Log Fano varieties, Mori Dream Spaces and moduli spaces of rational curves

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Introduction

The goal of the minimal model program is to construct a birational model of any complex projective variety which is as simple as possible in a suitable sense. This subject has its origins in the classical birational geometry of surfaces studied by the Italian school. In 1988 S. Mori extended the concept of minimal model to 3-folds by allowing suitable singularities on them. In 2010 there was a great breakthrough in the minimal model theory when C. Birkar, P. Cascini, C. Hacon and J. McKernan proved the existence of minimal models for varieties of log general type.
Mori Dream Spaces, introduced by Y. Hu and S. Keel in 2002, form a class of algebraic varieties that behave very well from the point of view of Mori's minimal model program. They can be algebraically characterized as varieties whose total coordinate ring, called the *Cox ring*, is finitely generated.
In addition to this algebraic characterization there are several algebraic varieties characterized by some positivity property of the anti-canonical divisor, such as *weak Fano* and *log Fano* varieties, that turn out to be Mori Dream Spaces, see Chapter 3 for details.
Chapter 5 is an introduction to moduli spaces of curves. The search for an object parametrizing n-pointed genus g smooth curves is a very classical problem in algebraic geometry. In **[DM]** P. Deligne and D. Mumford proved that there exists an irreducible scheme $M_{g,n}$ coarsely representing the moduli functor of n-pointed genus g smooth curves. Furthermore they provided a compactification $\overline{M}_{g,n}$ of $M_{g,n}$ adding Deligne-Mumford stable curves as boundary points and pointed out that the obstructions to representing the moduli functor of Deligne-Mumford stable curves in the category of schemes came from automorphisms of the curves. However this moduli functor can be represented in the category of algebraic stacks. Indeed there exists a smooth Deligne-Mumford algebraic stack $\overline{\mathcal{M}}_{g,n}$ parametrizing Deligne-Mumford stable curves. The stack $\overline{\mathcal{M}}_{g,n}$ and its coarse moduli space $\overline{M}_{g,n}$ are among the most studied objects in algebraic geometry.
In **[Ha]** B. Hassett introduced new compactifications $\overline{\mathcal{M}}_{g,A[n]}$ of the moduli stack $\mathcal{M}_{g,n}$ and $\overline{M}_{g,A[n]}$ for the coarse moduli space $M_{g,n}$, by assigning rational weights $A = (a_1, ..., a_n)$, $0 < a_i \leq 1$ to the markings. In genus zero some of these spaces appear as intermediate steps of the blow-up construction of $\overline{M}_{0,n}$ developed by M. Kapranov in **[Ka]**, while in higher genus they may be related to the Log Minimal Model Program on $\overline{M}_{g,n}$.
The aim of these notes is to give an introduction to Mori Dream Spaces, weak Fano and log Fano varieties and to moduli spaces of rational curves. In Chapter 4 we will focus on some particular and well understood examples of Mori Dream Space arising as blow-ups of projective spaces in points, and we will discuss their relations with some moduli spaces of weighted rational curves.
Finally, after discussing a famous conjecture by Y. Hu and S. Keel **[HK]**, predicting that $\overline{M}_{0,n}$ is a Mori Dream Space, we will summarize the main ideas of a paper by A. M. Castravet and I. Tevelev **[CT2]**. In this paper the authors prove that $\overline{M}_{0,n}$ is not a Mori Dream Space for $n > 133$.

CHAPTER 1

Singularities

Canonical singularities appear as singularities of the canonical model of a projective variety, and terminal singularities are special cases that appear as singularities of minimal models. Terminal singularities are important in the minimal model program because smooth minimal models do not always exist, and thus one must allow certain singularities, namely the terminal singularities. For instance, two-dimensional terminal singularities are smooth. The singular locus of a variety with at most terminal singularities has codimension at least three. In particular for curves and surfaces all terminal singularities are smooth. For 3-folds terminal singularities are isolated and have been classified by *S. Mori*.
Surface canonical singularities are exactly the *du Val singularities*, and are analytically isomorphic to quotients of $\mathbb{C}^2$ by finite subgroups of $SL_2(\mathbb{C})$.

Cyclic quotient singularities. Any cyclic quotient singularity is of the form $\mathbb{A}^n/\mu_r$, where μ_r is the group of r-roots of unit. The action $\mu_r \curvearrowright \mathbb{A}^n$ can be diagonalized, and then written in the form

$$\begin{array}{ccc} \mu_r \times \mathbb{A}^n & \longrightarrow & \mathbb{A}^n \\ (\epsilon, x_1, ..., x_n) & \longmapsto & (\epsilon^{a_1}x_1, ..., \epsilon^{a_n}x_n) \end{array}$$

for some $a_1, ..., a_n \in \mathbb{Z}/\mathbb{Z}_r$. The singularity is thus determined by the numbers $r, a_1, ..., a_n$. Following the notation set by *M. Reid* in **[Re]**, we denote by $\frac{1}{r}(a_1, ..., a_n)$ this type of singularity.

EXAMPLE 0.1. Let us consider the action:

$$\begin{array}{ccc} \mu_2 \times \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \\ (\epsilon, x_0, x_1) & \longmapsto & (\epsilon x_0, \epsilon x_1) \end{array}$$

The ring of invariants is given by:

$$k[x_0^2, x_0x_1, x_1^2] \cong k[y_0, y_1, y_2]/(y_0y_2 - y_1^2)$$

and we see that the singularity $X = \mathbb{A}^2/\mu_2$ corresponds to the vertex v of the affine cone

$$X = \mathrm{Spec}(k[x_0^2, x_0x_1, x_1^2] \cong k[y_0, y_1, y_2]/(y_0y_2 - y_1^2))$$

that is the vertex of a quadric cone $Q \subset \mathbb{P}^2$ or equivalently the singularity $\frac{1}{2}(1, 1)$ of the weighted projective plane $\mathbb{P}(1, 1, 2)$. Now, $dx_0 \wedge dx_1$ is a basis of $\bigwedge^2 \Omega_{\mathbb{A}^2}$, and $(dx_0 \wedge dx_1)^{\otimes 2}$ is invariant under the action. The form

$$\omega = \frac{(dy_0 \wedge dy_1)^{\otimes 2}}{y_0^2} \in (\bigwedge^2 \Omega_{k(X)})^{\otimes 2}$$

is a basis of $(\bigwedge^2 \Omega_X)^{\otimes 2}$ because the quotient map $\pi : \mathbb{A}^2 \to X$ is étale on $X \setminus \{v\}$, and $\pi^*\omega = 4(dx_0 \wedge dx_1)^{\otimes 2}$.

Blowing-up the vertex v we get a resolution $f : Y \to X$. If $[\lambda_0 : \lambda_1 : \lambda_2]$ are homogeneous coordinates on $\mathbb{P}^2$ then the equations of Y in $\mathbb{A}^3 \times \mathbb{P}^2$ are:

$$\begin{cases} y_0\lambda_1 - y_1\lambda_0 = 0, \\ y_0\lambda_2 - y_2\lambda_0 = 0, \\ y_1\lambda_2 - y_2\lambda_1 = 0, \\ y_0y_2 - y_1^2. \end{cases}$$

Therefore, $y_1 = \frac{\lambda_1}{\lambda_0}y_0$, and $\frac{\lambda_2}{\lambda_1} = \frac{\lambda_1}{\lambda_0}$ yields $y_2 = \frac{\lambda_1}{\lambda_0}y_1 = (\frac{\lambda_1}{\lambda_0})^2 y_0$. Then, in Y we have an affine chart isomorphic to $\mathbb{A}^2$ with coordinates (y_0, t) where the resolution is given by $(y_0, t) \mapsto (y_0, y_0t, y_0t^2)$, with $t = \frac{\lambda_1}{\lambda_0}$, and the exceptional divisor E over v is given by $\{y_0 = 0\}$. We have

$$f^*\omega = (dy_0 \wedge dt)^{\otimes 2}.$$

Therefore, $f^*\omega$ has neither a pole nor a zero along E, and we may write $K_Y = f^*K_X$.

EXAMPLE 0.2. Let us consider the action:

$$\begin{array}{ccc} \mu_3 \times \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \\ (\epsilon, x_0, x_1) & \longmapsto & (\epsilon x_0, \epsilon x_1) \end{array}$$

The ring of invariants is given by:

$$k[x_0^3, x_0^2x_1, x_0x_1^2, x_1^3] \cong k[y_0, y_1, y_2, y_3]/(y_0y_3 - y_1y_2, y_0y_2 - y_1^2, y_1y_3 - y_2^2)$$

and we see that the singularity $X = \mathbb{A}^2/\mu_3$ corresponds to the vertex v of the affine cone

$$X = \operatorname{Spec}(k[y_0, y_1, y_2, y_3]/(y_0y_3 - y_1y_2, y_0y_2 - y_1^2, y_1y_3 - y_2^2))$$

over a twisted cubic $C \subset \mathbb{P}^3$. Now, $dx_0 \wedge dx_1$ is a basis of $\bigwedge^2 \Omega_{\mathbb{A}^2}$, and $(dx_0 \wedge dx_1)^{\otimes 3}$ is invariant under the action. The form

$$\omega = \frac{(dy_0 \wedge dy_1)^{\otimes 3}}{y_0^4} \in (\bigwedge^2 \Omega_{k(X)})^{\otimes 3}$$

is a basis of $(\bigwedge^2 \Omega_X)^{\otimes 3}$ because the quotient map $\pi : \mathbb{A}^2 \to X$ is étale on $X \setminus \{v\}$, and

$$\pi^*\omega = \frac{(3x_0^4(dx_0 \wedge dx_1))^{\otimes 3}}{x_0^{12}} = 27(dx_0 \wedge dx_1)^{\otimes 3}.$$

Blowing-up the vertex v we get a resolution $f : Y \to X$, and we have an affine chart isomorphic to $\mathbb{A}^2$ with coordinates (y_0, t) where the resolution is given by $(y_0, t) \mapsto (y_0, y_0t, y_0t^2, y_0t^3)$, and the exceptional divisor E over v is given by $\{y_0 = 0\}$. We have

$$f^*\omega = \frac{(dy_0 \wedge (y_0dt + tdy_0))^{\otimes 3}}{y_0^4} = \frac{(dy_0 \wedge dt)^{\otimes 3}}{y_0}.$$

Therefore, $f^*\omega$ has a pole along E, and we may write $K_Y = f^*K_X - \frac{1}{3}E$.

EXAMPLE 0.3. Now, let us consider the action:

$$\begin{array}{ccc} \mu_2 \times \mathbb{A}^3 & \longrightarrow & \mathbb{A}^3 \\ (\epsilon, x_0, x_1, x_2) & \longmapsto & (\epsilon x_0, \epsilon x_1, \epsilon x_2) \end{array}$$

The ring of invariants is given by:

$$k[x_0^2, x_0x_1, x_0x_2, x_1^2, x_1x_2, x_2^2] \cong \frac{k[y_0, y_1, y_2, y_3, y_4, y_5]}{(y_0y_3 - y_1^2, y_0y_4 - y_1y_2, y_0y_5 - y_2^2, y_1y_4 - y_2y_3, y_1y_5 - y_2y_4, y_3y_5 - y_4^2)}$$

The singularity $X = \mathbb{A}^3/\mu_2$ corresponds to the vertex v of the affine cone X over a Veronese surface $V \subset \mathbb{P}^5$. The differential form $dx_0 \wedge dx_1 \wedge dx_2$ is a basis of $\bigwedge^3 \Omega_{\mathbb{A}^3}$, and $(dx_0 \wedge dx_1 \wedge dx_2)^{\otimes 2}$ is invariant under the action. The form

$$\omega = \frac{(dy_0 \wedge dy_1 \wedge dy_2)^{\otimes 2}}{y_0^3} \in (\bigwedge^3 \Omega_{k(X)})^{\otimes 2}$$

is a basis of $(\bigwedge^3 \Omega_X)^{\otimes 2}$ because the quotient map $\pi : \mathbb{A}^3 \to X$ is étale on $X \setminus \{v\}$, and

$$\pi^*\omega = \frac{(4x_0^6(dx_0 \wedge dx_1 \wedge dx_2))^{\otimes 2}}{x_0^6} = 4(dx_0 \wedge dx_1 \wedge dx_2)^{\otimes 2}.$$

Blowing-up the vertex v we get a resolution $f : Y \to X$, and we have an affine chart isomorphic to $\mathbb{A}^3$ with coordinates (y_0, s, t) where the resolution is given by $(y_0, s, t) \mapsto (y_0, y_0 s, y_0 t, y_0 s^2, y_0 st, y_0 t^2)$, and the exceptional divisor E over v is given by $\{y_0 = 0\}$. We have

$$f^*\omega = y_0(dy_0 \wedge ds \wedge dt)^{\otimes 2}.$$

Therefore, $f^*\omega$ has a zero along E, and we may write $K_Y = f^*K_X + \frac{1}{2}E$.

