MORI DREAM SPACES, LOG FANO VARIETIES AND MODULI SPACES OF RATIONAL CURVES

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Abstract. The notion of Mori Dream Space was introduced by Y. Hu and S. Keel in [HK]. This denomination is motivated by the fact that these spaces behave in the best possible way from the point of view of Mori's minimal model program. We recall the definition of Mori Dream Space, and their main properties in relation to Fano and log Fano varieties. After discussing a famous conjecture by Y. Hu and S. Keel, predicting that $\overline{M}_{0,n}$ is a Mori Dream Space, we summarize the main ideas of a recent paper by A. M. Castravet and I. Tevelev [CT2]. In this paper the authors prove that $\overline{M}_{0,n}$ is not a Mori Dream Space for $n > 133$.

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Acknowledgments. I thank *Cinzia Casagrande* for helpful comments on a preliminary version of these notes. In particular, for Remark 2.9, and for pointing me out [BCHM, Corollary 1.3.2] and [BL].

1. Mori dream spaces

Let X be a normal projective variety. We denote by $\mathrm{N}^1(X)$ the real vector space of Cartier divisors and by $\rho_X = \dim(\mathrm{N}^1(X))$ the Picard number of X.

- The *effective cone* $\mathrm{Eff}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of effective divisors. In general it is not a closed cone.
- The *nef cone* $\mathrm{Nef}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of divisors D such that $D \cdot C \geq 0$ for any curve $C \subset X$. It is closed, but in general it is neither polyhedral nor rational.
- A divisor $D \subset X$ is called *movable* if its stable base locus is in codimension greater or equal that two. The *movable cone* $\mathrm{Mov}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of movable divisors. In general, it is not closed.

Date: April 28, 2014.

A *small* $\mathbb{Q}$*-factorial transformation* of X is a birational map $f : X \dashrightarrow Y$ to another normal $\mathbb{Q}$-factorial projective variety Y, such that f is an isomorphism in codimension one. The exponential exact sequence

$$0 \mapsto \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^* \mapsto 0$$

induces the following exact sequence in cohomology

$$0 \mapsto H^1(X,\mathbb{Z}) \rightarrow H^1(X,\mathcal{O}_X) \rightarrow H^1(X,\mathcal{O}_X^*) \rightarrow H^2(X,\mathbb{Z}) \rightarrow H^2(X,\mathcal{O}_X).$$

The complex torus $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is the *Picard variety* of X. This variety $\mathrm{Pic}^0(X)$ is the connected component of the identity of $\mathrm{Pic}(X) \cong H^1(X,\mathcal{O}_X^*)$ and it is an abelian variety. The image of $\mathrm{Pic}(X)$ inside $H^2(X,\mathbb{Z})$ is isomorphic to $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. The group $\mathrm{NS}(X) \cong \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is a finitely generated abelian group called the *Néron-Severi group*. The group $\mathrm{NS}(X)$ parametrizes divisor on X modulo numerical equivalence.

Example 1.1. Let us consider a smooth projective curve X of genus g. That is X is a compact Riemann surface with g handles. Then $H^0(X,\mathbb{Z}) \cong H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ because X is connected, and $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Since $H^0(X,\mathcal{O}_X) \cong \mathbb{C}^g$ we have $\mathrm{Pic}^0(X) \cong \mathbb{C}^g/\mathbb{Z}^{2g} \cong \mathrm{Jac}(X)$, the Jacobian variety of X. In this case the degree gives an isomorphism $\mathrm{NS}(X) \cong \mathbb{Z}$.

Definition 1.2. A normal projective variety X is a *Mori Dream Space* if

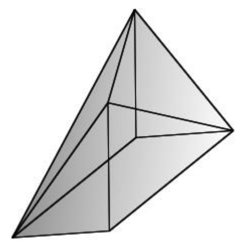
(a) X is $\mathbb{Q}$-factorial and $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathbb{Q}}$;
(b) $\mathrm{Nef}(X)$ is generated by finitely many semi-ample line bundles;
(c) there exist finitely many small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$ such that each X_i satisfies (a), (b), and $\mathrm{Mov}(X)$ us the union of $f_i^*\,\mathrm{Nef}(X_i)$.

Remark 1.3. Condition (a) is equivalent to the finite generation of $\mathrm{Pic}(X)$ which is equivalent to $h^1(X,\mathcal{O}_X) = 0$. Note that if X is a Mori Dream Space then the X_i are Mori Dream Spaces as well.

- A normal $\mathbb{Q}$-factorial projective variety of Picard number is one is a Mori Dream Space if and only if $\mathrm{Pic}(X)$ is finitely generated.
- Let X be a normal $\mathbb{Q}$-factorial projective surface satisfying (a), (b), then $\mathrm{Nef}(X) = \mathrm{Mov}(X)$ and, by taking Id_X, we see that (c) is satisfied as well.
- Any projective $\mathbb{Q}$-factorial toric variety and any smooth Fano variety is a Mori Dream Space.
- If X is a smooth rational surface and $-K_X$ is big the X is a Mori Dream Space.
- A smooth $K3$ surface is a Mori Dream Space if and only if its automorphism group is finite.

