degree one. Therefore, $f_1 = ax + by$ with $(a,b) \neq (0,0)$, and $\frac{\partial f}{\partial x}(0,0) = a$, $\frac{\partial f}{\partial y}(0,0) = b$. Therefore $\mu_p(Y) = 1$ if and only if $p \in Y$ is smooth.
Consider the nodal cubic $Y = \{x^3 + x^2 - y^2\}$. We have $f_3 = x^3$ and $f_2 = x^2 - y^2$. Therefore $\mu_p(Y) = 2$ and tangent directions of Y in p are the lines $\{x - y = 0\}$ and $\{x + y = 0\}$.

Exercise 2. [**Har**, Exercise 5.6]

(a) Consider the cusp $Y = \{x^3 - y^2 - x^4 - y^4 = 0\} \subset \mathbb{A}^2$. Substituting $y = ux$ we get

$$x^2(x - u^2 - x^2 - u^4x^2) = 0.$$

The curve $E = \{x = 0, y = 0\}$ is the exceptional divisor, while $\widetilde{Y} = \{x - u^2 - x^2 - u^4x^2 = y - ux = 0\}$ is the strict transform of Y. Taking the Jacobian matrix of $\widetilde{Y}$ we see that $\widetilde{Y}$ is smooth. Note that $E \cap \widetilde{Y} = (0,0,0)$ and the intersection multiplicity is two.
Now, consider the nodal curve $Y = \{xy - x^6 - y^6 = 0\} \subset \mathbb{A}^2$. Taking $y = ux$ we get

$$x^2(u - x^4 - u^6x^4) = 0.$$

Therefore $E = \{x = 0, y = 0\}$, and $\widetilde{Y} = \{y - ux = u - x^4 - u^6x^4 = 0\}$ is smooth.

(b) We may write the equation of the curve Y as $\{f(x,y) + xy = 0\}$, where f has terms of degree greater or equal that two. So that Y has s node in the origin. Let $X = \{xu - yv\} \subset \mathbb{A}^2 \times \mathbb{P}^1$ be the blow-up of $\mathbb{A}^2$ in the origin. Consider the chart $\{v \neq 0\}$. Then we get

$$f(x, ux) + x^2u = x^2(g(x, ux) + u) = 0.$$

The curve $E = \{x = 0, y = 0\}$ is the exceptional divisor, while the curve $\widetilde{Y} = \{y - ux = x^2(g(x,ux) + u) = 0\}$ is the strict transform of Y. Using the Jacobian criterion is is easy to see that $\widetilde{Y}$ is smooth. Furthermore $E \cap \widetilde{Y} = (0,0,0)$. The same argument works on the chart $\{u \neq 0\}$. Then $\widetilde{Y} \subset X$ is smooth and intersects E in two distinct points. This reflects the fact that Y has two distinct tangent direction at the origin.

(c) Consider the tacnode $Y = \{x^2 - x^4 - y^4 = 0\}$. In the affine chart $\{v \neq 0\}$ we have that the strict transform $\widetilde{Y}$ is defined by $\{y - ux = x^2 + x^2u^4 - 1 = 0\}$. Therefore $E \cap \widetilde{Y} = \emptyset$. In the chart $\{u \neq 0\}$ we have $\widetilde{Y} = \{x - vy = y^2v^4 + y^2 - v^2 = 0\}$, and $E \cap \widetilde{Y} = (0,0,0)$ with intersection multiplicity two. Note that the term of lowest degree in $y^2v^4 + y^2 - v^2$ is $y^2 - v^2 = (y - v)(y + v)$. Therefore, the origin is a node of $\widetilde{Y}$. By (b) we can resolve the singularity by blowing-up another time.

(d) Consider the higher cusp $Y = \{y^3 - x^5 = 0\}$. Note that the origin is a triple point with a unique triple tangent direction. Substituting $y = ux$ we get $x^3(u^3 - x^2) = 0$. Therefore, the strict transform $\widetilde{Y} = \{y - ux = u^3 - x^2 = 0\}$ intersects the exceptional divisor with multiplicity three in the origin. Note that $\widetilde{Y} = \{y - ux = u^3 - x^2 = 0\}$ is a cusp. By (a) we resolve the singularity by blowing-up the cusp.

Exercise 3. [**Har**, Exercise 5.8] A change of coordinates is an automorphism of $\mathbb{P}^n$. Therefore is sends smooth points of Y to smooth points. Therefore, we may assume that p lies in the affine chart $\{x_0 \neq 0\}$. The affine Jacobian is the $t \times n$ matrix obtained by deleting the first column of the projective Jacobian. Note that this column is

$$(\frac{\partial f_1}{\partial x_0}, ..., \frac{\partial f_t}{\partial x_0}).$$

By Euler's lemma

$$x_0 \frac{\partial f_j}{\partial x_0} = df_j - \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}$$

for any $j = 1, ..., t$. If $p = (a_0 : ... : a_n)$, since $p \in Y$ we get $f_j(p) = 0$ and

$$a_0 \frac{\partial f_j}{\partial x_0} = -\sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(p).$$

We see that the first column is a linear combination of the others n columns. Therefore, the rank of the projective Jacobian is equal to the rank of the affine Jacobian. By the affine Jacobian criterion we conclude that $p \in Y$ is smooth if and only if $\mathrm{rank}(Jac(f_1, ..., f_t)) = n - r$.

Exercise 4. [**Har**, Exercise 5.11] The projection from $p = [0 : 0 : 0 : 1]$ to $\{w = 0\}$ is the map

$$\begin{array}{ccc} \phi : Y & \longrightarrow & \mathbb{P}^2 \\ [x : y : z : w] & \longmapsto & [x : y : z] \end{array}$$

Note that

$$y^2z - x^3 + xz^2 = (x + z)(x^2 - xz - yw) + y(yz - xw - zw).$$

Therefore $\phi(Y) \subset \overline{Y} = \{y^2z - x^3 + xz^2 = 0\}$. Now, since ϕ is not constant $\phi(Y)$ is a curve, and since $\overline{Y}$ is irreducible we get $\phi(Y) = \overline{Y}$. The inverse of ϕ is given by

$$\begin{array}{ccc} \psi : \mathbb{P}^2 & \longrightarrow & Y \subset \mathbb{P}^3 \\ [x : y : z] & \longmapsto & [x : y : z : \frac{yz}{x+z}] = [x : y : z : \frac{x(x-z)}{y}] \end{array}$$

Note that ψ is defined on $\overline{Y} \setminus \{[1 : 0 : -1]\}$. This reflects the fact that the line spanned by $[1 : 0 : -1 : 0]$ and $[0 : 0 : 0 : 1]$ intersects Y in three points.

Exercise 4. [**Har**, Exercise 5.12]

(a) Over an algebraically closed field of characteristic different from two quadratics forms are classified by the rank. Then, we can write any homogeneous polynomial of degree two as
$$f = x_0^2 + ... + x_r^2$$
with $0 \leq r \leq n$.

(b) If $r \leq 1$ then, either $f = x_0^2$ and the quadric is a double hyperplane of $f = x_0^2 + x_1^2$ and the quadric is the union of two plane. In the first case the quadric is non-reduced, in the second it is reducible.
Assume f reducible. Then either $f = l^2$ or $f = lm$ where l, m are linear forms. Up to a change of coordinates we may write $f = x_0^2$ and $f = x_0x_1$. In the first case $r = 0$. In the second case $f = \frac{1}{4}((x_0 + x_1)^2 - (x_0 - x_1)^2)$ and $r = 1$.

(c) The singular locus of $Q = Z(x_0^2 + ... + x_r^2)$ is given by $\{x_0 = ... = x_r = 0\}$. Then, $\mathrm{Sing}(Q)$ is a linear subspace of dimension $n - r$.