DEFINITION 0.4. *A normal variety X is terminal (canonical) if K_X is $\mathbb{Q}$-Cartier and there exists a resolution $f : Y \to X$ such that*

$$K_Y = f^*K_X + \sum_i a_i E_i$$

with $a_i > 0$ ($a_i \geq 0$). The rational numbers a_i are called discrepancies.

For instance, the quadric cone in Example 2.17 is canonical but not terminal, the cone over the twisted cubic in Example 2.18 is not even canonical, and the cone over the Veronese surface in Example 0.3 is terminal.
A projective variety X has canonical singularities if it is normal, some power of the canonical bundle of the smooth locus of X extends to a line bundle on V, and X has the same plurigenera as any resolution of its singularities.
A normal projective variety X has terminal singularities, if some power of the canonical line bundle of the smooth locus of X extends to a line bundle on X, and the pullback of any section of $\omega_X^{\otimes m}$ vanishes along any codimension one component of the exceptional locus of a resolution of the singularities of X.

EXAMPLE 0.5. Let S be a terminal projective surface, and let $f : Y \to S$ be a resolution of S. Then

$$K_Y = f^*K_S + \sum_i a_i E_i$$

with $a_i > 0$. By Grauert-Mumford theorem [**BPV**, Theorem 2.1] the intersection matrix of the E_i's is negative definite. Therefore, there exists an E_j such that

$$E_j \cdot (\sum_i a_i E_i) < 0.$$

Let us check this in the case of two components E_1, E_2. The general case will be clear. The intersection matrix

$$I = \begin{pmatrix} E_1^2 & E_1E_2 \\ E_1E_2 & E_2^2 \end{pmatrix}$$

is negative definite. In particular, if for the vector $a = (a_1, a_2)$ we have

$$a \cdot I \cdot a^t = a_1^2 E_1^2 + 2a_1 a_2 E_1 E_2 + a_2^2 E_2^2 < 0.$$

On the other hand

$$a_1^2 E_1^2 + 2a_1 a_2 E_1 E_2 + a_2^2 E_2^2 = a_1 E_1(a_1 E_1 + a_2 E_2) + a_2 E_2(a_1 E_1 + a_2 E_2) < 0.$$

Since $a_1, a_2 > 0$ the last inequality yields either $E_1(a_1E_1 + a_2E_2) < 0$ or $E_2(a_1E_1 + a_2E_2) < 0$. Furthermore $E_j^2 < 0$. We conclude that there exists an E_j such that $E_j \cdot (\sum_i a_i E_i) < 0$ and $E_j^2 < 0$. By adjunction on the curve E_j we get

$$2g(E_j) - 2 = K_Y \cdot E_j + E_j^2 < 0.$$

Therefore, $g(E_j) = 0$ and $K_Y \cdot E_j + E_j^2 = -2$. This forces, $K_Y \cdot E_j = E_j^2 = -1$. By Castelnuovo contractibility criterion [**Har**, Theorem 5.7] we can contract E_j on a smooth surface. Proceeding recursively we get that S is smooth. Therefore, a surface is terminal if and only if it is smooth.
Now, let S be a surface with canonical singularities, and let $f : Y \to S$ be a minimal resolution that is there are no (-1)-curves contracted by f. We may write $K_Y = f^*K_S + \sum_i a_i E_i$ with $a_i \geq 0$. If S is not smooth we have $a_i = 0$, and

$$K_Y = f^*K_S.$$

If E is a curve contracted by f we get $K_Y \cdot E = 0$ and $E^2 < 0$. This imply $2g(E) - 2 = K_Y \cdot E + E^2 = E^2 < 0$, which in turn yields $g(E) = 0$ and $E^2 = -2$. Since the intersection matrix is negative definite $(E_i + E_j)^2 < 0$, and hence $E_i \cdot E_j \leq 1$. Therefore, any contracted fiber of f is a tree of rational curves corresponding to one of the Dynkin diagrams: A_n, D_n, E_6, E_7, and E_8. Canonical surface singularities are the so called Rational Double Points, also known as Du Val singularities or ADE singularities.

1. Singularities of Pairs

Let us consider a $\mathbb{Q}$-Weil divisor $D = \sum_i d_i D_i$ on a normal variety X. We assume that the D_i's are distinct. We want to give a reasonable notion of singularities of the pair (X, D). We require that $K_X + D$ is $\mathbb{Q}$-Cartier. Then for a resolution $f : Y \to X$ we have the formula

$$K_Y = f^*(K_X + D) + \sum_i a_i E_i - \widetilde{D},$$

where $\widetilde{D}$ is the strict transform. Even when X is smooth D could be very singular. A resolution of X is meaningless for the pair (X, D).

DEFINITION 1.1. *A divisor $D = \sum_i D_i$ on a smooth variety X is simple normal crossing if D is reduced, any component D_i of D is smooth, and D is locally defined in a neighborhood of any point by an equation in local analytic coordinates of the type*

$$z_1 \cdot \ldots \cdot z_k = 0$$

with $k \leq \dim(X)$.

Roughly speaking the singularities of D should locally look no worse that those of a union of coordinate hyperplanes.

EXAMPLE 1.2. Let $D = \sum_i D_i$ where the D_i's are hyperplanes in $\mathbb{P}^n$, and let $p_i \in \mathbb{P}^{n*}$ be the point corresponding to D_i. Then D is simple normal crossing if and only if the p_i's are in linear general position.

The following is a consequence of Hironaka's theorem on resolution **[Hi]** of singularities.

THEOREM 1.3. *Let X be an irreducible algebraic variety over $\mathbb{C}$, and let $D \subset X$ be an effective Cartier divisor on X.*

- *There exists a projective birational morphism $f : Y \to X$, where X is smooth and $f^{-1}D \cup \mathrm{Exc}(f)$ is simple normal crossing. The morphism f is called a log resolution of the pair (X, D).*
- *The smooth variety Y can be constructed as a sequence of blow-ups along smooth centers supported in the singular loci of D and X. In particular f is an isomorphism over $X \setminus (\mathrm{Sing}(X) \cup \mathrm{Sing}(D))$.*

We will need many times the following result.

PROPOSITION 1.4. *Let X be a smooth variety, $Z \subset X$ a smooth subvariety with $\mathrm{codim}_Z(Y) = c \geq 2$, and $\pi : Y \to X$ the blow-up of X along Z with exceptional divisor E. Then*

$$\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}.$$

Furthermore,

$$K_Y = \pi^* K_X + (c-1)E.$$

PROOF. Let us consider the map

$$\begin{array}{rcl} \psi : \mathbb{Z} & \to & \mathrm{Pic}(Y) \\ n & \longmapsto & nE \end{array}$$

By [**Har**, Proposition 6.5] we have an exact sequence

$$\mathbb{Z} \to \mathrm{Pic}(Y) \to \mathrm{Pic}(Y \setminus E) \mapsto 0.$$

Let us assume that $nE \sim 0$ for some $n \neq 0$. Then there exists $f \in k(Y)$ with a zero of order n along E. Since π is surjective and birational, the function f induces a function $g \in k(X)$ having only a zero of order n a long Z. A contradiction because $c = \mathrm{codim}_Z(Y) \geq 2$. Therefore we have the exact sequence

$$0 \mapsto \mathbb{Z} \to \mathrm{Pic}(Y) \to \mathrm{Pic}(Y \setminus E) \mapsto 0. \tag{1.1}$$

Since π is an isomorphism outside E we have $\mathrm{Pic}(Y \setminus E) \cong \mathrm{Pic}(X \setminus Z)$, furthermore $c \geq 2$ yields $\mathrm{Pic}(X \setminus Z) \cong \mathrm{Pic}(X)$, and

$$\mathrm{Pic}(Y \setminus E) \cong \mathrm{Pic}(X \setminus Z) \cong \mathrm{Pic}(X).$$

Therefore, the pull-back map $\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ gives a section of the second map in the exact sequence 1.1. This implies that the sequence 1.1 splits and $\mathrm{Pic}(Y) \cong \mathrm{Pic}(X) \oplus \mathbb{Z}$.
Now, we may write $K_Y = \pi^* D + qE$ for some $D \in \mathrm{Pic}(X)$. The isomorphism $X \setminus Z \cong Y \setminus E$ yields $K_{Y|Y\setminus E} \cong K_{X|X\setminus Z}$. Since $\mathrm{Pic}(X \setminus Z) \cong \mathrm{Pic}(X)$ we get $D = K_X$, and $K_Y = \pi^* K_X + qE$.
Now, our aim is to determine the integer q. By adjunction and using $\mathcal{O}_Y(E)_{|E} = \mathcal{O}_E(-1)$ we get

$$K_E \cong (K_Y + E)_{|E} \cong (\pi^* K_X + (q+1)E)_{|E} = \pi^* K_X - (q+1)E.$$

Let $F = z \times_Z E$ be the fiber over a point $z \in Z$. Then

$$\omega_F = \pi_1^* \omega_z \otimes \pi_2^* \omega_E = \pi_1^* \omega_z \otimes \pi_2^* (\pi^* \omega_X \otimes \mathcal{O}_E(-q-1)) = \pi_2^* (\pi^* \omega_X \otimes \mathcal{O}_E(-q-1)).$$

Now, a differential form on Y that is the pullback of a differential form on X must vanish on E. In particular $\pi_2^*(\pi^* \omega_X)$ is trivial, and

$$\omega_F \cong \pi_2^* (\mathcal{O}_E(-q-1)) \cong \mathcal{O}_F(-q-1).$$

On the other hand $F \cong \mathbb{P}^{c-1}$. Therefore, $\omega_F \cong \mathcal{O}_F(-c)$ implies $q = c-1$. □

EXAMPLE 1.5. Let $Z \subset \mathbb{P}^n$ be a smooth variety of codimension c, $\pi : Y \to \mathbb{P}^n$ the blow-up of Z, H the pullback of the hyperplane class of $\mathbb{P}^n$ and E the exceptional divisor. Then

$$K_Y = (-n-1)H + (c-1)E.$$

Now, let us assume that X and D are both smooth and consider $(1+\epsilon)D$. The $Id_X : X \to X$ is a log resolution and

$$K_X = Id_X^*(K_X + (1+\epsilon)D) - (1+\epsilon)D.$$

Let $\pi_1 : X_1 \to X$ be the blow-up of a codimension two smooth subvariety $Z_1 \subset D$. Then

$$K_{X_1} = \pi_1^*(K_X + (1+\epsilon)D) - \epsilon E_1 - (1+\epsilon)D_1$$

where D_1 is the strict transform of D. Now, let $f : X_2 \to X_1$ be the blow-up of $D_1 \cap E_1$, and $\pi_2 = f \circ \pi_1$. Then

$$K_{X_2} = \pi_2^*(K_X + (1+\epsilon)D) - 2\epsilon E_2 - \epsilon E_1 - (1+\epsilon)D_2$$

Proceeding like this we see that starting with a discrepancy less than -1 we can produce arbitrarily negative discrepancies. This motivates the following definition.

DEFINITION 1.6. *Let X be a normal variety and $D = \sum_j d_j D_j$ be a $\mathbb{Q}$-Weil divisor. Assume that $K_X + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of the pair (X, D) and write*

$$K_Y = f^*(K_X + D) + \sum_i a_i E_i - \widetilde{D}.$$

The pair (X, D) is

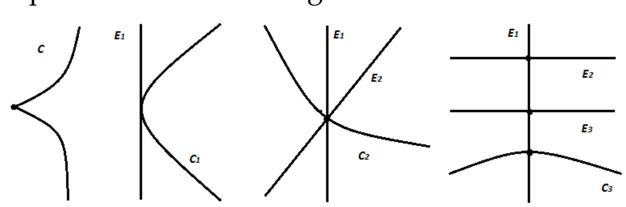
terminal	*if*	$a_i > 0$ *for any* i,
canonical	*if*	$a_i \geq 0$ *for any* i,
klt	*if*	$a_i > -1$ *and* $d_j < 1$ *for any* i, j,
plt	*if*	$a_i > -1$ *for any* i,
lc	*if*	$a_i \geq -1$ *for any* i.

Here klt, plt, lc stands for Kawamata log terminal, purely log terminal, and log canonical respectively.

EXAMPLE 1.7. Assume that D is a simple normal crossing divisor, and that X is smooth. Then Id_X is a log resolution. If $0 < \epsilon < 1$ is a rational number then we have $K_X = Id_X^*(K_X + \epsilon D) - \epsilon D$. The pair $(X, \epsilon D)$ is Kawamata log terminal.
Let $D \subset \mathbb{P}^2$ an irreducible curve with one node, and let $f : Y \to \mathbb{P}^2$ be the blow-up of the node. Then $f^{-1}D \cup E$ is simple normal crossing. Furthermore $K_Y = f^*K_{\mathbb{P}^2} + E$ and $f^*D = \widetilde{D} + 2E$ where $\widetilde{D}$ is the strict transform of D, yield

$$K_Y = f^*(K_{\mathbb{P}^2} + D) - \widetilde{D} - E.$$

Therefore the pair $(\mathbb{P}^2, D)$ is log canonical.
Now, let us consider a cusp $D \subset \mathbb{P}^2$ to have a log resolution we have to blow-up three times.