Example 1.4. Let X be the blow-up of $\mathbb{P}^3$ at two distinct points x_1, x_2. Let H be the pullback of the hyperplane section and E_1, E_2 the two exceptional divisors. The anti-canonical divisor of X is $-K_X = 4H - 2E_1 - 2E_2$. If L is the strict transform of the line $\langle x_1, x_2\rangle$ we have $-K_X \cdot L = 0$. Therefore X is not Fano. The Picard group of X is generated by H, E_1, E_1 and $\rho_X = 3$. Clearly X is a toric variety. Therefore it is a Mori Dream Space.

The following is the polyhedron of X in $\mathbb{R}^3$.



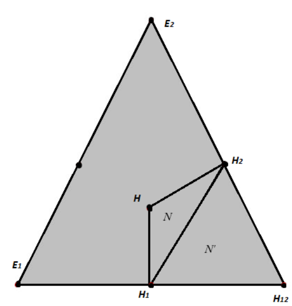
Let $|\mathcal{I}_{x_1,x_2}(2)|$ be the linear system of quadrics in $\mathbb{P}^3$ through x_1, x_2. The corresponding linear system on X induces an morphism

$$\begin{tikzcd} X \arrow[d, "\epsilon"'] \arrow[dr, "f"] & \\ \mathbb{P}^3 \arrow[r, dashed] & Y \subset \mathbb{P}^7 \end{tikzcd}$$

contracting L. Since the normal bundle of L is $\mathcal{O}_L(-1)^{\oplus 2}$ the singular point $f(L) \in f(X) = Y$ is a node. Furthermore f is a small contraction and $f(X)$ is not $\mathbb{Q}$-factorial. Let us blow-up the curve L and let Z be the blow-up. The exceptional divisor is isomorphic two $\mathbb{P}^1 \times \mathbb{P}^1$. By contracting one ruling we get X. On the other hand by contracting the other ruling we find another smooth variety $X^{'}$. The birational map $g : X \dashrightarrow X^{'}$ is the flip of f. The situation is summarized in the following diagram.

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & X^{'} \arrow[dl] \\ & Y & \end{tikzcd}$$

The following is a section of $\mathrm{Eff}(X)$.



Let L be the strict transform of a general line and R_1, R_2 the classes of a line in the exceptional divisors E_1, E_2. Then the strict transform of the line through x_1, x_2 is given by $C = L - E_1 - E_2$. Now, let H_1, H_2, H_{12} be strict transforms of planes through x_1, x_2 and containing the line $\langle x_1, x_2 \rangle$ respectively. Consider $D = aH_{12} + bH_1 + cH_2$. We have $D \cdot C = -a$. Therefore $D \cdot C$ is always less or equal that zero and its zero if and only if $a = 0$. On the other hand after the contraction of C any divisor of this form becomes nef.

The variety X has exactly two small $\mathbb{Q}$-factorial transformations: the identity and the flip g. Furthermore we have $\mathrm{Mov}(X) = \mathrm{Nef}(X) \cup g^* \mathrm{Nef}(X^{'})$. In the picture $\mathrm{Nef}(X)$ is the cone generated by H, H_1, H_2, and $\mathrm{Nef}(X^{'})$ is the cone generated by $H_{1,2}, H_1, H_2$.

We recall two important facts about Mori Dream Space.

Proposition 1.5. *Let X a be a Mori Dream Space.*

- *Any normal projective variety Y which is a small $\mathbb{Q}$-factorial modification of X is a Mori Dream Space. Furthermore the f_i of Definition 1.2 are the only small $\mathbb{Q}$-factorial transformations of X,* [HK, Proposition 1.11].
- *If there is a surjective morphism $X \to Y$ on a normal $\mathbb{Q}$-factorial projective variety Y, then Y is a Mori Dream Space,* [Ok, Theorem 1.1].

Definition 1.6. Let Γ be a semigroup of Weil divisors on X. We can consider the Γ-graded ring:

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D)).$$

If the divisor class group $\mathrm{Cl}(X)$ is finitely generated and Γ is a group of Weil divisors such that $\Gamma_{\mathbb{Q}} \cong \mathrm{Cl}(X)_{\mathbb{Q}}$ then the ring $R_X(\Gamma)$ is denoted by $\mathrm{Cox}(X)$, and called the *Cox ring* of X.

Remark 1.7. Let X be a normal and $\mathbb{Q}$-factorial projective variety with finitely generated and free Picard group and Picard number ρ_X. Let $D_1, ..., D_{\rho_X}$ be a basis of Cartier divisors of $\mathrm{Pic}(X)$. Then

$$\mathrm{Cox}(X) = \bigoplus_{m_1,...,m_{\rho_X} \in \mathbb{Z}} H^0(X, \sum_{i=1}^{\rho_X} m_i D_i).$$

Different choices of divisors $D_1, ..., D_{\rho_X}$ yield isomorphic algebras.