(d) If $r < n$ and $Q = Z(x_0^2 + ... + x_r^2)$, then the polynomial $f = x_0^2 + ... + x_r^2$ defines a smooth quadric $Q^{'} \subset \mathbb{P}^r$. Any line generated by a point in $Q^{'}$ and a point in $\mathrm{Sing}(Q)$ intersects Q in at least three points counted with multiplicity because any point of $\mathrm{Sing}(Q)$ is a double point of Q. Then any such line is contained in Q, and Q is the cone over $Q^{'}$ with vertex $\mathrm{Sing}(Q)$.

CHAPTER 6

Non-singular Curves

Exercise 1. [**Har**, Exercise 6.4] Let f be a non-constant rational function on Y. Then, f yields a non-constant rational map $\phi : Y \dashrightarrow \mathbb{P}^1$, defined by $\phi(y) = f(y)$. Furthermore, since Y is a smooth projective curve the rational map ϕ extend to a morphism $\phi : Y \to \mathbb{P}^1$. Now, ϕ is non-constant we have that ϕ is surjective, and it induces an inclusion of fields $k(\mathbb{P}^1) \to k(Y)$. Both $k(\mathbb{P}^1)$ and $k(Y)$ are finite algebraic extensions of transcendence degree one of k we conclude that $k(Y)$ is a finite algebraic extension of $k(\mathbb{P}^1)$. Therefore, ϕ is finite. Another way to see this last fact is the following. Let $p \in \mathbb{P}^1$ be a point. Then $\phi^{-1}(p) \subseteq Y$ is closed. Since ϕ is not constant $\phi^{-1}(p) \neq Y$. Then, $\phi^{-1}(p)$ is a proper closed subset of a curve, therefore it is a finite set of points counted with multiplicity.

Exercise 2. [**Har**, Exercise 6.6] Let us consider the fractional linear transformation:

$$\begin{array}{cccc} \phi : \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\} & \longrightarrow & \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\} \\ x & \longmapsto & \frac{ax+b}{cx+d} \end{array}$$

with $a, b, c, d \in k$, $ad - bc \neq 0$. The inverse of ϕ is given by

$$\begin{array}{cccc} \phi^{-1} : \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\} & \longrightarrow & \mathbb{P}^1 = \mathbb{A}^1 \cup \{\infty\} \\ x & \longmapsto & \frac{1}{ad-bc}\frac{dx-b}{a-cx} \end{array}$$

Therefore, ϕ is an automorphism of $\mathbb{P}^1$.
Any automorphism ϕ of $\mathbb{P}^1$ induces an automorphism ϕ^* of $k(x) \cong k(\mathbb{P}^1)$ given by

$$\begin{array}{ccc} \phi^* : k(x) & \longrightarrow & k(x) \\ f & \longmapsto & f \circ \phi \end{array}$$

On the other hand, an automorphism of $k(x)$ induces a birational automorphism of $\mathbb{P}^1$. Since $\mathbb{P}^1$ is a smooth curve such a birational automorphism is indeed an automorphism.
Now, let ψ be an automorphism of $k(x)$. Then $\psi(x) = \frac{p(x)}{q(x)}$ where p and q do not have common factors. If either $\deg(p) \geq 2$ or $\deg(q) \geq 2$ then ψ can not be linear. Therefore, $p(x) = ax + b$ and $g(x) = cx + d$. Finally, since p and q do not have common factors their resultant is not zero, that is $ad - bc \neq 0$. We conclude that:

$$\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{Aut}(k(x)) \cong PGL(1).$$

CHAPTER 7

Intersections in Projective Space

Exercise 1. [**Har**, Exercise 7.1] Let $\nu_d : \mathbb{P}^n \to \mathbb{P}^n$ be the Veronese embedding of degree d, and let V_d^n be the Veronese variety. Since the embedding ν_d is defined by taking all the possible monomials of degree d in the homogeneous coordinates of $\mathbb{P}^n$ we see that degree l homogeneous polynomials on V_d^n correspond to degree ld homogeneous polynomials on $\mathbb{P}^n$. Then

$$h_{V_d^n}(l) = \dim(S(V_d^n)_l) = \binom{ld+n}{n} = \frac{d^n}{n!}l^n + ...$$

In particular, $\dim(V_d^n) = n$ and $\deg(V_d^n) = d^n$.
Now, let $\sigma_{n,m} : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ be the Segre embedding. In this case polynomials of degree l on the Segre variety $\Sigma_{n,m}$ corresponds to polynomials of bi-degree (l,l) on $\mathbb{P}^n \times \mathbb{P}^m$. Therefore,

$$h_{\Sigma_{n,m}}(l) = \dim(S(\Sigma_{n,m})_l) = \binom{l+n}{n} \cdot \binom{l+m}{m} = \frac{1}{(n+m)!}\binom{n+m}{m} l^{n+m} + ...$$

Hence, $\dim(\Sigma_{n,m}) = n+m$, and $\deg(\Sigma_{n,m}) = \binom{n+m}{m}$.

Exercise 1. [**Har**, Exercise 7.2]

(a) The Hilbert polynomial of $\mathbb{P}^n$ is given by $h_{\mathbb{P}^n}(l) = \dim(S_l) = \binom{n+l}{n}$. Then $p_a(\mathbb{P}^n) = (-1)^n(h_{\mathbb{P}^n}(l)(0) - 1) = (-1)^n(\binom{n}{n} - 1) = 0$.

(b) Let $Y = Z(f)$ be a plane curve of degree d. From the exact sequence

$$0 \mapsto S(-d) \to S \to S/(f) \mapsto 0$$

we get

$$h_Y(l) = \binom{l+2}{2} - \binom{l-d+2}{2}.$$

Therefore $h_Y(0) = 1 - \binom{2-d}{2} = 1 - \frac{1}{2}(d-1)(d-2)$. Then

$$p_a(Y) = \frac{1}{2}(d-1)(d-2).$$

(c) If $Y = Z(f)$ is an hypersurface of degree d we still have the exact sequence

$$0 \mapsto S(-d) \to S \to S/(f) \mapsto 0$$

and

$$h_Y(l) = \binom{l+n}{n} - \binom{l-d+n}{n}.$$

Then

$$p_a(Y) = (-1)^n\binom{n-d}{n} = (-1)^n\frac{(n-d)...(1-d)}{n!} = \frac{(d-1)...(d-n)}{n!} = \binom{d-1}{n}.$$

(c) If $Y = S_1 \cap S_2$ with $S_i = Z(f_i)$ from the exact sequence
$$0 \mapsto S/(f_1 f_2) \to S/(f_1) \oplus S/(f_2) \to S/(f_1, f_2) \mapsto 0$$
we get $h_Y = h_{S_1} + h_{S_2} - h_{S_1 \cup S_2}$. If $\deg(S_1) = a$ and $\deg(S_2) = b$ we get
$$p_a(Y) = \binom{3-a}{3} + \binom{3-b}{3} - \binom{3-a-b}{3} = \frac{1}{2}ab(a+b-4) + 1.$$

(d) We have $S(Y \times Z) = S(Y) \otimes S(Z)$. Therefore $h_{Y \times Z} = h_Y h_Z$, and $p_a(X \times Y) = (-1)^{r+s}(h_Y(0)h_Z(0) - 1)$ because $\dim(X \times Y) = r + s$. Then
$$\begin{aligned} p_a(X \times Y) &= (-1)^{r+s}((h_Y(0)-1)(h_Z(0)-1) + (h_Y(0)-1) + (h_Z(0)-1)) \\ &= p_a(Y)p_a(Z) + (-1)^s p_a(Y) + (-1)^r p_a(Z). \end{aligned}$$

Exercise 2. [**Har**, Exercise 7.4] Since $\mathrm{Sing}(Y)$ is a closed proper subset of Y the lines passing through singular points of Y defines a closed subset $Z_1 \subset (\mathbb{P}^2)^*$. The tangent lines two Y are a closed subset $Y^* \subset (\mathbb{P}^2)^*$. The closed subset Y^* is the dual curve of Y. By Bézout's theorem any line which is either tangent to Y or passing through a singular point of Y intersects Y is exactly $\deg(Y)$ distinct points. Therefore any line in the open subset $\mathcal{U} = (\mathbb{P}^2)^* \setminus (Z_1 \cup Y^*)$ has the required property.