Let $\epsilon_1 : X_1 \to \mathbb{P}^2$ be the first blow-up. We have $K_{X_1} = \epsilon_1^* K_{\mathbb{P}^2} + E_1$ and $C_1 = \epsilon_1^* C - 2E_1$. If $\epsilon_2 : X_2 \to X_1$ is the second blow-up we have $K_{X_2} = \epsilon_2^*(\epsilon_1^* K_{\mathbb{P}^2} + E_1) + E_2 = \epsilon_2^*\epsilon_1^* K_{\mathbb{P}^2} + E_1 + 2E_2$

and $C_2 = \epsilon_2^* C_1 - E_2 = \epsilon_2^*\epsilon_1^* C - 2E_1 - 3E_2$. Finally, let $\epsilon_3 : X_3 \to X_2$ be the third blow-up. Then $K_{X_3} = \epsilon_3^*\epsilon_2^*\epsilon_1^* K_{\mathbb{P}^2} + E_1 + 2E_2 + 4E_3$ and $C_3 = \epsilon_3^* C_2 - E_3 = \epsilon_3^*\epsilon_2^*\epsilon_1^* C - 2E_1 - 3E_2 - 6E_3$. Let $\epsilon = \epsilon_1 \circ \epsilon_2 \circ \epsilon_3$. Summing up we have

$$\begin{aligned} K_{X_3} &= \epsilon^* K_{\mathbb{P}^2} + E_1 + 2E_2 + 4E_3, \\ C_3 &= \epsilon^* C - 2E_1 - 3E_2 - 6E_3. \end{aligned}$$

Therefore we get

$$K_{X_3} = \epsilon^*(K_{\mathbb{P}^2} + C) - C_3 - E_1 - E_2 - 2E_3.$$

In particular, $a_i(E_3, \mathbb{P}^2, D) = -2$ and $(\mathbb{P}^2, D)$ is not log canonical.

Now, let us consider a slightly more complicated example.

EXAMPLE 1.8. Let us consider the cubic surface

$$S = \{x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0\} \subset \mathbb{P}^3.$$

known as the Cayley nodal cubic surface. By taking partial derivatives it is easy to see that the singular locus of S consists of the four coordinates points of $\mathbb{P}^3$, and that each of them is a point of multiplicity two for S. Let us consider the point $p = [1 : 0 : 0 : 0]$. In the chart $\mathcal{U}_0 := \{x_0 \neq 0\}$ the equation of S is given by $\{x_1x_2 + x_1x_3 + x_2x_3 + x_1x_2x_3 = 0\}$. Therefore, the projective tangent cone of S in p is the conic $\{x_1x_2 + x_1x_3 + x_2x_3 = 0\} \subset \mathbb{P}^3$. Since this conic is smooth p is an ordinary double point. We conclude that the fundamental points of $\mathbb{P}^3$ are ordinary singularities for S, and hence S can be resolved simply by blowing-up these four points. Now, let $\pi : Y \to \mathbb{P}^3$ be the blow-up with exceptional divisors $E_1, ..., E_4$. Then we may write

$$K_Y = \pi^* K_{\mathbb{P}^3} + 2(E_1 + E_2 + E_3 + E_4),$$

and

$$\epsilon\widetilde{D} = \pi^*(\epsilon D) - 2\epsilon(E_1 + E_2 + E_3 + E_4).$$

Therefore

$$K_Y = \pi^*(K_{\mathbb{P}^3} + \epsilon D) + (2 - 2\epsilon)(E_1 + E_2 + E_3 + E_4),$$

and since $2 - 2\epsilon > -1$ if and only if $\epsilon < \frac{3}{2}$ we get that $(\mathbb{P}^3, \epsilon S)$ is klt if and only if $\epsilon < 1$.

CHAPTER 2

Secant Varieties

We recall some definitions and basic facts concerning secant varieties.

DEFINITION 0.1. *Let $X \subset \mathbb{P}^N$ be an irreducible and reduced non-degenerate variety. We will denote by*

$$\Gamma_h(X) \subset X \times ... \times X \times \mathbb{G}(h-1, N),$$

the reduced closure of the graph of $\alpha : X \times ... \times X \dashrightarrow \mathbb{G}(h-1, N)$, taking h general points to their linear span $\langle x_1, ..., x_h \rangle$.

Therefore, $\Gamma_h(X)$ is irreducible and reduced of dimension hn. Let us call $\pi_2 : \Gamma_h(X) \to \mathbb{G}(h-1, N)$ the natural projection, and set $\mathcal{S}_h(X) := \pi_2(\Gamma_h(X)) \subset \mathbb{G}(h-1, N)$. The variety $\mathcal{S}_h(X)$ is irreducible and reduced of dimension hn as well. Finally, let us define $\mathcal{I}_h := \{(x, \Lambda) \mid x \in \Lambda\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N)$, with projections π_h and ψ_h onto the factors.

DEFINITION 0.2. *Let $X \subset \mathbb{P}^N$ be an irreducible and reduced, non degenerate variety. We call the abstract h-Secant variety* the irreducible and reduced $(hn+h-1)$-dimensional variety

$$\mathrm{Sec}_h(X) := (\psi_h)^{-1}(\mathcal{S}_h(X)) \subset \mathcal{I}_h.$$

We call the *h-Secant variety*

$$\mathbb{S}ec_h(X) := \pi_h(Sec_h(X)) \subset \mathbb{P}^N.$$

The variety X is said to be h-defective if $\delta_h = nh + h - 1 - \dim \mathbb{S}ec_h(X) > 0$. In this case δ_h is called the h-secant defect of X.

Let us consider some simple example.

EXAMPLE 0.3. Let $C \subseteq \mathbb{P}^3$ be the twisted cubic curve and let $p \in \mathbb{P}^3$ be a general point. There exists a line L passing thorough p and secant to C. Indeed, if a such line does not exist then the projection of C in $\mathbb{P}^2$ from p would be a smooth plane cubic $\overline{C}$ isomorphic to C. However, $g(C) = 0$ and $g(\overline{C}) = 1$, a contradiction. Let us assume that there are two distinct lines L, R secant to C through p. Then for the plane $H = \langle L, R \rangle$ we have $H \cdot C \geq 4$, a contradiction because $\deg(C) = 3$ and C is not contained in a plane.
Hence, a general point $p \in \mathbb{P}^3$ lies on a unique secant line to C. We conclude that $\mathbb{S}ec_2(C)$ is the whole of $\mathbb{P}^3$.

EXAMPLE 0.4. Let $X = \nu(\mathbb{P}^2) \subseteq \mathbb{P}^5$ be the Veronese surface. Let $u \in \mathbb{P}^5$ be a point lying on a secant line to X. We write the secant line as $\langle \nu(p), \nu(q) \rangle$ with $p, q \in \mathbb{P}^2$. The line $L = \langle p, q \rangle \subseteq \mathbb{P}^2$ is mapped via the Veronese embedding ν to a conic $C \subseteq X$. Since $u \in \langle \nu(p), \nu(q) \rangle$ and $\nu(p)$, $\nu(q)$ lie in C the point u lies on the plane H spanned by C. All lines passing through u and contained in H intersect C in two points and so are secant lines of X. We see that the general point of $\mathbb{S}ec_2(X)$ lies on a 1-dimensional family of secant lines. So $\dim(\mathbb{S}ec_2(X)) = 4$. There is another way to see this fact. The points of $\mathbb{S}ec_2(X)$ represent conics which can be written as sum of two squares, that is conics of rank equal either 1 or 2. So we can describe $\mathbb{S}ec_2(X) \subseteq \mathbb{P}^5$ as the determinantal variety defined by

$$\det \begin{pmatrix} X_0 & X_3 & X_4 \\ X_3 & X_1 & X_5 \\ X_4 & X_5 & X_2 \end{pmatrix} = 0$$

Therefore, $\mathbb{S}ec_2(X)$ is a cubic hypersurface in $\mathbb{P}^5$.

EXAMPLE 0.5. Let $\mathbb{G}(1,n) \subseteq \mathbb{P}^N$, with $N = \binom{n+1}{2} - 1$, be the Grassmannians of lines of $\mathbb{P}^n$ and let $p \in Sec_2(\mathbb{G}(1,n))$ be a point, and let $L = \langle u, v \rangle$ be a secant line through p. The points u, v represent two lines R_1, R_2 in $\mathbb{P}^n$. Now, two general lines span a 3-plane $H \subset \mathbb{P}^n$. The lines contained in H are parametrized by the Grassmannian $\mathbb{G}(1,3) \subseteq \mathbb{G}(1,n)$.
Now $\dim(\mathbb{G}(1,3)) = 4$, $\deg(\mathbb{G}(1,3)) = 4$ and $\mathbb{G}(1,3)$ spans a 5-plane $E \subseteq \mathbb{P}^N$. All the lines in E and passing through p intersect $\mathbb{G}(1,3)$ in two points because $\deg(\mathbb{G}(1,3)) = 2$. We see that any point $p \in \mathbb{S}ec_2(\mathbb{G}(1,n))$ lies on a 4-dimensional family of secant lines. Therefore

$$\dim(\mathbb{S}ec_2(\mathbb{G}(1,n))) = 2\dim(\mathbb{G}(1,n)) + 1 - 4 = 4n - 7,$$

$$\delta(\mathbb{G}(1,n)) = 2\dim(\mathbb{G}(1,n)) + 1 - 4n + 7 = 4.$$

1. Terracini's Lemma

Terracini's Lemma **[Te]** is a fundamental result for the computation of the dimension of $\mathbb{S}ec_h(X)$. The leading idea is quite simple: let $p \in \mathbb{S}ec_h(X)$ be a general points. assume $p \in \langle x_1, ..., x_h \rangle$. Then a tangent vector to $\mathbb{S}ec_h(X)$ at p can be interpreted as an infinitesimal direction of $\mathbb{S}ec_h(X)$ in p. This should correspond then to an infinitesimal movement of the x_i's in X, that is to a set of tangent vectors to X at the x_i's.

THEOREM 1.1. *(Terracini's Lemma* **[Te]***) Let $X \subset \mathbb{P}^N$ be a non-degenerate variety over a field of characteristic zero. Let $p \in \mathbb{S}ec_h(X)$ be a point, lying in the linear span of $x_1, ..., x_h \in X$. Then*

$$\mathbb{T}_p\mathbb{S}ec_h(X) \supseteq \langle \mathbb{T}_{x_1}X, ..., \mathbb{T}_{x_h}X \rangle .$$

Furthermore, if $p \in \mathbb{S}ec_h(X)$ is general we have

$$\mathbb{T}_p\mathbb{S}ec_h(X) = \langle \mathbb{T}_{x_1}X, ..., \mathbb{T}_{x_h}X \rangle .$$

Alexander-Hirshowitz Theorem. A variation on the Waring problem (coming from a question in number theory stated by *E. Waring* in 1770, see **[Wa]** (which states that every integer is a sum of at most 9 positive cubes) asked which is the minimum positive integer h such that the generic polynomial of degree d on $\mathbb{P}^n$ admits a decomposition as a sum of h d-powers of linear forms:

PROBLEM 1.2. *(Waring problem - first formulation) Given a general homogeneous polynomial $F \in k[x_0, ..., x_n]_d$ what is the minimum positive integer h such that F admits a decomposition as a sum of h d-powers of linear forms ?*

In 1995 *J. Alexander* and *A. Hirshowitz* solved completely this problem over an algebraically closed base field k of characteristic zero, see **[AH]**. They proved that the minimum integer h is the expected one $h = \lfloor \frac{1}{n+1}\binom{n+d}{d} \rfloor$, except in the following cases:

(1.1)

n	d	h
n	2	$2 \leq h \leq n$
2	4	5
3	4	9
4	3	7
4	4	14

Now, let $\nu : \mathbb{P}^n \to \mathbb{P}^N$ be the d-Veronese embedding, and let $V_d^n = \nu(\mathbb{P}^n)$ be its image. The minimum positive integer h such that the generic polynomial of degree d on $\mathbb{P}^n$ admits a decomposition as a sum of h d-powers of linear forms is indeed the minimum positive integer h such that $\mathbb{S}ec_h(V_d^n) = \mathbb{P}^N$. Therefore, Waring problem for the general homogeneous polynomial can be restated as follows:

PROBLEM 1.3. *(Waring problem - second formulation) Given a pair of positive integers n, d what is the minimum positive integer h such that $\mathbb{S}ec_h(V_d^n) = \mathbb{P}^N$?*

Recall that the expect dimension of $\mathbb{S}ec_h(V_d^n)$ is

$$\mathrm{expdim}(\mathbb{S}ec_h(V_d^n)) = \min\{hn + h - 1, N\}.$$

Therefore, by Alexander-Hirshowitz Theorem the expected dimension of $\mathbb{S}ec_h(V_d^n)$ is its actual dimension with the exceptions in Table 1.1.
Finally, by Theorem 1.1, we may give a third interpretation of the Waring problem. Let $p \in \mathbb{S}ec_h(V_d^n)$ be a general point, and let $s = \dim(\mathbb{T}_p\mathbb{S}ec_h(V_d^n)) = \dim(\mathbb{S}ec_h(V_d^n))$. Let $\mathcal{H} \subseteq \mathcal{O}_{\mathbb{P}^N}(1)$ be the linear system of hyperplanes of $\mathbb{P}^N$ containing $\mathbb{T}_p\mathbb{S}ec_h(V_d^n)$. By Theorem 1.1 a general element of $\mathcal{H}$ correspond to an hypersurface of degree d in $\mathbb{P}^n$ having double points at $p_1 = \nu^{-1}(x_1), ..., p_h = \nu^{-1}(x_h)$. Note that any double point imposes at most $n+1$ independent conditions, namely the vanishing of the first partial derivatives of the polynomials defining the hypersurface of $\mathbb{P}^n$. Therefore, the expected codimension of the $\mathcal{H}$ is:

$$\mathrm{expcodim}(\mathcal{H}) = \min\left\{h(n+1), \binom{n+d}{d}\right\}.$$

PROBLEM 1.4. *(Waring problem - third formulation) Given a pair of positive integers n, d, let $\mathcal{H}$ be the linear system of hypersurfaces of degree d in $\mathbb{P}^n$ having double points at h general points. In which cases the expected codimension of $\mathcal{H}$ coincides with is actual codimension ?*

Indeed this third formulation is the one taken into account by Alexander and Hirshowitz. Finally, by Alexander-Hirshowitz Theorem the expected codimension of $\mathcal{H}$ is the actual one with the exception in Table 1.1.