For the details of the proof of the following Theorem we refer to [HK, Proposition 2.9].

Theorem 1.8. *A $\mathbb{Q}$-factorial projective variety X with $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathbb{Q}}$ is a Mori Dream Space if and only if $\mathrm{Cox}(X)$ is finitely generated. In this case X is a GIT quotient of the affine variety $Y = \mathrm{Spec}(\mathrm{Cox}(X))$ by a torus of dimension ρ_X.*

Proof. Let X be a Mori Dream Space. Then the effective cone is rational and polyhedral and we have a decomposition:

$$\mathrm{Eff}(X) = \bigcup_{i=1}^{k} P_i$$

where the P_i's are rational polyhedra. Furthermore there are finitely many rational maps $f_i : X \dashrightarrow X_i$ such that if $D \in \mathrm{Eff}(X)$ then $f_D = f_i$ for some $i = 1, ..., k$. Let us take $D_1, ..., D_h$ divisors generating the cone P_i. The cone $R_X(D_1, ..., D_h)$ does not change by replacing X with X_i and $D_1, ..., D_h$ by the corresponding divisors $D_{1,i}, ..., D_{h,i}$ on X_i. On X_i the divisors $D_{1,i}, ..., D_{h,i}$ are semi-ample. Then $R_{X_i}(D_{1,i}, ..., D_{h,i})$, and hence $R_X(D_1, ..., D_h)$ are finitely generated.

Now, let us assume that $\mathrm{Cox}(X)$ is finitely generated. Then we have an equivariant embedding, with respect a torus G, of $Y = \mathrm{Spec}(\mathrm{Cox}(X))$ is $\mathbb{A}^n$. Taking the GIT quotient we have an embedding $Y \subseteq Q = \mathbb{A}^n /\!/ G$. Since G is a torus Q is a toric variety and hence a

Mori Dream Space. Furthermore if $r : X \dashrightarrow Y$ is a rational map then there is a rational map of toric varieties $t : M \dashrightarrow N$ inducing r by restriction. Therefore X is a Mori Dream Space. $\square$

2. Weak Fano and log Fano varieties

Let $D = \sum_{i=1}^{k} d_i D_i$ be a *simple normal crossing* $\mathbb{Q}$-divisor on a normal variety X. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Then for a resolution $f : Y \to X$ of X we can write

$$K_Y = f^*(K_X + D) + \sum a_i E_i,$$

where the E_i are either f-exceptional or a strict transforms of the D_i.

Definition 2.1. A *log resolution* of the pair (X, D) is a birational surjective morphism $f : Y \to X$ such that Y is smooth and $f^{-1}D \cup \mathrm{Exc}(f)$ is a simple normal crossing $\mathbb{Q}$-divisor.

By [Hi] a log resolution always exists.

Definition 2.2. Let $D = \sum_{i=1}^{k} d_i D_i$ be a *simple normal crossing* $\mathbb{Q}$-divisor on a normal variety X. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier and let $f : Y \to X$ be a log resolution with

$$K_Y = f^*(K_X + D) + \sum a(E_i, X, D) E_i.$$

We call:

$$discrep(X, D) = min_{E_i}\{a(E_i, X, D) \mid E_i \text{ is } f-\text{ exceptional}\},$$

and

$$totaldiscrep(X, D) = min_{E_i}\{a(E_i, X, D)\}.$$

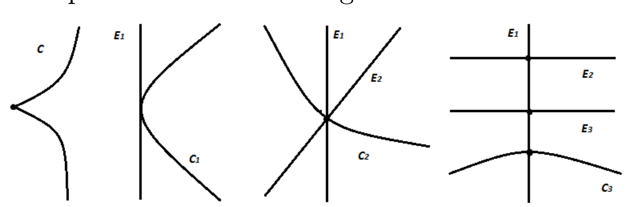
We say that the pair (X, D) is

- terminal if $discrep(X, D) > 0$;
- canonical if $discrep(X, D) \geq 0$;
- Kawamata log terminal (klt) if $discrep(X, D) > -1$ and $d_i < 1$ for any $i = 1, ..., k$;
- purely log terminal (plt) if $discrep(X, D) > -1$;
- log canonical (lc) if $discrep(X, D) \geq -1$.

Example 2.3. Assume that D is a simple normal crossing divisor, and that X is smooth. Then Id_X is a log resolution. If $0 < \epsilon \ll 1$ is a rational number then we have $K_X = Id_X^*(K_X + \epsilon D) - \epsilon D$. The pair $(X, \epsilon D)$ is Kawamata log terminal.
Let $D \subset \mathbb{P}^2$ an irreducible curve with one node, and let $f : Y \to \mathbb{P}^2$ be the blow-up of the node. Then $f^{-1}D \cup E$ is simple normal crossing. Furthermore $K_Y = f^* K_{\mathbb{P}^2} + E$ and $f^*D = \widetilde{D} + 2E$ where $\widetilde{D}$ is the strict transform of D, yield

$$K_Y = f^*(K_{\mathbb{P}^2} + D) - \widetilde{D} - E.$$

Therefore the pair $(\mathbb{P}^2, D)$ is log canonical.
Now, let us consider a cusp $D \subset \mathbb{P}^2$ to have a log resolution we have to blow-up three times.