Exercise 3. [**Har**, Exercise 7.5]

(a) Assume that there is point $p \in Y$ of multiplicity greater or equal that $d = \deg(Y) > 1$. Let $q \in Y$ be another point. Then the line $L = \langle p, q \rangle$ intersects Y in at least $d+1$ points counted with multiplicity. Since $\deg(Y) = d$, by Bézout's theorem we have $L \subset Y$. A contradiction, because Y is irreducible and $\deg(Y) \geq 2$.

(b) Since Y is irreducible of degree d, by Bézout's theorem any line passing through the point p of multiplicity d and another point $q \in Y$ is not contained in Y and does not intersect Y in any other point. Therefore, the projection $\pi_p : Y \dashrightarrow \mathbb{P}^1$ from p is birational.

Exercise 4. [**Har**, Exercise 7.6] If $Y = Y_1 \cup Y_2$ has two components then $\deg(Y) = deg(Y_1) + \deg(Y_2) = 1$. Therefore Y is irreducible.
Assume $\dim(Y) = 1$. Consider two points $p, q \in Y$. By Bézout's theorem any hyperplane passing through p, q contain Y. Therefore Y is the intersection of these hyperplanes, that is Y is the line spanned by p and q.
If $\dim(Y) = r$ consider a general hyperplane section $Y_H = Y \cap H$. Then $\dim(Y_H) = r - 1$ and $\deg(Y_H) = 1$. By induction hypothesis we have that Y_H is linear. Now, take a point $p \in Y \setminus Y_H$. Any line spanned by p and a point in Y_H intersects Y in at least two points. Since $\deg(Y) = 1$ by Bézout's theorem any such line is contained in Y. Therefore Y is a cone over the linear subspace Y_H. Then Y itself is a linear subspace of dimension r.

CHAPTER 8

Blow-ups

Exercise 1. **[PhE]** Let $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ be the surface $\{x_0y_1 - x_1y_0 = 0\}$. Prove that X is not isomorphic to $\mathbb{P}^2$.

The surface $X \subset \mathbb{P}^2 \times \mathbb{P}^1$ is the blow-up of $\mathbb{P}^2$ in $p = [0 : 0 : 1]$. Consider two lines $L, R \subset \mathbb{P}^2$ through p. Then, their strict transforms $\widetilde{L}, \widetilde{R}$ via the blow-up map $\pi_X \to \mathbb{P}^2$ do not intersect. On the other hand any two curves in $\mathbb{P}^2$ intersect. So X can not be isomorphic to $\mathbb{P}^2$.

Exercise 2. Let $Q \subset \mathbb{P}^3$ be a smooth quadric and let $p \in Q$ be a point. Prove that Bl_pQ is isomorphic to $B_{q_1,q_2}\mathbb{P}^2$ where $q_1, q_2 \in \mathbb{P}^2$ are two distinct points.

We may assume $Q = \{x_0x_3 - x_1x_2 = 0\} \subset \mathbb{P}^3$, and $p = [0 : 0 : 0 : 1]$. Let $\pi : Q \dashrightarrow H \cong \mathbb{P}^2$ be the projection from p. Note that π is birational. If y_0, y_1, y_2 are homogeneous coordinates on $\mathbb{P}^2$ then the graph Γ_π of π is given by

$$\{x_0y_1 - x_1y_0 = x_1y_2 - x_2y_1 = x_0x_3 - x_1x_2 = 0\} \subset \mathbb{P}^3 \times \mathbb{P}^2.$$

Let $\pi_1 : \Gamma_\pi \to Q$ be the projection. From these equations we see that $\pi_1 : \Gamma_\pi \to Q$ is the blow-up of Q in p.
Now, let $\pi_2 : \Gamma_\pi \to \mathbb{P}^2$ be the second projection. The exceptional divisor $E = \{x_1 = y_0 = 0\}$ is mapped via π_2 to the line $\{y_0 = 0\}$. The intersection $\mathbb{T}_pQ \cap Q$ is the union of the two lines $L = \{x_0 = x_1 = 0\}$ and $R = \{x_0 = x_2 = 0\}$. Let $\widetilde{L}$ and $\widetilde{R}$ be the strict transforms of L and R via π_1. Then $\pi_2(\widetilde{L}) = [0 : 0 : 1] = q_1$ and $\pi_2(\widetilde{R}) = [0 : 1 : 0] = q_2$.
Now let $f : BL_{q_1,q_2}\mathbb{P}^2 \to \mathbb{P}^2$ be the blow-up of $\mathbb{P}^2$ in q_1, q_2. Consider the rational map

$$\begin{array}{cccc} g : \mathbb{P}^2 & \dashrightarrow & \mathbb{P}^3 \times \mathbb{P}^2 \\ [y_0 : y_1 : y_2] & \longmapsto & ([y_0^2 : y_0y_1 : y_0y_2 : y_1y_2], [y_0 : y_1 : y_2]). \end{array}$$

Since $I(\{q_1, q_2\}) = (y_0^2 : y_0y_1 : y_0y_2 : y_1y_2)$ the map g is the inverse of f. On the other hand g is the inverse of π_2 as well. Therefore π_2 and f are two morphisms coinciding on an open subset. We conclude that $\pi_2 = f$ and $\Gamma_\pi \cong Bl_{q_1,q_2}\mathbb{P}^2$. Finally $\Gamma_\pi \cong Bl_{q_1,q_2}\mathbb{P}^2 \cong Bl_pQ$.

CHAPTER 9

Grassmannians

Exercise 1. Consider the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$ parametrizing lines in $\mathbb{P}^4$. Find five polynomials in the homogeneous coordinates of $\mathbb{P}^9$ vanishing on $\mathbb{G}(1,4)$.

Let L be a line in $\mathbb{P}^4$ generated be the two points $[u_0 : u_1 : u_2 : u_3 : u_4]$ and $[v_0 : v_1 : v_2 : v_3 : v_4]$. Then the Plücker embedding

$$p : \mathbb{G}(1,4) \to \mathbb{P}^9$$

is given by mapping L to

$$[u_0v_1 - u_1v_0 : u_0v_2 - u_2v_0 : u_0v_3 - u_3v_0 : u_0v_4 - u_4v_0 : u_1v_2 - u_2v_1 :$$

$$u_1v_3 - u_3v_1 : u_1v_4 - u_4v_1 : u_2v_3 - u_3v_2 : u_2v_4 - u_4v_2 : u_3v_4 - u_4v_3].$$

Let $X_{0,1}, ..., X_{3,4}$ be the homogeneous coordinates on $\mathbb{P}^9$. Then, among the coordinates of the Plücker embedding there are the following relations:

$$\begin{gathered}
X_{0,1}X_{2,3} - X_{0,2}X_{1,3} + X_{0,3}X_{1,2} = 0,\\
X_{0,1}X_{2,4} - X_{0,2}X_{1,4} + X_{0,4}X_{1,2} = 0,\\
X_{0,1}X_{3,4} - X_{0,3}X_{1,4} + X_{0,4}X_{1,3} = 0,\\
X_{0,2}X_{3,4} - X_{0,3}X_{2,4} + X_{0,4}X_{2,3} = 0,\\
X_{1,2}X_{3,4} - X_{1,3}X_{2,4} + X_{1,4}X_{2,3} = 0.
\end{gathered}$$

Exercise 2. Let L, R be two lines in $\mathbb{P}^3$, and let $l, r \in \mathbb{G}(1,3)$ be the corresponding points. Prove that $L \cap R \neq \emptyset$ if and only if the line joining l and r is contained in $\mathbb{G}(1,3)$.