2. Equations for secant varieties of Veronese varieties

Let $\nu : \mathbb{P}^n \to \mathbb{P}^{N_d}$ be the d-Veronese embedding, and let $V_d^n = \nu(\mathbb{P}^n)$ be its image. Let $[F] \in \mathbb{P}^N = \mathrm{Proj}(k[x_0, ..., x_n]_d)$ be a degree d homogeneous polynomial. Fixed a positive integer h such that $\mathbb{S}ec_h(V_d^n) \neq \mathbb{P}^N$ we want to determine whether $[F] \in \mathbb{S}ec_h(V_d^n)$. We begin with the following simple observation:

REMARK 2.1. If $F = \sum_{i=1}^h \lambda_i L_i^d$ then its partial derivatives of order l lie in the linear space $\langle L_1^{d-l}, ..., L_h^{d-l}\rangle$ for any $l = 1, ..., d-1$.

The partial derivatives of order l are $\binom{n+l}{l}$ homogeneous polynomials of degree $d-l$, so the previous observation is meaningful when $h < \binom{n+l}{l}$ and $h < \binom{d-l+n}{n}$. The latter condition ensures that $\langle L_1^{d-l}, ..., L_h^{d-l}\rangle$ is a proper subspace of the projective space $\mathbb{P}^{N_{d-l}}$ parametrizing homogeneous polynomials of degree $d-l$.
Consider the partial derivatives $F^l_{l_0,...,l_n} := \frac{\partial^l F}{\partial x_0^{l_0},...,\partial x_n^{l_n}}$ and the incidence variety

$$\begin{tikzcd}[column sep=small] & \mathcal{I}_{l,h} = \{(F,H) \mid \in F^l_{l_0,...,l_n} \in H,\ \forall\, l_0+...+l_n = l\} \subset \mathbb{P}^N \times \mathbb{G}(h-1, N_{d-l}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathbb{P}^N & & \mathbb{G}(h-1, N_{d-l}) \end{tikzcd}$$

where $\mathbb{S}_h V^n_{d-l} \subseteq \mathbb{G}(h-1, N_{d-l})$ is the abstract h-secant variety of V^n_{d-l}. Note that when $h < \binom{n+l}{l}$ the map π_1 is generically injective. Let $X_{l,h} = \pi_1(\mathcal{I}_{l,h}) \subseteq \mathbb{P}^N$ be its image, note that $X_{l,h}$ is irreducible. By remark 2.1 we get $\mathbb{S}ec_h(V^n_d) \subseteq X_{l,h}$. By construction $X_{l,h}$ is not too difficult to describe, so we want to find cases when the equality holds in order to get a simple criterion to establish whether $[F] \in \mathbb{S}ec_h(V^n_d)$.

REMARK 2.2. The equality holds trivially when $d = 2$. Let $F \in k[x_0, ..., x_n]_2$ be a polynomial and let $\mathcal{M}_F$ the matrix of the quadratic symmetric form associated to F. Then $F \in \mathbb{S}ec_h(V^n_2)$ if and only if $\operatorname{rank}(\mathcal{M}_F) \leq h$. But the rows of $\mathcal{M}_F$ are exactly the partial derivatives of F.

Consider the partial derivatives $F_1, ..., F_m \in k[x_0, ..., x_n]_{d-l}$ of order l of F. Let $\phi : \mathbb{P}^n \times \mathbb{P}^{N_{d-l}} \to \mathbb{P}^M$ be the Segre-Veronese embedding induced by $\mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^{N_{d-l}}}(d-l, 1)$, and let $\Sigma_{d-l,1}$ be its image.

PROPOSITION 2.3. *If the partial derivatives $F_1, ..., F_m$ lie in a $(h-1)$-plane $H \subset \mathbb{P}^{N_{d-l}}$ which is h-secant to the Veronese variety $V^n_{d-l} \subset \mathbb{P}^{N_{d-l}}$, with $h-1 < N_{d-l}$, then $[F] \in \mathbb{S}ec_h(\Sigma_{d-l,1})$.*

PROOF. By assumption $F^l_{l_0,...,l_n} = \sum_{i=1}^h \lambda_i^{l_0,...,l_n} L_i^{d-l}$. Recursively applying Euler formula we get $F = P_1 L_1^{d-l} + ... + P_h L_h^{d-l}$ where $P_i \in k[x_0, ..., x_n]_l$, and this means that $[F] \in \mathbb{S}ec_h(\Sigma_{d-l,1})$. □

REMARK 2.4. Suppose that $F_{x_0}, ..., F_{x_n} \in k[x_0, ..., x_n]_{d-1}$ are the partial derivatives of a homogeneous polynomial $F \in k[x_0, ..., x_n]_d$. Furthermore suppose that $F_{x_i} \in \langle L_1^{d-1}, ..., L_h^{d-1}\rangle$ for any i. By Euler formula we get

$$F = P_1 L_1^{d-1} + ... + P_h L_h^{d-1},$$

where the P_i's are linear forms, i.e. $F \in \mathbb{S}ec_h(\Sigma_{d-1,1})$. Since $F \in \mathbb{P}^N$ by hypothesis we have $F \in \mathbb{S}ec_h(\Sigma_{d-1,1}) \cap \mathbb{P}^N$. Consider the following two statements

(i) $\mathbb{S}ec_h(\Sigma_{d-1,1}) \cap \mathbb{P}^N = \mathbb{S}ec_h(V^n_d)$;
(ii) $F_{x_i} \in \langle L_1^{d-1}, ..., L_h^{d-1}\rangle$ for any $i = 0, ..., n$, implies $[F] \in \mathbb{S}ec_h(V^n_d)$.

From the above discussion we deduce that (i) implies (ii).

The Case $n = 1$**.** We begin with the simplest case $n = 1$. We denote by $C_d \subset \mathbb{P}^d$ the degree d rational normal curve, in this case $\mathbb{S}ec_h(C_d) \neq \mathbb{P}^d$ if and only if $h \leq \frac{d}{2}$.

LEMMA 2.5. *Let $F = \sum_{i+j=d} \alpha_{i,j} x_0^i x_1^j \in k[x_0, x_1]_d$ be a homogeneous polynomial, and let $c = c(\alpha_{i,j})$ be the coefficient of x_0^h in the partial derivative $\frac{\partial^{d-h} F}{\partial x_0^m \partial x_1^s}$, with $h \geq 1$. Then $c = C \cdot \alpha_{d-s,s}$, where C is a constant.*

PROOF. Since the only monomial of F producing c is $x_0^{d-s}x_1^s$ the assertion follows. $\square$

THEOREM 2.6. *For any $h \leq \frac{d}{2}$ we have $\mathcal{S}ec_h(C_d) = X_{d-h,h}$. Consequently if the partial derivatives of order $d-h$ of a homogeneous polynomial $F \in k[x_0,x_1]_d$ lie in a hyperplane of $\mathbb{P}^h$ then $[F]$ lies in $\mathcal{S}ec_h(C_d)$.*

PROOF. The partial derivatives of order $d-h$ of F are $d-h+1$ homogeneous polynomials of degree h. If $F = \sum_{i=1}^{h} \lambda_i L_i^d$ the partial derivatives lie in $\langle L_1^h, ..., L_h^h \rangle$ which is a hyperplane h-secant to C_h, but $\deg(C_h) = h$ and the latter condition is irrelevant. Let H be a general hyperplane in $\mathbb{P}^h$, forcing the partial derivatives of a degree d polynomial $G = \sum_{i+j=d} \alpha_{i,j} x_0^i x_1^j \in k[x_0,x_1]_d$ to lie in H gives $d-h+1$ linear equations in the coefficients of G. Without loss of generality we can suppose H to be the defined by the vanishing of the first homogeneous coordinate on $\mathbb{P}^h$, then by 2.5 the fiber of π_2 is the linear subspace of $\mathbb{P}^N$ defined by

$$\pi_2^{-1}(H) = \{\alpha_{d-s,s} = 0,\ \forall\ s = 0,...,d-h\}.$$

The equations of $\pi_2^{-1}(H)$ are independent so

$$\dim(\pi_2^{-1}(H)) = d - (d-h+1) = h-1,$$

and the dimension of $X_{d-h,h}$ is

$$\dim(X_{d-h,h}) = \dim(\mathcal{I}_{d-h,h}) = h-1+h = 2h-1.$$

Finally $\dim(\mathcal{S}ec_h(C_d)) = h+h-1 = 2h-1$ yields $\mathcal{S}ec_h(C_d) = X_{d-h,h}$. $\square$

REMARK 2.7. The partial derivatives of order $d-h$ of a homogeneous polynomial $F \in k[x_0,x_1]_d$ depend on $d+1$ parameters. We consider the matrix $\mathcal{M}_{d,h}$ whose lines are the partial derivatives. From 2.6 we get equations for $\mathcal{S}ec_h(C_d)$ imposing $\operatorname{rank}(\mathcal{M}_{d,h}) \leq h$, that is the classical determinantal description of $\mathcal{S}ec_h(C_d)$.

The Case $h \leq n$. Now we consider the variety $X_{d-1,h}$. The partial derivatives of order $d-1$ of F are linear forms i.e. points in $(\mathbb{P}^n)^*$, so we restrict our attention on the case $h \leq n$ to have significant constraints. First we compute the dimension of the general fiber of $\pi_2 : \mathcal{I}_{d-1,h} \rightarrow \mathbb{G}(h-1,n)$.

THEOREM 2.8. *The fiber of $\pi_2 : \mathcal{I}_{d-1,h} \rightarrow \mathbb{G}(h-1,n)$ on a general $(h-1)$-plane $H \in \mathbb{G}(h-1,n)$ is a linear subspace of $\mathbb{P}^N$ of dimension*

$$\dim(\pi_2^{-1}(H)) = \binom{d+h-1}{d} - 1.$$

Furthermore the dimension of X_{d-1} is given by

$$\dim(X_{d-1,h}) = h(n-h+1) + \binom{d+h-1}{d} - 1.$$

PROOF. We can suppose $H = \{X_0 = ... = X_{n-h} = 0\}$, where $\{X_0, ..., X_n\}$ are homogeneous coordinates on $\mathbb{P}^n$. We write a general polynomial $[F] \in \mathbb{P}^N$ in the form

$$F = \sum_{i_0+...+i_n=d} \alpha_{i_0,...,i_n} x_0^{i_0} ... x_n^{i_n}.$$

The fiber $\pi_2^{-1}(H)$ is the linear subspace of $\mathbb{P}^N$ defined by the vanishing of the coefficients of $x_0, ..., x_{n-h}$ in the derivatives of F. Many of these equations are redundant, the difficulty is in counting the exact number of independent equations. We prove that this number is $\binom{d+n-1}{d-1} + \binom{d+n-1}{d} -$