Let $\epsilon_1 : X_1 \to \mathbb{P}^2$ be the first blow-up. We have $K_{X_1} = \epsilon_1^* K_{\mathbb{P}^2} + E_1$ and $C_1 = \epsilon_1^* C - 2E_1$. If $\epsilon_2 : X_2 \to X_1$ is the second blow-up we have $K_{X_2} = \epsilon_2^*(\epsilon_1^* K_{\mathbb{P}^2} + E_1) + E_2 = \epsilon_2^*\epsilon_1^* K_{\mathbb{P}^2} + E_1 + 2E_2$ and $C_2 = \epsilon_2^* C_1 - E_2 = \epsilon_2^*\epsilon_1^* C - 2E_1 - 3E_2$. Finally, let $\epsilon_3 : X_3 \to X_2$ be the third blow-up. Then $K_{X_3} = \epsilon_3^*\epsilon_2^*\epsilon_1^* K_{\mathbb{P}^2} + E_1 + 2E_2 + 4E_3$ and $C_3 = \epsilon_3^* C_2 - E_3 = \epsilon_3^*\epsilon_2^*\epsilon_1^* C - 2E_1 - 3E_2 - 6E_3$. Let $\epsilon = \epsilon_1 \circ \epsilon_2 \circ \epsilon_3$. Summing up we have

$$\begin{aligned} K_{X_3} &= \epsilon^* K_{\mathbb{P}^2} + E_1 + 2E_2 + 4E_3, \\ C_3 &= \epsilon^* C - 2E_1 - 3E_2 - 6E_3. \end{aligned}$$

Therefore we get

$$K_{X_3} = \epsilon^*(K_{\mathbb{P}^2} + C) - C_3 - E_1 - E_2 - 2E_3.$$

In particular $discrep(\mathbb{P}^2, D) = a(E_3, \mathbb{P}^2, D) = -2$ and $(\mathbb{P}^2, D)$ is not log canonical.

Definition 2.4. Let X be a smooth projective variety. We say that X is:

- *weak Fano* if $-K_X$ is nef and big,
- *log Fano* if there exists a divisor D such that $-(K_X + D)$ is ample and the pair (X, D) is Kawamata log terminal. In particular if $D = 0$ we have terminal Fano varieties.

The Picard group of a Fano variety $\mathrm{Pic}(X) = H^2(X, \mathbb{Z})$ is always finitely generated. Any toric variety is log Fano. Let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Then X is log Fano if and only if $d \leq n$.

Lemma 2.5. *Let D be a nef and big divisor on an irreducible projective variety X. Then there exist an effective divisor E and a rational number $0 < \epsilon \ll 1$ such that $D - \epsilon E$ is ample.*

Proof. Let D be a nef and big divisor. Since D is big, by [La, Corollary 2.2.6], there exist an ample divisor A, an effective divisor E, and a positive integer k such that $kD \equiv A + E$. If $h > k$ we can write $hD \equiv (h-k)D + A + E$. The divisor $D^{'} = (h-k)D + A$ is a sum of a nef and an ample divisor. Therefore $D^{'}$ is ample. If $\epsilon = \frac{1}{h}$ we get that

$$D - \epsilon E \equiv \epsilon D^{'}$$

is ample. □

Proposition 2.6. *Let X be normal, irreducible, projective variety with at most klt singularities. If X is weak Fano then X is log Fano.*

Proof. Since X is weak Fano $-K_X$ is nef and big. By Lemma 2.5 there exists an effective divisor D and a rational number $0 < \epsilon \ll 1$ such that $-K_X - \epsilon D = -(K_X + \epsilon D)$ is ample. The pair $(X, \epsilon D)$ is klt for $\epsilon \ll 1$ because X has at most klt singularities. □

Remark 2.7. The converse of Proposition 2.6 is false. For instance the Hirzebruch surface $X_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ is a toric surface and hence log Fano. The anti-canonical divisor is $-K_{X_e} = -2C_0 - (2+e)F$, where C_0 is the section and F is the fiber. Therefore $-K_{X_e} \cdot C_0 = 2C_0^2 + 2 + e = -e + 2$, and $-K_{X_e}$ is not nef for $e > 2$. We conclude that for any $e > 2$ the Hirzebruch surface X_e is log Fano but not weak Fano.

The bridge between Mori Dream Spaces and log Fano varieties is the content of the following proposition.

Proposition 2.8. [BCHM, Corollary 1.3.2] *Let X be a smooth projective variety. If X is log Fano then X is a Mori Dream Space .*

Remark 2.9. Let X be a Mori Dream Space with big and movable anti-canonical divisor. Then X is not necessarily log Fano [CG, Example 5.1]. Indeed X admits a small $\mathbb{Q}$-factorial modification Y such that $-K_Y$ is nef and big, but Y could have bad singularities. In particular the variety X in [CG, Example 5.1] is not rationally connected. Therefore it is not log Fano.