Let H_L, H_R be the planes in V^4 corresponding to L and R. If $L \cap R \neq \emptyset$ then H_L and H_R share a non-zero vector $u \in H_L \cap H_R$. Let $\{u_1, u\}$ and $\{u_2, u\}$ be basis of H_L and H_R respectively. Therefore the corresponding points in $\mathbb{G}(1,3)$ are $u_1 \wedge u$ and $u_2 \wedge u$. So the line spanned by $u_1 \wedge u$ and $u_2 \wedge u$ is $\mathbb{P}(W)$ where $W = \langle u_1 \wedge u, u_2 \wedge u \rangle \subset \bigwedge^2 V$. Now, note that any vector in W is of the form

$$\alpha(u_1 \wedge u) + \beta(u_2 \wedge u) = u \wedge (\alpha u_1 + \beta u_2).$$

Therefore, any point in $\mathbb{P}(W)$ corresponds to a decomposable 2-vector, that is $\mathbb{P}(W) \subset \mathbb{G}(1,3)$.
Now, assume $L \cap R \neq \emptyset$. Then $H_L \cap H_R = \{0\}$. Let $\{u_1, u_2\}$ and $\{v_1, v_2\}$ be basis of H_L and H_R respectively. So $\{u_1, u_2, v_1, v_2\}$ is a basis of V, and $u_1 \wedge u_2 \wedge v_1 \wedge v_2 \neq 0$. In this case the line $\mathbb{P}(W)$ is generated by $u_1 \wedge u_2$ and $v_1 \wedge v_2$, that is a point on $\mathbb{P}(W)$ is of the form

$$\alpha(u_1 \wedge u_2) + \beta(v_1 \wedge v_2).$$

Now,

$$(\alpha(u_1 \wedge u_2) + \beta(v_1 \wedge v_2)) \wedge (\alpha(u_1 \wedge u_2) + \beta(v_1 \wedge v_2)) = 2\alpha\beta(u_1 \wedge u_2 \wedge v_1 \wedge v_2).$$

Note that, if $v = (\alpha(u_1 \wedge u_2) + \beta(v_1 \wedge v_2)) \in \mathbb{G}(1,3)$ then v is decomposable. So $v = w_1 \wedge w_2$ yields $v \wedge v = 0$.
On the other hand

$$(\alpha(u_1 \wedge u_2) + \beta(v_1 \wedge v_2)) \wedge (\alpha(u_1 \wedge u_2) + \beta(v_1 \wedge v_2)) = 2\alpha\beta(u_1 \wedge u_2 \wedge v_1 \wedge v_2) = 0$$

if and only if either $\alpha = 0$ or $\beta = 0$. Then, the line $\mathbb{P}(W)$ is not contained in $\mathbb{G}(1,3)$.

Exercise 3. Let $p \in \mathbb{P}^3$ be a point, and $H \subset \mathbb{P}^3$ a plane containing p. Let $\Sigma_{p,H} \subset \mathbb{G}(1,3)$ be the locus parametrizing lines in H passing through p. Prove that the image of $\Sigma_{p,H}$ via the Plücker embedding is a line in $\mathbb{P}^5$. Conversely, prove that any line contained in $\mathbb{G}(1,3) \subset \mathbb{P}^5$ is of the form $\Sigma_{p,H}$.

Let $u \in V$ be a representative for p, and let $\{u, v, w\}$ be a basis of W, where $\mathbb{P}(W) = H$. Then the lines in H through p corresponds to the subspaces of W spanned by u and a vector of the form $\alpha u + \beta v + \gamma w$. Now

$$w \wedge (\alpha u + \beta v + \gamma w) = \beta(u \wedge v) + \gamma(u \wedge w).$$

Therefore, lines in H passing through p corresponds to the line in $\mathbb{G}(1,3)$ spanned by $u \wedge v$ and $u \wedge w$.
Now, let T be a line in $\mathbb{G}(1,3)$, and let $r, s \in T$ be two points. Then the lines $L, R \subset \mathbb{P}^3$ corresponding to r, s intersects by Exercise 2. Let Π be the plane spanned by L, R. Since L, R generate Π and l, r generate T, a line in Π through $L \cap R$ corresponds to a point of T.

Exercise 4. For any point $p \in \mathbb{P}^3$ be $\Sigma_p \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ be the locus parametrizing lines in $\mathbb{P}^3$ through p. Similarly, for any plane $H \subset \mathbb{P}^3$ be $\Sigma_H \subset \mathbb{G}(1,3) \subset \mathbb{P}^5$ be the locus parametrizing lines in $\mathbb{P}^3$ contained in H. Prove that both Σ_p and Σ_H are mapped to planes of $\mathbb{P}^5$ via the Plücker embedding. Conversely, prove that any plane in $\mathbb{G}(1,4) \subset \mathbb{P}^5$ is of the form Σ_p of Σ_H.

Let $u \in V$ be a vector representing $p \in \mathbb{P}^3$. Then the lines through p are represented by 2-vectors of the form $u \wedge v$. Let $\{u, u_1, u_2, u_3\}$ be a basis of V. Then we may write $v = \alpha u + \beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3$, and

$$u \wedge v = \alpha_1(u \wedge u_1) + \alpha_2(u \wedge u_2) + \alpha_3(u \wedge u_3).$$

Therefore, lines through p are represented by the points of the plane spanned by $u \wedge u_1$, $u \wedge u_2$ and $u \wedge u_3$.
Now, the lines contained in the plane $H \subset \mathbb{P}^3$, by duality corresponds to the lines in $\mathbb{P}^{3^*}$ through the point H^*. Therefore they are parametrized by a plane in $\mathbb{G}(1,3)$ by the first part of the exercise.
Now, take a plane Π in $\mathbb{G}(1,3)$ and three points l, r, s in this plane that do not lie on the same line. Let $L, R, S \subset \mathbb{P}^3$ be the corresponding lines. Since the three lines joining l,r and s are on the same plane contained in $\mathbb{G}(1,3)$ they intersect and they are contained in $\mathbb{G}(1,3)$. By Exercise 2 the lines L, R, S intersect. We have two cases.

- $L \cap R \cap S = \{p\}$. In this case Π parametrizes lines in $\mathbb{P}^3$ through p.

- L, R and S intersect in three distinct points. Let u, v, w be three representative vectors for these three points. Then L, R, S are represented by $v \wedge w$, $u \wedge w$ and $u \wedge v$. Then a point on the plane Π is of the form
$$\alpha(v \wedge w) + \beta(u \wedge w) + \gamma(u \wedge v).$$
Therefore L, R, S lie in the plane $\mathbb{P}(H)$, where $H = \langle u, v, w \rangle$.

Exercise 5. Let $Q \subset \mathbb{P}^3$ be a smooth quadric. Prove that the two families of lines in Q are mapped via the Plücker embedding to two plane conics in $\mathbb{G}(1,3) \subset \mathbb{P}^5$ lying in two complementary planes.

We can assume that $Q = \{xw - yz = 0\} \subset \mathbb{P}^3$ is the image of the Segre embedding
$$\begin{array}{rcc} s : \mathbb{P}^1 \times \mathbb{P}^1 & \longrightarrow & \mathbb{P}^3 \\ ([u,v],[s,t]) & \longmapsto & [us : ut : vs : vt] \end{array}$$
In order to parametrize the first family of lines we can consider for each $[u : v] \in \mathbb{P}^1$ the line $L_{u,v}$ spanned by the two points $s([u,v],[1,0]) = [u : 0 : v : 0]$ and $s([u,v],[0,1]) = [0 : u : 0 : v]$. Under the Plücker embedding $L_{u,v}$ is mapped to the point
$$[u^2 : 0 : uv : -uv : 0 : v^2].$$
If $X_0, ..., X_5$ are the homogeneous coordinates on $\mathbb{P}^5$ we see that the set of points of the form $[u^2 : 0 : uv : -uv : 0 : v^2]$ is defined by $\{X_1 = X_4 = X_2 + X_3 = X_5X_0 - X_2^2 = 0\} \subset \mathbb{G}(1,3)$. Therefore the lines of the first family are parametrized by a smooth conic in the plane $H_1 = \{X_1 = X_4 = X_2 + X_3 = 0\}$.
In the same way the lines of the second family correspond to points of the form
$$[0 : s^2 : st : st : t^2 : 0].$$
Therefore, the lines of the second family are parametrized by the smooth conic given by $\{X_0 = X_5 = X_2 - X_3 = X_2^2 - X_1X_4 = 0\}$ in the plane $H_2 = \{X_0 = X_5 = X_2 - X_3 = 0\}$. Finally, $H_1 \cap H_2 = \emptyset$.