$\binom{d+h-1}{d}$ by induction on $n-h$. If $n-h=0$ then H is an hyperplane and the condition on the derivatives are all independent, so the number of conditions is exactly the number of derivatives $\binom{d-1+n}{d-1}$. Furthermore our formula for $n-h=0$ gives $\binom{d+n-1}{d-1}+\binom{d+n-1}{d}-\binom{d+n-1}{d}=\binom{d+n-1}{d-1}$, and the case $n-h=0$ is verified. Consider now the general case, let $\overline{H}=\{X_0=...=X_{n-h-1}=0\}$, let C_{n-h-1} the number of independent conditions obtained forcing the partial derivatives to lie in $\overline{H}$. Adding the condition $\{X_{n-h}=0\}$ gives new equations coming from the coefficients of the form $\alpha_{0,...,0,i_{n-h},i_{n-h+1},...,i_n}$, with $i_{n-h}\neq 0$. These correspond to monomials of degree d in the variables $x_{n-h},...,x_n$ that contain the variable x_{n-h}. Now the monomials of degree d not containing x_{n-h} are the monomials of degree d in $x_{n-h+1},...,x_n$. So in the final step we are adding

$$\binom{d+h}{d}-\binom{d+h-1}{d}$$

conditions. Then the number if independent equations is $C_{n-h}=C_{n-h-1}+\binom{d+h}{d}-\binom{d+h-1}{d}$, by induction hypothesis

$$C_{n-h-1}=\binom{d+n-1}{d-1}+\binom{d+n-1}{d}-\binom{d+n-(n-h-1)-1}{d}.$$

So $C_{n-h}=\binom{d+n-1}{d-1}+\binom{d+n-1}{d}-\binom{d+n-(n-h-1)-1}{d}+\binom{d+h}{d}-\binom{d+h-1}{d}=\binom{d+n-1}{d-1}+\binom{d+n-1}{d}-\binom{d+h-1}{d}$. Finally we have $\dim(X_{d-1,h})=\dim(\mathbb{G}(h-1,n))+\dim(\pi_2^{-1}(H))=h(n-h+1)+\binom{d+h-1}{d}-1$. $\square$

REMARK 2.9. Consider the case $d=2$. By Alexander-Hirshowitz theorem **[AH]**, $\mathbb{S}ec_h(V_2^n)\neq\mathbb{P}^N$ if and only if $h\leq n$. By Theorem 2.8 and Remark 2.2 we recover the effective dimension of $\mathbb{S}ec_h(V_2^n)$,

$$\dim(\mathbb{S}ec_h(V_2^n))=\frac{2nh-h^2+3h-2}{2},$$

and consequently the formula for the h-secant defect of V_2^n,

$$\delta_h(V_2^n)=\frac{h(h-1)}{2}.$$

At this point we have a complete description for polynomials of arbitrary degree in two variables and for polynomials of degree two in any number of variables. So we concentrate on the case $n\geq 2$ and $d\geq 3$.

THEOREM 2.10. *Let $n\geq 2, d\geq 3, h\leq n$ be positive integers. Then $\mathbb{S}ec_h(V_d^n)$ is a subvariety of $X_{d-1,h}$ of codimension*

$$\operatorname{codim}_{\mathbb{S}ec_h(V_d^n)}(X_{d-1,h})=\binom{d+h-1}{d}-h^2.$$

PROOF. Since $n\geq 2$, $d\geq 3$, and $h\leq n$, by Alexander-Hirshowitz theorem the effective dimension of $\mathbb{S}ec_h(V_d^n)$ is the expected one

$$\dim(\mathbb{S}ec_h(V_d^n))=\min\{hn+(h-1),N_d\}.$$

Furthermore $n\geq 2$, $d\geq 3$, $h\leq n$ implies $hn+(h-1)<N_d$. So

$$\dim(\mathbb{S}ec_h(V_d^n))=hn+(h-1).$$

Finally $\operatorname{codim}_{\mathbb{S}ec_h(V_d^n)}(X_{d-1,h})=h(n-h+1)+\binom{d+h-1}{d}-1-hn-(h-1)=\binom{d+h-1}{d}-h^2$. $\square$

COROLLARY 2.11. *If $d = 3$ then $\mathbb{S}ec_2(V_3^n) = X_{2,2}$ for any $n \geq 2$. Consequently if the second partial derivatives of a homogeneous polynomial $F \in k[x_0, ..., x_n]_3$ lie in a line of $\mathbb{P}^n$ then $[F]$ lies in $\mathbb{S}ec_2(V_3^n)$.*

PROOF. For $h = 2, d = 3$ we have $\binom{d+h-1}{d} - h^2 = 0$. We conclude by theorem 2.10. □

2.1. The variety $X_{l,h}$. Let's look closer at the variety $X_{l,h}$. This variety parametrizes polynomials $F \in k[x_0, ..., x_n]_d$ whose partial derivatives of order l span a $(h-1)$-plane. Let $\mathcal{M}_{l,h}$ be the $\binom{n+l}{l} \times \binom{n+d-l}{d-l}$ matrix whose lines are the l-th derivatives of $F = \sum_{i_0+...+i_n=d} \alpha_{i_0,...,i_n} x_0^{i_0} ... x_n^{i_n}$. Then $X_{l,h}$ is the determinantal variety defined in $\mathbb{P}^N$ by $\operatorname{rank}(\mathcal{M}_{l,h}) \leq h$, where the $\alpha_{i_0,...,i_n}$ are the homogeneous coordinates on $\mathbb{P}^N$. Let $\mathbb{P}^M$ be the projective space parametrizing $\binom{n+l}{l} \times \binom{n+d-l}{d-l}$ matrices, and let $M_h \subset \mathbb{P}^M$ be the variety of matrices of rank less or equal than h. Then M_h is an irreducible variety of dimension $M - \left(\binom{n+l}{l} - h\right) \cdot \left(\binom{n+d-l}{d-l} - h\right)$. Clearly the variety $X_{l,h}$ is a special linear section of M_h.

LEMMA 2.12. *The varieties $X_{l,h}$ and $X_{d-l,h}$ are isomorphic.*

PROOF. The matrix $\mathcal{M}_{d-l,h}$ whose lines are the $(d-l)$-th partial derivatives of F is the $\binom{n+d-l}{d-l} \times \binom{n+l}{l}$ matrix given by

$$\mathcal{M}_{d-l,h} = \mathcal{M}_{l,h}^t,$$

where $\mathcal{M}_{l,h}^t$ is the transposed matrix of $\mathcal{M}_{d-l,h}$. Then the assertion follows. □

PROPOSITION 2.13. *Consider the case $h \leq n$. The variety $X_{1,h}$ is irreducible.*

PROOF. By Lemma 2.12 it is equivalent to prove that $X_{d-1,h}$ is irreducible. Consider the map $\pi_2 : \mathcal{I}_{d-1,h} \to G(h-1, n)$. By Theorem 2.8 the general fiber of π_2 is a linear subspace of $\mathbb{P}^N$ of dimension $\dim(\pi_2^{-1}(H)) = \binom{d+h-1}{d} - 1$ and π_2 is surjective on $G(h-1, n)$, so $X_{d-1,h}$ is irreducible. □

In the cases $d = 2$ and $d = 3, h = 2$ we have that $\dim(X_{1,h}) = \dim(\mathbb{S}ec_h(V_d^n))$, since $X_{1,h}$ is irreducible we get $\mathbb{S}ec_h(V_d^n) = X_{1,h}$. So if the first partial derivatives of a polynomial F span a linear space of dimension $h-1$ then F can be decomposed into a sum of h powers of linear forms.

EXAMPLE 2.14. Consider a polynomial of degree three in three variables

$$F = a_0x^3 + a_1x^2y + a_2x^2z + a_3xy^2 + a_4xyz + a_5xz^2 + a_6y^3 + a_7y^2z + a_8yz^2 + a_9z^3.$$

The variety $X_{1,2}$ is defined by

$$\operatorname{rank}\begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix} = \operatorname{rank}\begin{pmatrix} 3a_0 & 2a_1 & 2a_2 & a_3 & a_4 & a_5 \\ a_1 & 2a_3 & a_4 & 3a_6 & 2a_7 & a_8 \\ a_2 & a_4 & 2a_5 & a_7 & 2a_8 & 3a_9 \end{pmatrix} \leq 2.$$

Consider the projective space $\mathbb{P}^{17}$ of 3×6 matrix with homogeneous coordinates

$$X_{0,0}, ..., X_{0,5}, X_{1,0}, ..., X_{1,5}, X_{2,0}, ..., X_{2,5}.$$

The determinantal variety M_2 defined by

$$\operatorname{rank}\begin{pmatrix} X_{0,0} & X_{0,1} & X_{0,2} & X_{0,3} & X_{0,4} & X_{0,5} \\ X_{1,0} & X_{1,1} & X_{1,2} & X_{1,3} & X_{1,4} & X_{1,5} \\ X_{2,0} & X_{2,1} & X_{2,2} & X_{2,3} & X_{2,4} & X_{2,5} \end{pmatrix} \leq 2$$

is irreducible of dimension $17-4=13$. The linear space

$$H := \begin{cases} 2X_{1,0} - X_{0,1} = 0, \\ 2X_{2,0} - X_{0,2} = 0, \\ 2X_{0,3} - X_{1,1} = 0, \\ X_{0,4} - X_{1,2} = 0, \\ 2X_{0,5} - X_{2,2} = 0, \\ 2X_{2,3} - X_{1,4} = 0, \\ 2X_{2,4} - X_{1,5} = 0, \\ X_{0,4} - X_{2,1} = 0. \end{cases}$$

cuts out on M_2 the variety $X_{1,2}$, which is irreducible of dimension $5 = \dim(\mathbb{S}ec(V_3^2))$.

REMARK 2.15. Considering a polynomial $F \in k[x,y,z]_4$ and proceeding as in example 2.14 one gets $\dim(X_{1,2}) = 6$, so

$$\mathbb{S}ec_2(V_4^2) \subsetneqq X_{1,2}.$$

PROPOSITION 2.16. *Let $d = 2k$ be an even integer such that $\binom{n+k}{k} \geq N_{d-k}$, where $N_{d-k} = \binom{d-k+n}{n} - 1$. The variety $X_{k,N_{d-k}}$ is an irreducible hypersurface of degree $\binom{n+k}{k}$ in $\mathbb{P}^N$.*

PROOF. The map $\pi_2 : \mathcal{I}_{k,N_{d-k}} \to \mathbb{G}(N_{d-k}-1, N_{d-k}) \cong \mathbb{P}^{N_{d-k}}$ is dominant, so $\mathcal{I}_{k,N_{d-k}}$ and $X_{k,N_{d-k}}$ are irreducible. The assertion follows observing that $X_{k,N_{d-k}}$ is defined by the vanishing of the determinant of a $\binom{n+k}{k} \times \binom{n+k}{k}$ matrix. □

Let us look at some consequences of the previous proposition.

EXAMPLE 2.17. Consider a polynomial

$$F = a_0x^4 + a_1x^3y + a_2x^3z + a_3x^2y^2 + a_4x^2yz + a_5x^2z^2 + a_6xy^3 + a_7xy^2z + a_8xyz^2$$
$$+ a_9xz^3 + a_{10}y^4 + a_{11}y^3z + a_{12}y^2z^2 + a_{13}yz^3 + a_{14}z^4.$$

The map $\pi_2 : \mathcal{I}_{2,4} \to \mathbb{G}(3,5)$ is dominant, so $X_{2,4}$ is irreducible. Let $Z_0, Z_1, Z_2, Z_3, Z_4, Z_5$ be homogeneous coordinates on $\mathbb{P}^5$ corresponding to $x^2, xy, xz, y^2, yz, z^2$ respectively. To compute the dimension of the general fiber of π_2 we can take the $3-plane$ $H = \{Z_0 = Z_3 = 0\}$ which intersect V_2^2 in a subscheme of dimension zero. Computing the second partial derivatives of F it turns out that

$$\pi_2^{-1}(H) = \{a_0 = a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_{10} = a_{11} = a_{12} = 0\}.$$

So $\dim(\pi_2^{-1}(H)) = 14 - 11 = 3$ and $\dim(X_{2,4}) = 3 + 8 = 11$. Since $\dim(\mathbb{S}ec_4V_4^2) = 11$ we get

$$\mathbb{S}ec_4V_4^2 = X_{2,4}.$$

Consider now $\pi_2 : \mathcal{I}_{2,5} \to \mathbb{P}^5$. This map is dominant, so $X_{2,5}$ is irreducible. We have $\dim(\pi_2^{-1}(H)) = 14 - 6 = 8$, where $H = \{Z_0 = 0\}$. So $\dim(X_{2,5}) = 13$ and

$$\mathbb{S}ec_5V_4^2 = X_{2,5}$$

is an hypersurface of degree 6 in $\mathbb{P}^{14}$.
Consider now the case $d = 4, n = 3, h = 9$ and the second partial derivatives. The map $\pi_2 : \mathcal{I}_{2,9} \to \mathbb{P}^9$ is dominant and $X_{2,9}$ is irreducible. The general fiber of π_2 has dimension 24. Then $\dim(X_{2,9}) = 24 + 9 = 33$ and

$$\mathbb{S}ec_9V_4^3 = X_{2,9}$$

is an hypersurface of degree 10 in $\mathbb{P}^{34}$.
Finally in the case $d=4, n=4, h=14$ as before one can verify that $X_{2,14}$ is irreducible of dimension 68, so

$$\mathbb{S}ec_{14}V_4^4 = X_{2,14}$$

is an hypersurface of degree 15 in $\mathbb{P}^{69}$.