Let X be the blow-up of $\mathbb{P}^3$ at seven general points $p_1, ..., p_7$. Then X is not toric. In what follows we give a direct proof of the fact that X is log Fano. We remark that, by [BL, Proposition 2.9], $X = Bl_7\mathbb{P}^3$ is weak Fano. Therefore, by Proposition 2.6 X is log Fano.

Lemma 2.10. *Let $p_1, ..., p_7 \in \mathbb{P}^3$ be general points. There are not irreducible quartic curves $C \subset \mathbb{P}^3$ such that $p_1 = \mathrm{Sing}(C)$ is a point of multiplicity two, and $p_2, ..., p_7 \in C$.*

Proof. Let us assume that such a quartic curve exists and consider the projection

$$\pi_{p_1} : C \dashrightarrow \mathbb{P}^2.$$

Since $\mathrm{mult}_{p_1} C = 2$ and C is irreducible the image $\overline{C} = \overline{\pi_{p_1}(C)}$ is a conic though the six general points $\pi_{p_1}(p_i)$ for $i = 2, ..., 7$. A contradiction. $\square$

Lemma 2.11. *Let $p_1, ..., p_7 \in \mathbb{P}^3$ be general points, $C \subset \mathbb{P}^3$ an irreducible curve of degree d, and $m_i = \mathrm{mult}_{p_i}(C)$ the multiplicity of C at p_i. If*

$$m_1 + ... + m_7 = 2d$$

then C is either a line through two of the p_i's or a twisted cubic through six of the p_i's.

Proof. Let us consider the case of a plane curve $C \subset \mathbb{P}^2$. We may assume that the plane containing C is generated by p_1, p_2, p_3. Therefore we have $m_1 + m_2 + m_3 = 2d$ and $m_i = 0$ for $i = 4, ..., 7$. Since C is irreducible, if one of the three lines $\langle p_i, p_j \rangle \subseteq C$ then $C = \langle p_i, p_j \rangle$. Therefore we may assume that m_1, m_2, m_3 are positive. We have $m_i + m_j \leq d$ for any $i \neq j$ otherwise the line $\langle p_i, p_j \rangle$ would be a component of C. Summing up these three inequalities we get $2(m_1 + m_2 + m_3) = 2(m_1 + ... + m_7) \leq 3d$ and so the contradiction $4d \leq 3d$. We conclude that if C is plane then C is a line through two of the p_i's.
Now, let us assume C to be non-degenerate. Let $p \in C$ be a general point. Then there is a pencil Λ of quadric surfaces passing through $p_1, ..., p_7$ and p. Let Q be such a quadric surface. Now, $C \cdot Q \geq m_1 + ... + m_7 + 1 = 2d + 1$ implies $C \subset Q$. In particular for $Q_1, Q_2 \in \Lambda$ we have $C \subset Q_1 \cap Q_2$ and this yields $d \leq 4$. Furthermore C is non-degenerate and irreducible. So $d = 3, 4$.
Let us assume $d = 3$. Then $m_1 + ... + m_7 = 6$. If $m_i \geq 2$ for some i then, for two general points $p, q \in C$, we have $C \cdot \langle p, q, p_i \rangle \geq 4$ and so the contradiction $C \subset \langle p, q, p_i \rangle$. Therefore $0 \leq m_i \leq 1$ for any i and since $m_1 + ... + m_7 = 6$ we get the seven twisted cubic through six of the p_i's.
Finally, let us assume $d = 4$. Then $m_1 + ... + m_7 = 8$. Suppose to have $m_i \geq 2$ and $m_j \geq 2$ for $i \neq j$ and let $p \in C$ be a general point. Then $C \cdot \langle p, p_i, p_j \rangle \geq 5$ and again we get the contradiction $C \subset \langle p, p_i, p_j \rangle$. So there exists at most one integer $m_i \geq 2$. Note that m_i has to be exactly equal to two otherwise C would be contained in a plane. Furthermore $m_1 + ... + m_7 = 8$ implies that there exists exactly one $m_i = 2$. We may assume that

$m_1 = 2$ and $m_2 = ... = m_7 = 1$. Thus C is a quartic rational curve with a singular point of multiplicity 2 at p_1 and passing through $p_2, ..., p_7$. By Lemma 2.10 such a curve does not exist. $\square$

Proposition 2.12. *Let X be the blow-up of $\mathbb{P}^3$ at seven general points $p_1, ..., p_7$. Then X is log Fano. In particular* $\operatorname{Cox}(X)$ *is finitely generated and X is a Mori dream space.*