Exercise 6. Let $\mathbb{G}(1,n)$ be the Grassmannian of lines in $\mathbb{P}^n$. Prove that through two general points of $\mathbb{G}(1,n)$ there is a smooth variety of dimension four and degree two.

Let $l, r \in \mathbb{G}(1,n)$ be two general points. These points corresponds to to two general lines $L, R \subset \mathbb{P}^n$. Since L and R are general they span a linear space $H \subset \mathbb{P}^n$ of dimension three, $H \cong \mathbb{P}^3$. The image of the Plücker embedding of $\mathbb{G}(1,n)$ restricted to the lines in H gives a $\mathbb{G}(1,3) \subseteq \mathbb{G}(1,n)$ and $l, r \in \mathbb{G}(1,3)$. Now, it is enough to observe that under the the Plücker embedding $\mathbb{G}(1,3)$ is a smooth quadric hypersurface in $\mathbb{P}^5$.

Exercise 7. Let L_1, L_2, L_3, L_4 be four general lines in $\mathbb{P}^3$. Consider
$$X = \{L \mid L \cap L_i \neq \emptyset \, \forall \, i = 1,2,3,4\} \subset \mathbb{G}(1,3) \subset \mathbb{P}^5.$$
Compute the dimension and the degree of X.

Consider the lines L_1, L_2, L_3. Since they are general these three lines are pairwise skew. By Exercise 9 of Section 2 there exists a unique smooth quadric surface $Q \subset \mathbb{P}^3$ containing L_1, L_2, L_3.

Since L_4 is general we have that $L_4 \cap Q = \{p, q\}$. Now, any line L intersecting L_1, L_2, L_3 is contained in Q. Therefore, in order to intersect L_4 this line L has to pass either through p or q. We conclude that there are two lines intersecting L_1, L_2, L_3, L_4. Therefore, $\dim(X) = 0$ and $\deg(X) = 2$.

CHAPTER 10

Secant Varieties

Exercise 1. Let $\nu : \mathbb{P}^2 \to \mathbb{P}^{14}$ be the degree four Veronese embedding. Let $V \subset \mathbb{P}^{14}$ be the corresponding Veronese variety, and $\mathcal{S}ec_5(V) \subseteq \mathbb{P}^{14}$ the 5-secant variety of V.

- Compute the expected dimension of $\mathcal{S}ec_5(V)$,
- Prove that there exists a non-zero homogeneous polynomial P of degree six in the homogeneous coordinates of $\mathbb{P}^{14}$ such that any polynomial $F \in \mathcal{S}ec_5(V) \subseteq \mathbb{P}^{14}$ is a zero of P. Conclude that V is 5-secant defective with secant defect $\delta_5(V) = 1$, and therefore that $\mathcal{S}ec_5(V)$ is an hypersurface in $\mathbb{P}^{14}$.

Finally, prove that $X := \{P = 0\} \subset \mathbb{P}^{14}$ is irreducible. Conclude that $\mathcal{S}ec_5(V) \subset \mathbb{P}^{14}$ is an irreducible hypersurface of degree six.

The expected dimension is

$$\text{expdim}(\mathcal{S}ec_5(V)) := \min\{5\dim(V) + 4, 14\} = 14.$$

Now, consider a general polynomial $F \in k[x, y, z]_4$:

$$\begin{aligned} F = \;& a_0x^4 + a_1x^3y + a_2x^3z + a_3x^2y^2 + a_4x^2yz + a_5x^2z^2 + a_6xy^3 + a_7xy^2z + a_8xyz^2 \\ & + a_9xz^3 + a_{10}y^4 + a_{11}y^3z + a_{12}y^2z^2 + a_{13}yz^3 + a_{14}z^4. \end{aligned}$$

If $F \in \mathcal{S}ec_5(V)$ is general then $F = L_1^4 + ... + L_5^4$ for some linear forms $L_1, ..., L_5$. Therefore, the second partial derivatives of F are six points in $\mathbb{P}^5 = \mathbb{P}(k[x, y, z]_2)$ lying on the hyperplane spanned by $L_1^2, ..., L_5^2$. Let M be the 6×6 matrix whose lines are the second partial derivative of F. Take $P = \det(M)$. Then P is a homogeneous polynomial of degree six in $a_0, ..., a_{14}$. Let $X := \{P = 0\}$. Then $\mathcal{S}ec_5(V) \subseteq X$. In particular $\delta_5(V) > 0$. On the other hand $\delta_5(V) < \dim(V) = 2$. Therefore, $\delta_5(V) = 1$. Therefore: It is easy to see that there are three partial derivatives of P that are independent. Therefore, the codimension of $\text{Sing}(X)$ in $\mathbb{P}^{14}$ is strictly greater that two, and X can not be reducible.
Finally, $\mathcal{S}ec_5(V) \subseteq X$ is an hypersurface in $\mathbb{P}^{14}$ as well, since X is irreducible we conclude that $\mathcal{S}ec_5(V) = X$ is an irreducible hypersurface of degree six.

Exercise 2. Prove that $n \times n$ symmetric matrices over a field k modulo scalar multiplication are parametrized by a projective space of dimension $N = \frac{n(n+1)}{2} - 1$.
For any $0 < k \leq n$ prove that the set

$$M_k = \{A \in \mathbb{P}^N \mid \text{rank}(A) \leq k\}$$

is an algebraic subvariety of $\mathbb{P}^N$.
Consider the incidence variety

$$\mathcal{I} := \{(A, H) \mid H \subseteq \ker(A)\} \subseteq \mathbb{P}^N \times G(n - k, n)$$

with projections $f : \mathcal{I} \to \mathbb{P}^N$, and $g : \mathcal{I} \to G(n-k, n)$. Using the theorem on the dimension of the fibers prove that

$$\dim(M_k) = \binom{k-1+2}{2} - 1 + k(n-k).$$

Let $\nu : \mathbb{P}^{n-1} \to \mathbb{P}^N$ be the degree two Veronese embedding, and let V_2^{n-1} be the corresponding Veronese variety. Note that $N = \binom{n-1+2}{2} - 1 = \frac{n(n+1)}{2} - 1$.
Prove that $\mathbb{S}ec_k(V_2^{n-1}) = M_k$. Conclude that the $(n-1)$-secant defect of V_2^{n-1} is $\delta_{n-1}(V_2^{n-1}) = 1$ for any $n \geq 3$.

A general $n \times n$ symmetric matrix is determined by $n + (n-1) + ... + 2 + 1 = \frac{n(n+1)}{2}$ parameters. Therefore, $n \times n$ symmetric matrices moduli scalar multiplication are parametrized by a projective space of dimension $N = \frac{n(n+1)}{2} - 1$.
The set M_k is the common zero locus of the $(k+1) \times (k+1)$ minors of A. These are homogeneous polynomials of degree $k+1$ on $\mathbb{P}^N$. Therefore, $M_k \subseteq \mathbb{P}^N$ is an algebraic subvariety.
Consider the incidence variety

$$\mathcal{I} := \{(A, H) \mid H \subseteq \ker(A)\} \subseteq \mathbb{P}^N \times G(n-k, n)$$

with projections $f : \mathcal{I} \to \mathbb{P}^N$, and $g : \mathcal{I} \to G(n-k, n)$. Fix $H \in G(n-k, n)$. Then, a matrix $A \in g^{-1}(H)$ corresponds two a quadratic form on a vector space of dimension k. These quadratic forms are parametrized by a projective space of dimension $\binom{k-1+2}{2} - 1$. By the theorem on the dimension of the fibers we have

$$\dim(\mathcal{I}) = \binom{k-1+2}{2} - 1 + k(n-k).$$

Now the second projection f is generically injective. Since $M_k = f(\mathcal{I})$ we conclude that

$$\dim(M_k) = \binom{k-1+2}{2} - 1 + k(n-k).$$

A general degree two polynomial $F \in \mathbb{S}ec_k(V_2^{n-1})$ can be written as $F = L_1^2 + ... + L_k^2$ for k linear forms. The same holds for a general polynomial in M_k. Therefore, M_k and $\mathbb{S}ec_k(V_2^{n-1})$ are both defined by the vanishing of the $(k+1) \times (k+1)$ minors of a general $n \times n$ symmetric matrix.
In particular

$$\dim(\mathbb{S}ec_k(V_2^{n-1})) = \dim(M_k) = \binom{k-1+2}{2} - 1 + k(n-k).$$

For $k = n-1$ and $n \geq 3$ we have $\delta_{n-1}(V_2^{n-1}) = N - \binom{n-1-1+2}{2} + 1 - (n-1)(n-(n-1)) = \frac{n(n+1)}{2} - \binom{n}{2} - (n-1) = 1$.