EXAMPLE 2.18. Consider now a polynomial $F \in k[x,y,z]_6$ and the partial derivative of order 3. For $h=8,9$ the map π_2 is dominant, so $X_{3,8}$ and $X_{3,9}$ are irreducible. First let us take $h=8$. Proceeding as before we get $\dim(\pi_2^{-1}(H)) = 27-19 = 8$ and $\dim(X_{3,8}) = 24$. So $\mathbb{S}ec_8V_6^2 \subset X_{3,8}$ is a divisor.
In the case $h=9$ we have $\dim(\pi_2^{-1}(H)) = 27-10 = 17$ and $\dim(X_{3,9}) = 17+9 = 26$. So

$$\mathbb{S}ec_9V_6^2 = X_{3,9}$$

is an hypersurface of degree 10 in $\mathbb{P}^{27}$.

2.2. The first secant variety of V_d^n. We focus on the case $h=2$. Without any assumptions on d and n we obtain set-theoretical equations for the first secant variety of V_d^n. In the proof we use all the time the equality

$$\sum_{k=0}^{n}\binom{d-1+k}{d-1} = \binom{d+n}{d},$$

which can be easily proved by induction on n. In **[Kan]** *V. Kanev*, adopting a different approach, proved that the same equations cut out the ideal of $\mathbb{S}ec_2(V_d^n)$.

THEOREM 2.19. *If $h=2$ for the first secant variety of V_d^n we have*

$$\mathbb{S}ec_2(V_d^n) = X_{2,d-2}$$

for any n and $d \geq 3$.

PROOF. Consider the diagram

$$\begin{array}{c} \mathcal{I}_{2,d-2} = \{(F,H) \mid \in F^{l}_{l_0,\dots,l_n} \in H,\ \forall\, l_0+\dots+l_n = d-2\} \subset \mathbb{P}^N \times \mathbb{G}(1,N_2) \\ \swarrow^{\pi_1} \qquad\qquad \searrow^{\pi_2} \\ \mathbb{P}^N \qquad\qquad\qquad \mathbb{G}(1,N_2) \end{array}$$

clearly $\mathbb{S}_2V_2^n \subseteq Im(\pi_2)$. Let $F \in k[x_0,\dots,x_n]_d$ be a polynomial whose partial derivatives of order $d-2$ lie on a line $H \subset \mathbb{P}^{N_2}$. The derivatives of order $d-3$ of F are cubic polynomials whose first partial derivatives are collinear. By 2.11 $X_{2,1} = X_{2,2} = \mathbb{S}ec_2V_3^n$, so if we denote by G a partial derivative of order $d-3$ of F we get a decomposition $G = L_1^3 + L_2^3$. Then $G_{x_0},\dots,G_{x_n}$ (which are partial derivatives of order $d-2$ of F) lie on the line $\langle L_1^2, L_2^2\rangle$, and so the line containing the partial derivative of order $d-2$ of F is exactly the secant line to V_2^n given by $\langle L_1^2, L_2^2\rangle$. This means that

$$\mathbb{S}_2V_2^n = Im(\pi_2).$$

Since the fibers of π_2 are linear spaces we conclude that $\mathcal{I}_{2,d-2}$ and $X_{2,d-2}$ are irreducible.
We compute now the dimension of the fiber of π_2. We fix on $\mathbb{P}^{N_2}$ homogeneous coordinates $Z_0,\dots,Z_{N_2}$ corresponding to the monomials in lexicographic order $x_0^2, x_0x_1,\dots,x_n^2$, and consider the line $H = \{Z_0 = Z_1 = \dots = Z_{N_2-2} = 0\}$.
First consider monomials containing x_0. Forcing the derivatives to lie in $\{Z_0 = 0\}$ we get $\binom{d-2+n}{n}$

conditions (the monomials containing x_0^2, whose number is equal to the number of degree $d-2$ monomials in $x_0, ..., x_n$). Imposing $\{Z_1 = 0\}$ we get $\binom{d-2+n-1}{n-1}$ conditions (the monomials containing x_0x_1, whose number is equal to the number of degree $d-2$ monomials in $x_1, ..., x_n$). Proceeding in this way when we force $\{Z_n = 0\}$ we get $\binom{d-2+n-n}{n-n} = 1$ condition (the monomials containing x_0x_n, whose number is equal to the number of degree $d-2$ monomials in x_n). Up to now we have

$$\sum_{k=0}^{n} \binom{d-2+k}{k} = \binom{d-1+n}{d-1}$$

conditions.
Consider now the monomials containing x_1. Forcing $\{Z_{n+1} = 0\}$ we get $\binom{d-2+n-1}{n-1}$ conditions (the monomials containing x_1^2, whose number is equal to the number of degree $d-2$ monomials in $x_1, ..., x_n$). Imposing $\{Z_{n+2} = 0\}$ we get $\binom{d-2+n-2}{n-2}$ conditions (the monomials containing x_1x_2, whose number is equal to the number of degree $d-2$ monomials in $x_2, ..., x_n$). Proceeding in this way we get

$$\sum_{k=0}^{n-1} \binom{d-2+k}{k} = \binom{d-1+n-1}{d-1}$$

conditions.
Proceeding in this way at the step x_{n-2} we have

$$\sum_{k=0}^{2} \binom{d-2+k}{k} = \binom{d-1+2}{d-1}$$

more conditions. At the step x_{n-1} we have only to force $\{Z_{N_2-2} = 0\}$, and we get $\binom{d-1}{1} = d-1$ conditions.
Summing up the fiber $\pi_2^{-1}(H)$ is a linear subspace of $\mathbb{P}^N$ defined by

$$\sum_{k=2}^{n} \binom{d-1+k}{d-1} + d - 1 = \sum_{k=0}^{n} \binom{d-1+k}{d-1} - 1 - d + d - 1 = \binom{d+n}{d} - 2.$$

So the fiber has dimension

$$\dim(\pi_2^{-1}(H)) = N - \binom{d+n}{d} + 2 = 1,$$

recalling that $N = \binom{d+n}{d} - 1$. Finally we look at the map $\pi_2 : \mathcal{I}_{2,d-2} \to \mathbb{S}_2V_2^n$, since π_2 is dominant we have

$$\dim(X_{2,d-2}) = \dim(\mathcal{I}_{2,d-2}) = 2n+1.$$

Since $\dim(\mathbb{S}ec_2V_d^n) = 2n+1$ the assertion follows. $\square$

2.3. The case $n = 2, h = 4$. In the same spirit of Theorem 2.19 we obtain the following result.

THEOREM 2.20. *If $n = 2, h = 4$ for the variety of 4-secant 3-planes of V_d^2 we have*

$$\mathbb{S}ec_4(V_d^2) = X_{4,\lfloor \frac{d}{2} \rfloor}$$

for any d positive integer.

PROOF. The case $d=4$ is the Example 2.17. Consider now the case $d=5$. The map $\pi_2 : \mathcal{I}_{4,3} \to \mathbb{G}(3,5)$ is dominant, so $X_{4,3}$ and hence $X_{4,2}$ are irreducible. Let $F \in k[x,y,z]_5$ be a polynomial, looking at the proof of theorem 2.19 we get that forcing the partial derivatives of order 3 of F to lie in $\{Z_0 = Z_3 = 0\}$ gives

$$\binom{5-2+2}{2} + \binom{5-2+2}{2} - \sharp\{\text{monomials containing } x^2y^2\} = 20-3 = 17$$

conditions. Since $\dim(X_{4,2}) = \dim(X_{4,3}) = 20 - 17 + \dim(\mathbb{G}(3,5)) = 11$ we conclude

$$\mathbb{S}ec_4(V_5^2) = X_{4,2}.$$

Consider the case $d = 6$ and the partial derivative of order 3. If the 3-th derivatives of F lie in a 3-plane then the first partial derivative of F are degree 5 polynomials whose second partial derivatives lie in a 3-plane. By the same trick of Theorem 2.19 we prove that the 3-plane containing the 3-th partial derivative has to be 4-secant to V_3^2. So $X_{4,3}$ is irreducible, and as usual by counting dimension we get the equality

$$\mathbb{S}ec_4(V_6^2) = X_{4,3}.$$

Now we treat the general case by induction on d. Let $F \in k[x,y,z]_d$ be a polynomial whose $\lfloor \frac{d}{2} \rfloor$-th derivative lies in a 3-plane. Then the first partial derivative of F are polynomials of degree $d-1$ whose $\lfloor \frac{d-1}{2} \rfloor$-th derivatives lie in a 3-plane. So F_x, F_y, F_z can be decomposed as sums of four powers of linear forms. As before we conclude that the map $\pi_2 : \mathcal{I}_{4,\lfloor \frac{d}{2} \rfloor} \to \mathbb{G}(3, N_{d-\lfloor \frac{d}{2} \rfloor})$ is dominant, so $X_{4,\lfloor \frac{d}{2} \rfloor}$ is irreducible. We conclude, by combinatorial computations similar to the previous one, computing $\dim(X_{4,\lfloor \frac{d}{2} \rfloor}) = \dim(\mathbb{S}ec_4(V_d^2))$. $\square$

REMARK 2.21. In a completely analogous way one can show that $\mathbb{S}ec_5(V_d^2)$ is defined by size 6 minors of the matrix of partial derivatives of order $\lfloor \frac{d}{2} \rfloor$ for $d = 4$ and $d \geq 6$.

Finally, we report part of a table in **[LO]** summarizing the known cases in which a secant of a Veronese variety coincides at least set theoretically with a catalecticant variety. Indeed in these cases the equations of catalecticants cut scheme theoretically the secant variety and in some cases even the ideal. We denote by $\mathcal{M}_l$ the matrix whose lines are the partial derivatives of order l of a homogeneous polynomial $F \in k[x_0, ..., x_n]_d$.

Secant	Catalecticant	Reference
$\mathbb{S}ec_h V_2^n$	$h+1$ minors of $\mathcal{M}_1$	Classical
$\mathbb{S}ec_h V_d^1$	$h+1$ minors of $\mathcal{M}_{\mathrm{d-h}}$	Iarrobino – Kanev and Th 2.6
$\mathbb{S}ec_2 V_d^n$	3 minors of $\mathcal{M}_{\mathrm{d-2}}$	Kanev and Th 2.19
$\mathbb{S}ec_4 V_d^2$	5 minors of $\mathcal{M}_{\lfloor \frac{d}{2} \rfloor}$	Schreier and Th 2.20
$\mathbb{S}ec_5 V_d^2,\ d=4,\ d \geq 6$	6 minors of $\mathcal{M}_{\lfloor \frac{d}{2} \rfloor}$	Th 3.2.1 **[BCS]**
$\mathbb{S}ec_6 V_d^2,\ d \geq 6$	7 minors of $\mathcal{M}_{\lfloor \frac{d}{2} \rfloor}$	Th 3.2.1 **[CGLM]**
$\mathbb{S}ec_9 V_6^2$	determinant of $\mathcal{M}_3$	Ex 2.18

3. Sigularities of Secant Varieties

We are particularly interested in secant varieties of rational normal curves. Just to get acquainted with describe in detail the variety of secant lines of the degree four rational normal curve $C \subset \mathbb{P}^4$.

EXAMPLE 3.1. Let $C \subset \mathbb{P}^4$ be a degree four rational normal curve. By [**Harr**, Proposition 9.7] $\mathbb{S}ec_2(C) \subset \mathbb{P}^4$ is the cubic hypersurface given by the vanishing of the determinant of

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

that is

$$\mathbb{S}ec_2(C) = \{F = x_0x_2x_4 - x_0x_3^2 - x_1^2x_4 + 2x_1x_2x_3 - x_2^3 = 0\}.$$

The partial derivatives of F are given by

$$\begin{aligned} \tfrac{\partial F}{\partial x_0} &= x_2x_4 - x_3^2, \\ \tfrac{\partial F}{\partial x_1} &= 2(x_2x_3 - x_1x_4), \\ \tfrac{\partial F}{\partial x_2} &= x_0x_4 - x_2^2 - 2(x_2^2 - x_1x_3), \\ \tfrac{\partial F}{\partial x_3} &= 2(x_1x_2 - x_0x_3), \\ \tfrac{\partial F}{\partial x_4} &= x_0x_2 - x_1^2. \end{aligned}$$

Note that all the derivatives are linear combination of 2×2 minors of the matrix M and they vanish simultaneously on C. Furthermore the second partial derivatives of F are 15 linear polynomials that are never simultaneously zero. To see this, it is enough to notice that

$$\frac{\partial^2 F}{\partial x_0 x_3} = -2x_3,\ \frac{\partial^2 F}{\partial x_4 x_1} = -2x_1,\ \frac{\partial^2 F}{\partial x_2^2} = -6x_2,\ \frac{\partial^2 F}{\partial x_4 x_2} = x_0,\ \frac{\partial^2 F}{\partial x_0 x_2} = x_4.$$

We conclude that $\deg(\mathbb{S}ec_2(C)) = 3$, $\mathrm{Sing}(\mathbb{S}ec_2(C)) = C$ and $\mathrm{mult}_C \mathbb{S}ec_2(C) = 2$.