Proof. The anti-canonical divisor of X is given by

$$-K_X = 4H - 2E_1 - ... - 2E_7 = 2(2H - E_1 - ... - E_7).$$

By Lemma 2.11 we know that $|-K_X|$ contracts just the strict transforms of the lines through two of the p_i's and of the twisted cubics through six of the p_i's.
Surfaces of degree k in $\mathbb{P}^3$ are parametrized by a vector space of dimension $\binom{k+3}{3}$. A point of multiplicity m imposes at most $\binom{m+2}{3}$ conditions. Let us fix a $k \gg 0$ and a $m > \frac{k}{2}$ such that

$$\binom{k+3}{3} - 7\binom{m+2}{3} > 0.$$

Then, by [Su, Proposition 11], there exists an irreducible surface $S \subset \mathbb{P}^3$ such that $\operatorname{Sing}(S) = \{p_1, ..., p_7\}$ and having ordinary singularities of multiplicity m at $p_1, ..., p_7$. Furthermore the general element in the linear system $|S|$ has this property.
Let $\widetilde{S} \subset X$ be the strict transform of S. Note that, being $p_1, ..., p_7$ ordinary singularities of S, the divisor $\widetilde{S}$ is smooth. Let $0 < \epsilon \ll 1$ be a rational number. Our aim is to prove that the divisor

$$D = -(K_X + \epsilon\widetilde{S})$$

is ample. Since $\widetilde{S} = kH - mE_1 - ... - mE_7$ we can write

$$D = (4 - \epsilon k)H + (\epsilon m - 2)E_1 + ... + (\epsilon m - 2)E_7.$$

Let L be the strict transform of a general line in $\mathbb{P}^3$ and R_i be the class of a line in the exceptional divisor E_i for $i = 1, ..., 7$. Let $C \subset X$ be an irreducible curve. We distinguish two cases.

- $C \subset E_i$ for some $i \in \{1, ..., 7\}$. We may assume $C \subset E_1$. Then $C = dR_1$ and

  $$D \cdot C = d(2 - \epsilon m) > 0$$

  being $\epsilon < \frac{m}{2}$.
- C is the strict transform of a curve in $\mathbb{P}^3$. Then $C = dL - m_1R_1 - ... - m_7R_7$ that is C comes from a curve of degree d in $\mathbb{P}^3$ having points of multiplicity $m_1, ..., m_7$ at $p_1, ..., p_7$. Then

  $$D \cdot C = d(4 - \epsilon k) - (m_1 + ... + m_7)(2 - \epsilon m).$$

  By the proof of Lemma 2.11 we get $(m_1 + ... + m_7) \leq 2d$. Since $\epsilon < \frac{m}{2}$ we have $(2 - \epsilon m) > 0$, and

  $$D \cdot C \geq d(4 - \epsilon k) - 2d(2 - \epsilon m) = \epsilon(2dm - kd).$$

  Now, $m > \frac{k}{2}$ implies $D \cdot C > 0$.

Finally we compute

$$D^3 = 8 - \epsilon^3(k^3 + 7m^3) + \epsilon^2(12k^2 - 42m^2) - \epsilon(48k + 84m) > 0$$

for ϵ sufficiently small. Note that we do not need to intersect D^2 with surfaces. Indeed, the base locus of $|D|$ zero dimensional we have $D^2 \cdot S = D \cdot C$, where S is an irreducible surface and C a curve numerically equivalent to $D \cdot S$ and meeting S properly. Therefore, by the first part of the proof, $D^2 \cdot S > 0$ for any irreducible surface $S \subset X$. Finally, by Nakai-Moishezon criterion [La, Theorem 1.2.19] we conclude that $D = -(K_X + \epsilon\widetilde{S})$ is ample. Recall that $\widetilde{S}$ is the strict transform of a surface having ordinary singularities at $p_1, ..., p_7$ and smooth everywhere else. Therefore $\widetilde{S}$ is a smooth divisor in the smooth 3-fold X, and the pair $(X, \epsilon\widetilde{S})$ is klt. We conclude that X is log Fano. $\square$

Remark 2.13. For a complete classification of Mori Dream Spaces obtained by blowing-up points in $\mathbb{P}^n$ see [CT2].

3. The moduli space of pointed rational curves

Let $\overline{M}_{0,n}$ the moduli space of n-pointed rational curves. In [Ka] M. Kapranov realized $\overline{M}_{0,n}$ as a blow-up of $\mathbb{P}^{n-3}$.

Construction 3.1. [Ka] Fixed $(n-1)$-points $p_1, ..., p_{n-1} \in \mathbb{P}^{n-3}$ in linear general position:

(1) Blow-up the points $p_1, ..., p_{n-2}$, then the lines $\langle p_i, p_j \rangle$ for $i, j = 1, ..., n-2$,..., the $(n-5)$-planes spanned by $n-4$ of these points.

(2) Blow-up p_{n-1}, the lines spanned by pairs of points including p_{n-1} but not p_{n-2},..., the $(n-5)$-planes spanned by $n-4$ of these points including p_{n-1} but not p_{n-2}.

$\vdots$

(r) Blow-up the linear spaces spanned by subsets $\{p_{n-1}, p_{n-2}, ..., p_{n-r+1}\}$ so that the order of the blow-ups is compatible with the partial order on the subsets given by inclusion, the $(r-1)$-planes spanned by r of these points including $p_{n-1}, p_{n-2}, ..., p_{n-r+1}$ but not p_{n-r},..., the $(n-5)$-planes spanned by $n-4$ of these points including $p_{n-1}, p_{n-2}, ..., p_{n-r+1}$ but not p_{n-r}.