Exercise 3. Let us fix two integers $h > 1$, $h \leq d \leq 2h-1$. Prove that under this numerical hypothesis a general homogeneous polynomial $F \in k[x, y]_d$ of degree d admits

a decomposition of the form $F = \lambda_1 L_1^d + ... + \lambda_h L_h^d$.
Now, fix a general $F \in k[x,y]_d$ and consider the variety

$$X(F,h) = \overline{\{\{L_1,...,L_h\} \mid F = \lambda_1 L_1^d + ... + \lambda_h L_h^d,\ L_i \in k[x,y]_1\}} \subseteq \mathbb{P}(k[x,y]_1)^h / S_h.$$

The $X(F,h)$ parametrizes all the decomposition of F as sums of d-powers of linear forms. Prove that for $h > 1$, $h \leq d \leq 2h-1$ the variety $X(F,h)$ is birational to $\mathbb{P}^{2h-d-1}$.
In particular, conclude that a general homogeneous polynomial $F \in k[x,y]_{2h-1}$ admits a unique decomposition in h powers of linear form. Finally deduce that if $C \subset \mathbb{P}^3$ is a twisted cubic and $p \in \mathbb{P}^3$ is a general point, then there exists a unique line secant to C passing through p.

A general homogeneous polynomial $F \in k[x,y]_d$ of degree d admits a decomposition of the form $F = \lambda_1 L_1^d + ... + \lambda_h L_h^d$ if and only if there exists a $(h-1)$-plane h-secant to the rational normal curve $C \subset \mathbb{P}^d$ passing through $F \in \mathbb{P}^d$. Now, $C \subset \mathbb{P}^d$ is a non-degenerate curve. Assume $d = 2h-1$ is odd, and fix $h = \frac{d+1}{2}$. Now, assume that through a general point $p \in \mathbb{S}ec_h(C)$ there are two distinct $(h-1)$-plane H_1, H_2 that are h-secant to C, say $H_1 \cap C = \{p_1,...,p_h\}$, and $H_2 \cap C = \{q_1,...,q_h\}$. Since H_1, H_2 are distinct we may have at most $p_1 = q_1$,..., $p_k = q_k$ with $k < h$. Since H_1, H_2 intersects in p as well, they span a linear space H of dimension $h-1+h-1-k = 2h-2-k$. Therefore $H \subset \mathbb{P}^d$ is a linear subspace of dimension $2h-2-k$ intersecting C in $2h-k$ points. Since $\deg(C) = d$ we found a contradiction. Then, trough a general point of $\mathbb{S}ec_h(C)$ there is at most an $(h-1)$-plane h-secant to C, and $\dim(\mathbb{S}ec_h(C)) = 2h-1$.
If $d = 2h$ is even then through a general point $p \in \mathbb{S}ec_{h+1}(C)$ there is a family of dimension exactly one of h-planes $(h+1)$-secant to C (just consider the partial derivatives of order $h-1$ of p interpreted as a degree d polynomial). Then $\dim(\mathbb{S}ec_{h+1}(C)) = h+1+h-1 = 2h$. Therefore, in any case $d \leq 2h-1$ implies

$$\dim(\mathbb{S}ec_h(C)) \geq \frac{d+1}{2} + \frac{d+1}{2} - 1 \geq d.$$

Then $\mathbb{S}ec_h(C) = \mathbb{P}^d$ and we are done.
Now, consider a decomposition $F = \lambda_1 L_1^d + ... + \lambda_h L_h^d$. Consider the partial derivatives of order $d-h$ of F. These are

$$\binom{d-h+1}{d-h} = d-h+1 \leq h$$

homogeneous polynomial of degree h. Furthermore, all these partial derivatives can be decomposed as a linear combination of $L_1^h,...,L_h^h$. This means that the linear space $H_\partial \subset \mathbb{P}^h$ of dimension $d-h$ spanned by the partial derivative is contained in the hyperplane $\langle L_1^h,...,L_h^h \rangle$. Now, note that the hyperplanes in $\mathbb{P}^h$ containing H_∂ are parametrized by $\mathbb{P}^{2h-d-1}$. Therefore we get a rational map

$$\begin{array}{cccc} \phi: & X(F,h) & \dashrightarrow & \mathbb{P}^{2h-d-1} \\ & \{L_1,...,L_h\} & \longmapsto & \langle L_1^h,...,L_h^h \rangle \end{array}$$

Now, a general hyperplane H containing H_∂ intersects the rational normal curve $C_h \subset \mathbb{P}^h$ of degree h in h points $l_1^h, ..., l_h^h$. Since $H_\partial \subseteq H = \langle l_1^h, ..., l_h^h \rangle$ these points yields a decomposition of all the partial derivatives of order h of F. This gives a decomposition $\{l_1^d, ..., l_h^d\}$ of F. Therefore, ϕ is dominant and generically injective, that is ϕ is birational.
If $d = 2h-1$ then $2h-d-1 = 0$, and $X(F, 2h-1)$ is a point. This means that F admits a unique decomposition in h powers of linear form (This could be deduced from the first part of the proof as well).
if $C \subset \mathbb{P}^3$ is a twisted cubic and $p \in \mathbb{P}^3$ is a general points, we may interpret p as a general $F \in k[x,y]_3$. A line secant to C and passing through p corresponds to a decomposition of F as a sum of $h = 2$ cubes of linear forms. In this case $d = 3 = 2h-1$. Then there exists a unique such decomposition. This means that there exists a unique secant line to C passing through p.

Exercise 4. Let $\Sigma = \sigma(\mathbb{P}_1^2 \times \mathbb{P}_2^2) \subset \mathbb{P}^8$ be the Segre embedding. Compute the dimension of $\mathbb{S}ec_2(\Sigma)$ and the secant defect $\delta_2(\Sigma)$.

The expected dimension of $\operatorname{expdim}(\mathbb{S}ec_2(\Sigma)) = \min\{2\dim(\Sigma)+1, 8\} = 8$. On the other hand we may interpret $\mathbb{P}^8$ as the space of 3×3 matrices modulo scalar multiplication. Then, Σ parametrizes rank one matrices, and $\mathbb{S}ec_2(\Sigma)$ parametrizes rank two matrices. Therefore, $\mathbb{S}ec_2(\Sigma) = \{\det(M) = 0\}$, where M is a general 3×3 matrix, is an hypersurface of degree three in $\mathbb{P}^8$. We conclude that $\dim(\mathbb{S}ec_2(\Sigma)) = 7$ and $\delta_2(\Sigma) = 9 - 7 = 2$.
We may argue also as follows: let $p \in \mathbb{S}ec_2(\Sigma)$ be a general point, and let $x = (x_1, x_2)$, $y = (y_1, y_2)$ be points spanning a secant line to Σ through p, where $x_1, y_1 \in \mathbb{P}_1^2$, and $x_2, y_2 \in \mathbb{P}_2^2$. Let $L_1 \subset \mathbb{P}_1^2$ be the line spanned by x_1, y_1, and $L_2 \subset \mathbb{P}_2^2$ be the line spanned by x_2, y_2. Then $\sigma(L_1 \times L_2)$ is a quadric surface Q through x, y. If H is the 3-plane spanned by Q, then $p \in H$ and any line through p in H is a secant line of Σ. Therefore, $\delta_2(\Sigma) \geq 2$. Clearly $\delta_2(\Sigma) < 3$. We conclude that $\delta_2(\Sigma) = 2$ and $\dim(\mathbb{S}ec_2(\Sigma)) = 7$.