PROPOSITION 3.2. *Let $C \subset \mathbb{P}^n$ be a degree n rational normal curve, and let k be an integer such that $1 \leq k \leq \frac{n}{2}$. Then*

$$\dim(\mathbb{S}ec_k(C)) = 2k - 1.$$

Furthermore

$$\deg(\mathbb{S}ec_k(C)) = \binom{n-k+1}{k}, \quad \mathrm{Sing}(\mathbb{S}ec_k(C)) = \mathbb{S}ec_{k-1}(C).$$

Finally, if $n = 2h$ is even then

$$\mathrm{mult}_{\mathbb{S}ec_{h-t}(C)} \mathbb{S}ec_h(C) = t + 1$$

for any $1 \leq t \leq h$.

PROOF. Since $C \subset \mathbb{P}^n$ is non-degenerate we have $\dim(\mathbb{S}ec_k(C)) = 2k - 1$. By [**EH**, Theorem 12.16] we get $\deg(\mathbb{S}ec_k(C)) = \binom{n-k+1}{k}$.
The rational normal curve $C \subset \mathbb{P}^n$ is given by the Veronese embedding induced by the line bundle $L = \mathcal{O}_{\mathbb{P}^1}(n)$ on $\mathbb{P}^1$. Note that if $k \leq \frac{n}{2}$ then $n - 2k - 1 \geq -1$. This yields

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n - 2k - 1)) = n - 2k = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) - (2k + 1).$$

Therefore, $C \subset \mathbb{P}^n$ is embedded by a $(2k+1)$-very ample line bundle. By [**Ve1**, Theorem 1.1] we have that $\mathbb{S}ec_k(C)$ is normal and $\mathrm{Sing}(\mathbb{S}ec_k(C)) = \mathbb{S}ec_{k-1}(C)$ for any $k \leq \frac{n}{2}$.

Now, let $n = 2h$ even. It is well know, see for instance [**Harr**, Proposition 9.7], that $\mathcal{S}ec_h(C) \subset \mathbb{P}^{2h}$ is the degree $h+1$ hypersurface given by the vanishing of the determinant of the matrix

$$M_h = \begin{pmatrix} x_0 & x_1 & x_2 & \dots & x_h \\ x_1 & x_2 & x_3 & \dots & x_{h+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{h-1} & x_h & x_{h+1} & \dots & x_{2h-1} \\ x_h & x_{h+1} & x_{h+2} & \dots & x_{2h} \end{pmatrix}$$

Let $F = \det(M_h)$. Then $\mathcal{S}ec_h(C) = \{F = 0\} \subset \mathbb{P}^{2h}$. Let M_i^j be the $h \times h$ minors of M_h produces by erasing in M_h a row and a column meeting in an entry of type x_j, for $j = 0, ..., 2h$. Let ρ_j be the number of such minors. Then

$$\frac{\partial F}{\partial x_j} = \sum_{i=1}^{\rho_j} \alpha_i^j \det(M_i^j).$$

Now, proceeding recursively we see that for any $1 \leq t \leq h$ the partial derivatives of order t of F are linear combinations of determinants of $(h+1-t) \times (h+1-t)$ minors of M_h. Again by [**Harr**, Proposition 9.7] such minors define $\mathcal{S}ec_{h-t}(C)$. Furthermore, since $\mathrm{Sing}(\mathcal{S}ec_k(C)) = \mathcal{S}ec_{k-1}(C)$ for any $k \leq \frac{n}{2}$, there is at least one partial derivative of order $t+1$ of F not vanishing on $\mathcal{S}ec_{h-t-1}(C)$. This means that $\mathrm{mult}_{\mathcal{S}ec_{h-t}(C)} \mathcal{S}ec_h(C) = t+1$ for any $1 \leq t \leq h$. □

REMARK 3.3. Let $n = 2h$ be even. By Proposition 6.7 we have that $\mathcal{S}ec_h(C) \subset \mathbb{P}^{2h}$ is an hypersurface of degree $h+1$, and $\mathrm{mult}_C \mathcal{S}ec_h(C) = h$.

The following proposition is just a particular instance of [**Be**, Theorem 1]. The general statement for smooth curves embedded via a $2h$-very ample line bundle can be found in [**Ve**, Theorem 3.1] as well.

PROPOSITION 3.4. *Let $C \subset \mathbb{P}^n$ be a degree n rational normal curve, and let h be the greatest integer such that $h \leq \frac{n}{2}$. Consider the following sequence of blow-ups:*

- $\pi_1 : X_1 \to \mathbb{P}^n$ *the blow-up of C,*
- $\pi_2 : X_2 \to X_1$ *the blow-up of the strict transform of $\mathcal{S}ec_2(C)$,*

$\vdots$

- $\pi_h : X_h \to X_{h-1}$ *the blow-up of the strict transform of $\mathcal{S}ec_h(C)$.*

Let $\pi : X \to \mathbb{P}^n$ be the composition of these blow-ups. Then, for any $k \leq h$ the strict transform of $\mathcal{S}ec_k(C)$ in X_{k-1} is smooth, irreducible and transverse to all exceptional divisors. In particular X is smooth and the divisor in X given by the union of the exceptional divisors is simple normal crossing.

PROOF. Since $h \leq \frac{n}{2}$ we have

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2h)) = n - 2h + 1 = h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n)) - 2h.$$

This means that $C \subset \mathbb{P}^n$ is embedded by a $2h$-very ample line bundle. To conclude it is enough to apply [**Be**, Theorem 1]. □

CHAPTER 3

Weak Fano varieties, log Fano varieties and Mori Dream Spaces

Let X be a normal projective variety. We denote by $\mathrm{N}^1(X)$ the real vector space of Cartier divisors and by $\rho_X = \dim(\mathrm{N}^1(X))$ the Picard number of X.

- The *effective cone* $\mathrm{Eff}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of effective divisors. In general it is not a closed cone.
- The *nef cone* $\mathrm{Nef}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of divisors D such that $D \cdot C \geq 0$ for any curve $C \subset X$. It is closed, but in general it is neither polyhedral nor rational.
- A divisor $D \subset X$ is called *movable* if its stable base locus is in codimension greater or equal that two. The *movable cone* $\mathrm{Mov}(X)$ is the convex cone in $\mathrm{N}^1(X)$ generated by classes of movable divisors. In general, it is not closed.

A *small* $\mathbb{Q}$*-factorial transformation* of X is a birational map $f : X \dashrightarrow Y$ to another normal $\mathbb{Q}$-factorial projective variety Y, such that f is an isomorphism in codimension one.
The exponential exact sequence

$$0 \mapsto \mathbb{Z} \to \mathcal{O}_X \to \mathcal{O}_X^* \mapsto 0$$

induces the following exact sequence in cohomology

$$0 \mapsto H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}_X) \to H^1(X,\mathcal{O}_X^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathcal{O}_X).$$

The complex torus $H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$ is the *Picard variety* of X. This variety $\mathrm{Pic}^0(X)$ is the connected component of the identity of $\mathrm{Pic}(X) \cong H^1(X,\mathcal{O}_X^*)$ and it is an abelian variety. The image of $\mathrm{Pic}(X)$ inside $H^2(X,\mathbb{Z})$ is isomorphic to $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$. The group $\mathrm{NS}(X) \cong \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is a finitely generated abelian group called the *Néron-Severi group*. The group $\mathrm{NS}(X)$ parametrizes divisor on X modulo numerical equivalence.

EXAMPLE 0.1. Let us consider a smooth projective curve X of genus g. That is X is a compact Riemann surface with g handles. Then $H^0(X,\mathbb{Z}) \cong H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ because X is connected, and $H^1(X,\mathbb{Z}) \cong \mathbb{Z}^{2g}$. Since $H^0(X,\mathcal{O}_X) \cong \mathbb{C}^g$ we have $\mathrm{Pic}^0(X) \cong \mathbb{C}^g/\mathbb{Z}^{2g} \cong \mathrm{Jac}(X)$, the Jacobian variety of X. In this case the degree gives an isomorphism $\mathrm{NS}(X) \cong \mathbb{Z}$.

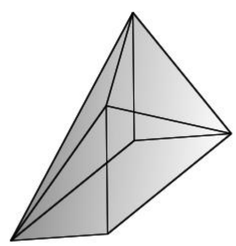
DEFINITION 0.2. *A normal projective variety X is a Mori Dream Space if*

(a) *X is $\mathbb{Q}$-factorial and* $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathbb{Q}}$;
(b) $\mathrm{Nef}(X)$ *is generated by finitely many semi-ample line bundles;*
(c) *there exist finitely many small $\mathbb{Q}$-factorial modifications $f_i : X \dashrightarrow X_i$ such that each X_i satisfies* (a), (b), *and* $\mathrm{Mov}(X)$ *us the union of* $f_i^* \mathrm{Nef}(X_i)$.

REMARK 0.3. Condition (a) is equivalent to the finite generation of $\mathrm{Pic}(X)$ which is equivalent to $h^1(X,\mathcal{O}_X) = 0$. Note that if X is a Mori Dream Space then the X_i are Mori Dream Spaces as well.

- A normal $\mathbb{Q}$-factorial projective variety of Picard number is one is a Mori Dream Space if and only if $\mathrm{Pic}(X)$ is finitely generated.
- Let X be a normal $\mathbb{Q}$-factorial projective surface satisfying (a), (b), then $\mathrm{Nef}(X) = \mathrm{Mov}(X)$ and, by taking Id_X, we see that (c) is satisfied as well.
- Any projective $\mathbb{Q}$-factorial toric variety and any smooth Fano variety is a Mori Dream Space.
- If X is a smooth rational surface and $-K_X$ is big the X is a Mori Dream Space.
- A smooth $K3$ surface is a Mori Dream Space if and only if its automorphism group is finite.

EXAMPLE 0.4. Let X be the blow-up of $\mathbb{P}^3$ at two distinct points x_1, x_2. Let H be the pullback of the hyperplane section and E_1, E_2 the two exceptional divisors. The anti-canonical divisor of X is $-K_X = 4H - 2E_1 - 2E_2$. If L is the strict transform of the line $\langle x_1, x_2 \rangle$ we have $-K_X \cdot L = 0$. Therefore X is not Fano. The Picard group of X is generated by H, E_1, E_1 and $\rho_X = 3$. Clearly X is a toric variety. Therefore it is a Mori Dream Space. The following is the polyhedron of X in $\mathbb{R}^3$.



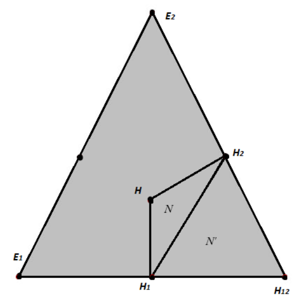
Let $|\mathcal{I}_{x_1,x_2}(2)|$ be the linear system of quadrics in $\mathbb{P}^3$ through x_1, x_2. The corresponding linear system on X induces an morphism

$$\begin{tikzcd} X \arrow[d, "\epsilon"'] \arrow[dr, "f"] & \\ \mathbb{P}^3 \arrow[r, dashed] & Y \subset \mathbb{P}^7 \end{tikzcd}$$

contracting L. Since the normal bundle of L is $\mathcal{O}_L(-1)^{\oplus 2}$ the singular point $f(L) \in f(X) = Y$ is a node. Furthermore f is a small contraction and $f(X)$ is not $\mathbb{Q}$-factorial. Let us blow-up the curve L and let Z be the blow-up. The exceptional divisor is isomorphic two $\mathbb{P}^1 \times \mathbb{P}^1$. By contracting one ruling we get X. On the other hand by contracting the other ruling we find another smooth variety $X^{'}$. The birational map $g : X \dashrightarrow X^{'}$ is the flip of f. The situation is summarized in the following diagram.

$$\begin{tikzcd} & Z \arrow[dl] \arrow[dr] & \\ X \arrow[rr, dashed, "g"] \arrow[dr, "f"'] & & X' \arrow[dl] \\ & Y & \end{tikzcd}$$

The following is a section of $\mathrm{Eff}(X)$.



Let L be the strict transform of a general line and R_1, R_2 the classes of a line in the exceptional divisors E_1, E_2. Then the strict transform of the line through x_1, x_2 is given by $C = L - E_1 - E_2$. Now, let H_1, H_2, H_{12} be strict transforms of planes through x_1, x_2 and containing the line $\langle x_1, x_2 \rangle$ respectively. Consider $D = aH_{12} + bH_1 + cH_2$. We have $D \cdot C = -a$. Therefore $D \cdot C$ is always less or equal that zero and its zero if and only if $a = 0$. On the other hand after the contraction of C any divisor of this form becomes nef.
The variety X has exactly two small $\mathbb{Q}$-factorial transformations: the identity and the flip g. Furthermore we have $\mathrm{Mov}(X) = \mathrm{Nef}(X) \cup g^* \mathrm{Nef}(X')$. In the picture $\mathrm{Nef}(X)$ is the cone generated by H, H_1, H_2, and $\mathrm{Nef}(X')$ is the cone generated by $H_{1,2}, H_1, H_2$.

We recall two important facts about Mori Dream Space.