$\vdots$

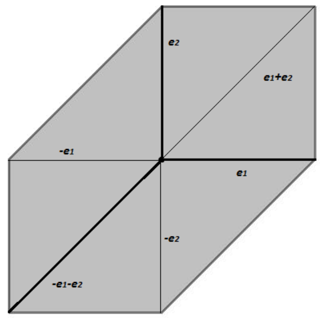
($n-3$) Blow-up the linear spaces spanned by subsets $\{p_{n-1}, p_{n-2}, ..., p_4\}$.

The composition of these blow-ups is the morphism $f_n : \overline{M}_{0,n} \to \mathbb{P}^{n-3}$ induced by the psi-class Ψ_n. In particular the variety obtained at the end of this sequence of blow-ups is isomorphic to $\overline{M}_{0,n}$.

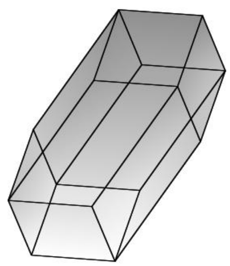
In [HK, Question 3.2] Y. Hu and S. Keel asked if $\overline{M}_{0,n}$ is a Mori Dream Space. If $n = 4, 5$ this is well known because $\overline{M}_{0,4} \cong \mathbb{P}^1$ and $\overline{M}_{0,5}$ is a Del Pezzo surface of degree five. By [HK] $\overline{M}_{0,n}$ is log Fano if and only if $n \leq 6$. In particular $\overline{M}_{0,6}$ is a Mori Dream Space. For $g \geq 1$ it is know that:

- in characteristic zero $\overline{M}_{g,n}$ is not a Mori Dream Space for $g \geq 3, n \geq 1$. This was proven in [Ke] by providing a nef but not semiample divisor on $\overline{M}_{g,n}$;
- in [CC] D. Chen and I. Coskun proved that $\overline{M}_{1,n}$ is not a Mori Dream Space for $n \geq 3$ because it has infinitely many extremal effective divisors.

Remark 3.2. The step $r = 1, s = n - 3$ of Construction 3.1 is the Losev-Manin's space $\overline{L}_{n-2}$ [Ha, Section 6.4]. This space is a toric variety of dimension $n - 3$. It is the last toric variety in Construction 3.1. For instance L_3 is a Del Pezzo surface of degree six. The following picture represents the corresponding polyhedron.



The space $\overline{L}_4$ is the blow-up of $\mathbb{P}^3$ at four general points and along the strict transform of the six lines joining them. The corresponding polyhedron is the following.



Note that both the polyhedra are very symmetric.

In a way $\overline{M}_{0,n}$ is very close to a toric variety. This is one of the reasons that led to conjecture that $\overline{M}_{0,n}$ is a Mori Dream Space.

Theorem 3.3. [CT1, Theorem 1.3] *Let $n = a + b + c + 8$ where a, b, c are positive coprime integers. If $Bl_e\overline{L}_{n-3}$ is a Mori Dream Space then $Bl_e\mathbb{P}(a, b, c)$ is a Mori Dream Space.*

Proof. Let $e_1, ..., e_{n-2}$ be vectors in $\mathbb{R}^{n-3}$ such that $e_1 + ... + e_{n-2} = 0$. Let N be the lattice generated by $e_1, ..., e_{n-2}$, and consider the fan Σ_{n-2} spanned by the primitive lattice vectors $\sum_{i\in I} e_i$ for each subset $I \subset S = \{1, ..., n - 2\}$ with $1 \leq |I| \leq n - 3$. The toric variety associated to this fan is the Losev-Manin space $\overline{L}_{n-2} = X(\Sigma_{n-2})$.
Let us consider a partition $S = S_1 \cup S_2 \cup S_3$ into subsets of order $a + 2, b + 2, c + 2$. Then $n = a + b + c + 8$. We fix $n_i \in S_i$ for $i = 1, 2, 3$, and consider the sublattice spanned by the vectors

$$e_{n_i} + e_r, \quad \text{for} \quad r \in S_i \setminus \{n_i\},\ i = 1, 2, 3. \tag{3.1}$$

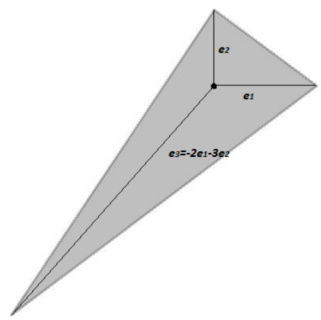
Let $N^{'} = N/N^{''}$ be the quotient and let $\pi : N \to N^{'}$ be the projection. Then $N^{'}$ is a lattice, it is spanned by the vectors $\pi(e_{n_i})$ for $i = 1, 2, 3$, and $a\pi(e_{n_1}) + b\pi(e_{n_2}) + c\pi(e_{n_3}) = 0$.

