Construction 0.1. Let $F \in k[x,y,z]_5$ be a generic homogeneous polynomial. By Hilbert's Theorem we know that there exists a unique 7-polyhedron $\{L_1,...,L_7\}$ for F. Following the idea of our proof we consider the third partial derivatives of F. These derivatives span a 5-plane $H_\partial \subseteq \mathbb{P}^9$ contained in the 6-plane $H_L =< L_1^3,...,L_7^3 >$. Consider the projection from H_∂

$$\pi : \mathbb{P}^9 \dashrightarrow \mathbb{P}^3.$$

If $V \subseteq \mathbb{P}^9$ is the Veronese variety then $\pi(V) = \overline{V} \subseteq \mathbb{P}^3$ is a surface and hence given by a single equation $\overline{V} = (f = 0)$. By Hilbert's Theorem $\overline{V}$ has a unique singular point of multiplicity 7, the point $P_L = \pi(H_L)$.
Now it is clear that $< P_L, H_\partial >= H_L$ and it is easy to find the 7-polyhedron of F computing the intersection

$$V \cdot H_L = \{L_1^3,...,L_7^3\}.$$

We give an example of the previous construction in an easiest case. We take $F \in k[x,y]_3$, by Sylvester's Theorem we know that there is a unique 2-polyhedron of F. We compute this polyhedron in a completely analogous way.

Example 1. *We consider the polynomial*

$$F = x^3 + x^2y - xy^2 + y^3 \in k[x,y]_3$$

i.e. the point $[F] = [1 : 1 : -1 : 1] \in \mathbb{P}^3$. *The projection from* $[F]$ *to the plane* $(X = 0) \cong \mathbb{P}^2$ *is given by*

$$\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2,\ [X : Y : Z : W] \mapsto [Y - X : X + Z : W - X].$$

Using the following sequence of MacAulay2 we compute the projection $C = \pi(X)$ *of the twisted cubic curve* X.
$Macaulay2, version 1.2$
$i1 : P3 = QQ[X,Y,Z,W]$
$o1 = P3$
$o1 : PolynomialRing$
$i2 : P1 = QQ[s,t]$
$o2 = P1$
$o2 : PolynomialRing$
$i3 : TC = map(P1, P3, s^3, 3*s^2*t, 3*s*t^2, t^3)$
$o3 = map(P1, P3, s^3, 3s^2t, 3s*t^2, t^3)$
$o3 : RingMap P1 <--- P3$
$i4 : ITC = kernel TC$
$o4 = ideal(Z^2 - 3Y*W, Y*Z - 9X*W, Y^2 - 3X*Z)$
$o4 : Ideal of P3$
$i5 : RTC = P3/ITC$
$o5 = RTC$
$o5 : QuotientRing$
$i6 : P2 = QQ[A,B,C]$
$o6 = P2$
$o6 : PolynomialRing$
$i7 : projmap = map(RTC, P2, Y - X, X + Z, W - X)$
$o7 = map(RTC, P2, -X + Y, X + Z, -X + W)$
$o7 : RingMap RTC <--- P2$
$i8 : I = kernel projmap$

$o8 = ideal(14A^3 + 15A^2 * B + 15A * B^2 - 13B^3 - 18A^2 * C + 45A * B * C - 18B^2 * C + 54A * C^2)$
$o8 : IdealofP2$

The latter is the equation of $C = \pi(X)$. Using Bertini we compute the singular point of C,

$$P = Sing(C) = [4 : -10 : 9].$$

The line generated by P and $[F]$ is given by the following equations

$$L = (6X - 10Y - 4Z = 5X - 9Y + 4W = 0).$$

We compute the intersection $X \cdot L$, where X is the twisted cubic curve, with Bertini and we find $L_1^3 = [-0.0515957 : 0.4157801 : -1.1168439 : 1]$ and $L_2^3 = [155.0515957 : 86.5842198 : 16.1168439 : 1]$. These points correspond to the linear forms

$$L_1 = -0.3722812x + y,\ L_2 = 5.3722813x + y.$$

Indeed we have

$$F = 0.99322 \cdot (-0.3722812x + y)^3 + 0.00678 \cdot (5.3722813x + y)^3.$$