Exercise 5. Let $X \subset \mathbb{P}^n$ be an irreducible curve. Then, $\dim(\mathbb{S}ec_2(X)) = 2$ implies that X is contained in a plane.

If $\dim(\mathbb{S}ec_2(X)) = 2$ then through a general point $p \in \mathbb{S}ec_2(X)$ there is a family of dimension one of secant lines to X. Since $p \in \mathbb{S}ec_2(X)$ is general it is smooth. Let $\mathbb{T}_p\mathbb{S}ec_2(X)$ be the tangent space of $\mathbb{S}ec_2(X)$ at p. Both $\mathbb{S}ec_2(X)$ and $\mathbb{T}_p\mathbb{S}ec_2(X)$ are of dimension two, and they intersect in a family of dimension one of lines. This forces $\mathbb{S}ec_2(X) = \mathbb{T}_p\mathbb{S}ec_2(X) \cong \mathbb{P}^2$, and X is contained in a plane.

Exercise 6. Let $p_1, ..., p_h \in \mathbb{P}^n$ be general points, and $\mathbb{P}(V_{n,d,h}) \subseteq \mathbb{P}(k[x_0, ..., x_n]_d) \cong \mathbb{P}^N$ be the projective space parametrizing degree d hypersurfaces in $\mathbb{P}^n$ having multiplicity two in $p_1, ..., p_h$. Compute the expected dimension of $\mathbb{P}(V_{n,d,h})$:

$$\operatorname{expdim}(\mathbb{P}(V_{n,d,h})) = \max\left\{\binom{n+d}{n} - h(n+1) - 1, -1\right\}.$$

Now, take $n = 4$, $d = 3$, and $h = 7$. Using the previous formula conclude that we expect that there is no cubic hypersurface in $\mathbb{P}^4$ having points of multiplicity two in seven general points.
Now, consider a polynomial $P(x_0, x_1) = (u_1x_0 - v_1x_1)(u_2x_0 - v_2x_1)...(u_{d+1}x_0 - v_{d+1}x_1)$ on

$\mathbb{P}^1$ with $d+1$ distinct zeros $[v_1, u_1], ..., [v_{d+1}, u_{d+1}] \in \mathbb{P}^1$ such that all the u_i's and the v_i's are not zero. Let $Q_i(x_0, x_1) = \frac{P(x_0,x_1)}{u_i x_0 - v_i x_1}$ for $i = 1, ..., d+1$.

- Prove that the $Q_i(x_0, x_1)$'s form a basis of $k[x_0, x_1]_d$,
- Conclude that the image of the map

$$\begin{array}{cccc} \nu : & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^d \\ & [x_0, x_1] & \longmapsto & [Q_1(x_0, x_1) : ... : Q_{d+1}(x_0, x_1)] \end{array}$$

  is a rational normal curve of degree d in $\mathbb{P}^d$ passing through the coordinate points of $\mathbb{P}^d$ and through the points $[u_2...u_{d+1} : ... : u_1...u_d]$, $[v_2...v_{d+1} : ... : v_1...v_d]$ (Note that these two points are not on the coordinate hyperplanes).
- Deduce that through $d+3$ general points in $\mathbb{P}^d$ there passes a unique rational normal curve of degree d (Here general means no $d+1$ lying in a hyperplane).

In particular, when $d = 4$ we get that there exists a degree four rational normal curve in $\mathbb{P}^4$ through any seven general points. Use this fact to deduce that there exists an irreducible cubic hypersurface in $\mathbb{P}^4$ with multiplicity two in seven general points. Hence $\dim(\mathbb{P}(V_{4,3,7})) \geq 0$, and $\mathrm{expdim}(\mathbb{P}(V_{4,3,7})) \neq \dim(\mathbb{P}(V_{4,3,7}))$.

For a hypersurface $X = Z(F) \subset \mathbb{P}^n$ having multiplicity two in h general points imposes at most $h(n+1)$ independent conditions, namely the vanishing of the $n+1$ partial derivatives of F in $p_1, ..., p_h$. Since $N = \binom{n+d}{n} - 1$ we get

$$\mathrm{expdim}(\mathbb{P}(V_{n,d,h})) = \max\left\{\binom{n+d}{n} - h(n+1) - 1, -1\right\}.$$

In particular, for $n = 4$, $d = 3$, and $h = 7$ we have

$$\mathrm{expdim}(\mathbb{P}(V_{4,3,7})) = \max\{35 - 35 - 1, -1\} = -1.$$

Then we expect $\mathbb{P}(V_{4,3,7})$ to be empty.
Assume there exists a linear relation $a_1 Q_1(x_0, x_1) + ... + a_{d+1} Q_{d+1}(x_0, x_1) \equiv 0$. Then, for $[x_0, x_1] = [v_i, u_i]$ we get $a_i Q_i(v_i, u_i) = 0$. Since $Q_i(v_i, u_i) \neq 0$ we get $a_i = 0$. Therefore, the Q_i's are linearly independent and since $\dim(k[x_0, x_1]_d) = d+1$ they form a basis of $k[x_0, x_1]_d$.
This means that there exists a linear transformation sending the basis formed by the Q_i's to the standard basis $x_0^d, x_0^{d-1}x_1, ..., x_1^d$ of $k[x_0, x_1]_d$. Such linear transformation induces an automorphism of $\mathbb{P}^d$ mapping the image of ν to the standard degree d rational normal curve of $\mathbb{P}^d$, then $\nu(\mathbb{P}^1)$ itself is a degree d rational normal curve. Now, note that $\nu([v_1, u_1]) = [1 : 0 : ... : 0]$,..., $\nu([v_{d+1} : u_{d+1}]) = [0 : ... : 0 : 1]$, $\nu([1 : 0]) = [u_2...u_{d+1} : ... : u_1...u_d]$, and $\nu([0 : 1]) = [v_2...v_{d+1} : ... : v_1...v_d]$. Since all the u_i's and the v_i's are not zero the last two points are not on any coordinate hyperplane.
Now, for any choice of $d+1$ points in $\mathbb{P}^d$ in general position there exists an automorphism of $\mathbb{P}^d$ mapping these points in the coordinate points. Furthermore, the points $[u_2...u_{d+1} : ... : u_1...u_d]$, $[v_2...v_{d+1} : ... : v_1...v_d]$ may be any points not on the coordinate hyperplanes. By the above construction the rational normal curve passing through these $d+3$ points is unique. We conclude that through $d+3$ general points in $\mathbb{P}^d$ there passes a unique rational normal curve of degree d.