Example 3.4. Take $a = 1, b = 2, c = 3$, and $S_1 = \{e_1, e_2, e_3\}, S_2 = \{e_4, e_5, e_6, e_7\}, S_3 = \{e_8, e_9, e_{10}, e_{11}, e_{12}\}$. The we take $e_{n_1} = e_1, e_{n_2} = e_4, e_{n_3} = e_8$. Clearly $N^{'} = N/N^{''}$ is generated by $\pi(e_1), \pi(e_4), \pi(e_8)$. Since $\pi(e_1) = -\pi(e_i)$ for $i = 2, 3$, $\pi(e_4) = -\pi(e_i)$ for

$i = 5, 6, 7$, and $\pi(e_8) = -\pi(e_i)$ for $i = 9, 10, 11, 12$, the relation $\sum_{i=1}^{12} e_i = 0$ gives $\pi(e_1) - \pi(e_1) - \pi(e_1) + \pi(e_4) - \pi(e_4) - 2\pi(e_4) + \pi(e_8) - \pi(e_8) - 3\pi(e_8) = -(\pi(e_1) + 2\pi(e_4) + 3\pi(e_8)) = 0$. Therefore

$$\pi(e_1) + 2\pi(e_4) + 3\pi(e_8) = 0.$$

It follows that the toric surface with lattice N' and rays spanned by $\pi(e_{n_i})$ for $i = 1, 2, 3$ is the weighted projective plane $\mathbb{P}(a, b, c)$. For instance the following is the fan of $\mathbb{P}(1, 2, 3)$.



Let N_j, for $j = 1, ..., n-4$, be the lattice obtained by taking the quotient of N by a sublattice spanned by the first $j-1$ vectors of the sequence 3.1. Let Γ_j be a sets of rays obtained by projecting the rays of the fan of $\overline{L}_{n-2}$, and $X_j = X(\Gamma_j)$. Mote that $N_{n-4} = N'$ and we have a regular map $X_{n-4} \to \mathbb{P}(a, b, c)$ obtained forgetting all vector of Γ_{n-4} except the $\pi(e_{n_i})$ for $i = 1, 2, 3$. Since this map is an isomorphism on the torus it induces a birational morphism $Bl_e X_{n-4} \to Bl_e \mathbb{P}(a, b, c)$, where e is the identity of the torus. In this way we get a sequence of toric morphism

$$X_1 \to X_2 \to ... \to X_{n-4} \to \mathbb{P}(a, b, c).$$

Note that X_1 has the same rays of $\overline{L}_{n-2}$ and therefore is a small modification of $\overline{L}_{n-2}$ which is an isomorphism on the torus. Then $Bl_e X_1$ is a small modification of $Bl_e \overline{L}_{n-2}$. $\square$

Next we consider the following theorem.

Theorem 3.5. [CT1, Theorem 1.1] *There exists a small $\mathbb{Q}$- factorial projective modification $\widetilde{L}_{n-2}$ of $Bl_e \overline{L}_{n-2}$, and surjective morphisms*

$$\widetilde{L}_{n-2} \to \overline{M}_{0,n} \to Bl_e \overline{L}_{n-3}.$$

In particular, by Proposition 1.5, if $\overline{M}_{0,n}$ is a Mori Dream Space then $Bl_e \overline{L}_{n-3}$ is a Mori Dream Space, if $Bl_e \overline{L}_{n-2}$ is a Mori Dream Space then $\overline{M}_{0,n}$ is a Mori Dream Space.

In particular, if $\overline{M}_{0,n}$ is a Mori Dream Space then $Bl_e \overline{L}_{n-2}$ is a Mori Dream Space, and by Theorem 3.3 $Bl_e \mathbb{P}(a, b, c)$ is a Mori Dream Space. Now, the key ingredient is the following result due to S. Goto, K. Nishida, and K. Watanabe.

Theorem 3.6. [GNW] *Assume* $\mathrm{char}(k) = 0$. *If* $(a, b, c) = (7h - 3, 5h^2 - 2h, 8h - 3)$, *with* $h \geq 4$ *and* $3 \nmid h$, *then* $Bl_e \mathbb{P}(a, b, c)$ *is not a Mori Dream Space.*

An immediate consequence of Theorems 3.3, 3.5 and 3.6 is the following.

Theorem 3.7. [CT1, Corollary 1.4] *Assume* $\mathrm{char}(k) = 0$. *Then $\overline{M}_{0,n}$ is not a Mori Dream Space for $n > 133$.*

Proof. We have $n(h) = a+b+c+8 = 7h-3+5h^2-2h+8h-3+8 = 5h^2+13h+2$. So $n(4) = 134$. Therefore $\overline{M}_{0,134}$ is not a Mori Dream Space. If $n > 135$ we have a surjective forgetful morphism $\pi_i : \overline{M}_{0,n} \rightarrow \overline{M}_{0,134}$. Therefore, by Proposition 1.5, $\overline{M}_{0,n}$ is not a Mori Dream Space for $n \geq 134$. $\square$

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