Now, we know that through seven general points $p_1, ..., p_7 \in \mathbb{P}^4$ there exists an unique rational normal curve C of degree four. We may assume that C is the image of

$$\begin{array}{cccc} \nu_4: & \mathbb{P}^1 & \longrightarrow & \mathbb{P}^4 \\ & [x_0, x_1] & \longmapsto & [x_0^4 : x_0^3 x_1 : x_0^2 x_1^2 : x_0 x_1^3 : x_1^4] \end{array}$$

$\mathbb{S}ec_2(C) \subset \mathbb{P}^4$ is the cubic hypersurface given by the vanishing of the determinant of

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

that is

$$\mathbb{S}ec_2(C) = \{F = x_0 x_2 x_4 - x_0 x_3^2 - x_1^2 x_4 + 2x_1 x_2 x_3 - x_2^3 = 0\}.$$

The partial derivatives of F are given by

$$\begin{aligned} \tfrac{\partial F}{\partial x_0} &= x_2 x_4 - x_3^2, \\ \tfrac{\partial F}{\partial x_1} &= 2(x_2 x_3 - x_1 x_4), \\ \tfrac{\partial F}{\partial x_2} &= x_0 x_4 - x_2^2 - 2(x_2^2 - x_1 x_3), \\ \tfrac{\partial F}{\partial x_3} &= 2(x_1 x_2 - x_0 x_3), \\ \tfrac{\partial F}{\partial x_4} &= x_0 x_2 - x_1^2. \end{aligned}$$

Note that all the derivatives are linear combination of 2×2 minors of the matrix M and they vanish simultaneously on C. Furthermore the second partial derivatives of F are 15 linear polynomials that are never simultaneously zero. To see this, it is enough to notice that

$$\frac{\partial^2 F}{\partial x_0 x_3} = -2x_3, \ \frac{\partial^2 F}{\partial x_4 x_1} = -2x_1, \ \frac{\partial^2 F}{\partial x_2^2} = -6x_2, \ \frac{\partial^2 F}{\partial x_4 x_2} = x_0, \ \frac{\partial^2 F}{\partial x_0 x_2} = x_4.$$

We conclude that $\deg(\mathbb{S}ec_2(C)) = 3$, $\mathrm{Sing}(\mathbb{S}ec_2(C)) = C$ and $\mathrm{mult}_C \, \mathbb{S}ec_2(C) = 2$.
In particular, since $p_1, ..., p_7 \in C$ we have $\mathrm{mult}_{p_i} \mathbb{S}ec_2(C) = 2$ for $i = 1, ..., 7$. Then the secant variety $\mathbb{S}ec_2(C) \subset \mathbb{P}^4$ is an irreducible cubic hypersurface having multiplicity two in seven general points. This yields $\dim(\mathbb{P}(V_{4,3,7})) \geq 0$, and $\mathrm{expdim}(\mathbb{P}(V_{4,3,7})) \neq \dim(\mathbb{P}(V_{4,3,7}))$.

CHAPTER 11

Riemann-Roch Theorem

Exercise 1. [**Har**, Exercise 1.1 - Chapter IV] We have two show that there exists a non-constant rational function $f \in K(X)$ such that $\operatorname{div}(f) + kP \geq 0$ for some $k \gg 0$, that is $h^0(X, kP) > 0$ for $k \gg 0$. By Riemann-Roch for the divisor $D = kP$ we have

$$h^0(X, kP) - h^0(X, K_X - kP) = \deg(kP) - g + 1 = k - g + 1.$$

Now, $\deg(K_X - kP) = 2g - 2 - k < 0$ for $k \gg 0$. Therefore, for $k \gg 0$ we have $h^0(X, K_X - kP) = 0$, and

$$h^0(X, kP) = \deg(kP) - g + 1 = k - g + 1 > 0.$$

Exercise 2. [**Har**, Exercise 1.2 - Chapter IV] By the previous exercise for any $P_i \in X$ there exists a rational function $f_i \in K(X)$ that is regular everywhere except at P_i. Finally, we take $f = f_1 + ... + f_r$.

Exercise 3. [**Har**, Exercise 1.5 - Chapter IV] By Riemann-Roch we have $h^0(X, D) = \deg(D) - g + 1 + h^0(K_X - D)$. Since D is effective we have that sections of $K_X - D$ are differential forms on X vanishing on the effective divisor D. Then

$$h^0(X, K_X - D) \leq h^0(X, K_X) = g.$$

This yields

$$h^0(X, D) = \deg(D) - g + 1 + h^0(K_X - D) \leq \deg(D) + 1.$$

If $D = 0$ then $h^0(X, K_X - D) = h^0(X, K_X) = g$ and the equality holds. If $g = 0$ then $K_X = -2P$ and $\deg(K_X - D) < 0$ yields $h^0(K_X - D) = 0$. Again the equality holds. On the other hand if $g > 0$ then $h^0(X, K_X - D) = h^0(X, K_X) = g$ yield that D is linearly equivalent to zero. Since D is effective we conclude that $D = 0$.

Exercise 4. [**Har**, Exercise 1.6 - Chapter IV] Let us consider $g + 1$ points $P_1, ..., P_{g+1} \in X$ and the divisor $D = \sum_{i=1}^{g+1} P_i$. By Riemann-Roch we have

$$h^0(X, D) = \deg(D) - g + 1 + h^0(X, K_X - D) \geq 2 + h^0(X, K_X - D) \geq 2.$$

Therefore, there exists a non-constant rational function $f \in K(X)$ having poles at most on some of the P_i's. Then, f induce a morphism $\overline{f} : X \to \mathbb{P}^1$ such that $\overline{f}^{-1}(\infty) \subseteq \{P_1, ..., P_{g+1}\}$. Hence $\deg(\overline{f}) \leq g + 1$.

Exercise 5. [**Har**, Exercise 1.7 - Chapter IV] A curve X is hyperelliptic if $g(X) \geq 2$ and there exists a finite morphism $f : X \to \mathbb{P}^1$ of degree two.

(a) Let X be a curve of genus two. Then, $\deg(K_X) = 2g - 2 = 2$, and $h^0(X, K_X) = g = 2$. Let $P \in X$ be a point. By Riemann-Roch
$$h^0(X, K_X - P) = 2g - 2 - 1 - g + 1 + h^0(X, P) = g - 2 + h^0(X, P) = h^0(X, P).$$
Now, since $P \in X$ is effective we have $h^0(X, P) \geq 1$. On the other hand $h^0(X, P) \leq 1$ because X is not rational. We conclude that
$$h^0(X, K_X - P) = h^0(X, P) = 1 = h^0(X, K_X) - 1.$$
Then $|K_X|$ is base point free and it induces a morphism $f_{K_X} : X \to \mathbb{P}^1$ of degree two.

(b) Let X be a smooth curve of bidegree $(g+1, 2)$ on a smooth quadric surface $Q \cong \mathbb{P}^1 \times \mathbb{P}^1$. Then
$$g(X) = 2(g+1) - (g+1) - 2 + 1 = 2g + 2 - g - 1 - 1 = g.$$
Let L and R be the two generators of $\mathrm{Pic}(Q)$. We may write $C \sim (g+1)L + 2R$. Then
$$C \cdot L = (g+1)L^2 + 2R \cdot L = 2.$$
This means that the restriction of the second projection $\pi_2 : Q \to \mathbb{P}^1$ defines a morphism $\pi_{2|X} : X \to \mathbb{P}^1$ of degree two.

Exercise 6. Let X be a smooth projective curve. Prove that X is rational if and only if $g(X) = 0$.

Assume that X is rational, that is X is birational to $\mathbb{P}^1$. Since X is smooth and projective we have that X is isomorphic to $\mathbb{P}^1$. Then $g(X) = g(\mathbb{P}^1) = 0$.
Conversely, assume that $g(X) = 0$. Let $P, Q \in X$ be two points with $P \neq Q$, and consider the divisor $D = P - Q$. We have $\deg(K_X - D) = 2g - 2 = -2 < 0$. Then $h^0(X, K_X - D) = 0$. By Riemann-Roch we get $h^0(X, D) = \deg(D) - g + 1 = 1$. On the other hand $\deg(D) = 0$ forces $D \sim 0$, that is $P \sim Q$. Then there exists a non-constant rational function $f \in k(X)$ such that $\mathrm{div}(f) = P - Q$. The rational function f induces a non-constant rational map $\phi : X \dashrightarrow \mathbb{P}^1$ such that $\phi^{-1}(0) = P$ and $\phi^{-1}(\infty) = Q$. Since ϕ is non-constant it is dominant. Furthermore, since $\phi^{-1}(0) = P$ the map ϕ is generically injective. This means that $\phi : X \dashrightarrow \mathbb{P}^1$ is birational.

Bibliography

[PhE] PH. ELLIA, *Un'introduzione light alla geometria algebrica*, IntroLight-GA-2006.pdf.

[Har] R. HARTSHORNE, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Ku] E. KUNZ, *Introduction to Commutative Algebra and Algebraic Geometry*, Birkhäuser, first edition, 1984.