PROPOSITION 0.5. *Let X a be a Mori Dream Space.*

- *Any normal projective variety Y which is a small $\mathbb{Q}$-factorial modification of X is a Mori Dream Space. Furthermore the f_i of Definition 0.2 are the only small $\mathbb{Q}$-factorial transformations of X,* [**HK**, Proposition 1.11].
- *If there is a surjective morphism $X \to Y$ on a normal $\mathbb{Q}$-factorial projective variety Y, then Y is a Mori Dream Space,* [**Ok**, Theorem 1.1].

DEFINITION 0.6. *Let Γ be a semigroup of Weil divisors on X. We can consider the Γ-graded ring:*

$$R_X(\Gamma) = \bigoplus_{D \in \Gamma} H^0(X, \mathcal{O}_X(D)).$$

If the divisor class group $\mathrm{Cl}(X)$ *is finitely generated and Γ is a group of Weil divisors such that* $\Gamma_{\mathbb{Q}} \cong \mathrm{Cl}(X)_{\mathbb{Q}}$ *then the ring $R_X(\Gamma)$ is denoted by* $\mathrm{Cox}(X)$*, and called the Cox ring of X.*

REMARK 0.7. Let X be a normal and $\mathbb{Q}$-factorial projective variety with finitely generated and free Picard group and Picard number ρ_X. Let $D_1, ..., D_{\rho_X}$ be a basis of Cartier divisors of $\mathrm{Pic}(X)$. Then

$$\mathrm{Cox}(X) = \bigoplus_{m_1,...,m_{\rho_X} \in \mathbb{Z}} H^0(X, \sum_{i=1}^{\rho_X} m_i D_i).$$

Different choices of divisors $D_1, ..., D_{\rho_X}$ yield isomorphic algebras.

For the details of the proof of the following Theorem we refer to [**HK**, Proposition 2.9].

THEOREM 0.8. *A $\mathbb{Q}$-factorial projective variety X with* $\mathrm{Pic}(X)_{\mathbb{Q}} \cong \mathrm{N}^1(X)_{\mathbb{Q}}$ *is a Mori Dream Space if and only if* $\mathrm{Cox}(X)$ *is finitely generated. In this case X is a GIT quotient of the affine variety* $Y = \mathrm{Spec}(\mathrm{Cox}(X))$ *by a torus of dimension ρ_X.*

PROOF. Let X be a Mori Dream Space. Then the effective cone is rational and polyhedral and we have a decomposition:

$$\mathrm{Eff}(X) = \bigcup_{i=1}^{k} P_i$$

where the P_i's are rational polyhedra. Furthermore there are finitely many rational maps $f_i : X \dashrightarrow X_i$ such that if $D \in \mathrm{Eff}(X)$ then $f_D = f_i$ for some $i = 1, ..., k$. Let us take $D_1, ..., D_h$ divisors generating the cone P_i. The cone $R_X(D_1, ..., D_h)$ does not change by replacing X with X_i and $D_1, ..., D_h$ by the corresponding divisors $D_{1,i}, ..., D_{h,i}$ on X_i. On X_i the divisors $D_{1,i}, ..., D_{h,i}$ are semi-ample. Then $R_{X_i}(D_{1,i}, ..., D_{h,i})$, and hence $R_X(D_1, ..., D_h)$ are finitely generated.
Now, let us assume that $\mathrm{Cox}(X)$ is finitely generated. Then we have an equivariant embedding, with respect a torus G, of $Y = \mathrm{Spec}(\mathrm{Cox}(X))$ is $\mathbb{A}^n$. Taking the GIT quotient we have an embedding $Y \subseteq Q = \mathbb{A}^n // G$. Since G is a torus Q is a toric variety and hence a Mori Dream Space. Furthermore if $r : X \dashrightarrow Y$ is a rational map then there is a rational map of toric varieties $t : M \dashrightarrow N$ inducing r by restriction. Therefore X is a Mori Dream Space. □

1. Weak Fano and log Fano varieties

DEFINITION 1.1. *Let X be a smooth projective variety. We say that X is:*

- *weak Fano if $-K_X$ is nef and big,*
- *log Fano if there exists an effective divisor D such that $-(K_X + D)$ is ample and the pair (X, D) is Kawamata log terminal. In particular if $D = 0$ we have terminal Fano varieties,*
- *weak log Fano if there exists an effective divisor D such that $-(K_X + D)$ is neg and big, and the pair (X, D) is Kawamata log terminal.*

For instance, any toric variety is log Fano, a smooth hypersurface $X \subset \mathbb{P}^n$ of degree d is log Fano if and only if $d \leq n$.
If X is a normal $\mathbb{Q}$ projective variety with $\rho(X) = 1$ then X is a Mori Dream Space if and only if $\mathrm{Pic}(X)$ is finitely generated. For instance, the only Mori Dream Space of dimension one is $\mathbb{P}^1$.
The bridge between Mori Dream Spaces and log Fano varieties is the content of the following proposition.

PROPOSITION 1.2. [**BCHM**, Corollary 1.3.2] *Let X be a smooth projective variety. If X is log Fano then X is a Mori Dream Space .*

REMARK 1.3. On the other hand a Mori Dream Space is not necessarily log Fano. Indeed, by Grothendieck-Lefschetz theorem if $X \subset \mathbb{P}^n$ is a general hypersurface and $n \geq 4$ then $\mathrm{Pic}(X) \cong \mathbb{Z}$ is generated by $X \cap H$ where H is a general hyperplane in $\mathbb{P}^n$. Therefore, X is a Mori Dream Space. On the other hand, if $d = \deg(X)$ then X is not rationally connected as soon as $d \geq n + 1$. In particular if $d \geq n + 1$ the hypersurface X is not log Fano.
By Noether-Lefschetz theorem we have $\mathrm{Pic}(S_d) \cong \mathbb{Z}$ and generated by the restriction of the hyperplane section of $\mathbb{P}^3$ for a general surface of degree $d \geq 4$ in $\mathbb{P}^3$. These give other examples of Mori Dream Spaces that are not log Fano.
Even when X is a Mori Dream Space with big and movable anti-canonical divisor it is not necessarily log Fano. Indeed we have the following:

PROPOSITION 1.4. [**CG**, Proposition 2.6] *Let X be a projective $\mathbb{Q}$-factorial variety which is a Mori Dream Space, and let $L_1, ..., L_m$ be ample line bundles on X. Then*

$$Y = \mathbb{P}(\bigoplus_{i=1}^{m} L_i)$$

is a Mori Dream Space.

Now, following [**CG**, Example 5.1] we consider a smooth projective variety X of general type such that $H^1(X, \mathcal{O}_X) = 0$ and $\rho(X) = 1$. Let $\mathcal{E} = L_1 \oplus L_2 \oplus (\omega_X^\vee \oplus L_1^\vee \oplus L_2^\vee)$, and $Y = \mathbb{P}(\mathcal{E})$. Then $-K_Y$ is big and movable. On the other hand if Y would be rationally connected then X would be rationally connected as well. A contradiction because X is of general type. Therefore Y is not rationally connected and in particular it is not log Fano.

The following is an important result in order to achieve, among other things, an useful characterization of big divisors.

LEMMA 1.5. *(Kodaira's Lemma) Let D and E be respectively a big and an effective Cartier divisor on a projective variety X. Then*

$$H^0(X, mD - E) \neq 0$$

for $m \gg 0$.

PROOF. Since D is big there exists a constant $c > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq c \cdot m^{\dim(X)}$ for $m \gg 0$. On the other hand $\dim(E) = \dim(X) - 1$ implies that $h^0(X, \mathcal{O}_E(mD))$ grows at most like $m^{\dim(X)-1}$, and $h^0(X, \mathcal{O}_X(mD)) > h^0(X, \mathcal{O}_E(mD))$ for $m \gg 0$.
Now, let us consider the following exact sequence:

$$0 \mapsto \mathcal{O}_X(mD - E) \to \mathcal{O}_X(mD) \to \mathcal{O}_E(mD) \mapsto 0.$$

By taking cohomology we get

$$h^0(X, \mathcal{O}_X(mD - E)) \geq h^0(X, \mathcal{O}_X(mD)) - h^0(X, \mathcal{O}_E(mD)) > 0$$

for $m \gg 0$. $\square$

LEMMA 1.6. *Let D be a divisor on an irreducible projective variety X then D is big if and only if for any integer ample divisor A on X there exist an integer m and an effective divisor E such that $mD \sim_{lin} A + E$.*

PROOF. Assume that D is big and consider $mD - rA$ with $r \gg 0$. Then rA and $(r-1)A$ are both effective and by Lemma 1.5 we get $H^0(X, mD - rA) \neq 0$. Therefore, there exists an effective divisor E such that $mD - rA \sim_{lin} E$. That is

$$mD \sim_{lin} A + (r-1)A + E = A + E^{'}$$

where $E^{'} = (r-1)A + E$ is effective.
Now, let $mD \sim_{lin} A + E$ with A ample and E effective. Therefore, possibly passing to an higher multiple, we have $r \cdot mD \sim_{lin} rA + rE$ with $H = rA$ very ample, and rE effective. Then

$$\mathrm{kod}(X, D) \geq \mathrm{kod}(X, H) = \dim(X)$$

and D is big. $\square$

REMARK 1.7. Note that in the proof of Lemma we have to consider a multiple of A in order to have an effective divisor. To see this for instance consider three general points $p_1, p_2, p_3 \in C$ where C is a smooth curve of genus $g = 2$. The divisor $D = p_1 + p_2 - p_3$ is ample, indeed $\deg(5D) = 5 = 2g + 1$ and by [**Har**, Corollary 3.2] $5D$ is very ample. Then D is ample. Now, let us consider $D^{'} = p_1 + p_2$. Then $\deg(K_C - D^{'}) = 0$. If $h^0(K_C - D^{'}) \neq 0$ then $\deg(K_C - D^{'}) = 0$ yields $K_C - D^{'} \sim 0$ and $h^0(K_C - D^{'}) = 1$. On the other hand $h^0(K_C) = 2$, and since p_1, p_2 are general they impose independent conditions to the differential forms on C, that is $h^0(K_C - D^{'}) = 0$. By Riemann-Roch this gives $h^0(p_1 + p_2) = 1$. Now, assume that $h^0(p_1 + p_2 - p_3) \neq 0$. The inclusion $H^0(C, p_1 + p_2 - p_3) \subseteq H^0(C, p_1 + p_2)$ forces $H^0(C, p_1 + p_2 - p_3) = H^0(C, p_1 + p_2)$, that is any global section $s \in H^0(C, p_1 + p_2) \cong k$ vanishes at p_3. Therefore s is zero because it is constant. This implies $h^0(p_1 + p_2) = 0$, a contradiction. We conclude that $H^0(C, p_1 + p_2 - p_3) = 0$, that is there is no effective divisor on C linearly equivalent to $p_1 + p_2 - p_3$.

LEMMA 1.8. *Let D be a nef and big divisor on an irreducible projective variety X. Then there exist an effective divisor E such that $D - \epsilon E$ is ample for* $0 < \epsilon \ll 1$.

PROOF. Let D be a nef and big divisor. Since D is big, by Lemma 1.7, there exist an ample divisor A, an effective divisor E, and a positive integer k such that $kD \equiv A + E$. If $h > k$ we can write $hD \equiv (h-k)D + A + E$. The divisor $D^{'} = (h-k)D + A$ is a sum of a nef and an ample divisor. Therefore $D^{'}$ is ample. If $\epsilon = \frac{1}{h}$ we get that

$$D - \epsilon E \equiv \epsilon D^{'}$$

is ample. □

PROPOSITION 1.9. *Let X be normal, irreducible, projective variety with at most klt singularities. If X is weak Fano then X is log Fano.*

PROOF. Since X is weak Fano $-K_X$ is nef and big. By Lemma 1.8 there exists an effective divisor D and a rational number $0 < \epsilon \ll 1$ such that $-K_X - \epsilon D = -(K_X + \epsilon D)$ is ample. The pair $(X, \epsilon D)$ is klt for $\epsilon \ll 1$ because X has at most klt singularities. □

REMARK 1.10. The converse of Proposition 1.9 is false. For instance the Hirzebruch surface $X_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$ is a toric surface and hence log Fano. The anti-canonical divisor is $-K_{X_e} = 2C_0 + (2+e)F$, where C_0 is the section and F is the fiber. Therefore $-K_{X_e} \cdot C_0 = 2C_0^2 + 2 + e = -e + 2$, and $-K_{X_e}$ is not nef for $e > 2$. We conclude that for any $e > 2$ the Hirzebruch surface X_e is log Fano but not weak Fano.

It is quite easy to see that projective toric varieties are log Fano.

LEMMA 1.11. *Let $D = \sum_i d_i D_i$ be a $\mathbb{Q}$-divisor on a normal projective variety X such that $d_i < 1$ and the pair $(X, \lceil D \rceil)$ is lc. Then (X, D) is klt.*

PROOF. Let $f : Y \to X$ be a log resolution of the pair $(X, \lceil D \rceil)$. We have

$$K_Y = f^*K_X + \sum_i a_i E_i$$

and

$$\lceil \widetilde{D} \rceil = f^*\lceil D \rceil - \sum_i b_i E_i$$

where $\lceil\widetilde{D}\rceil$ is the strict transform of $\lceil D\rceil$. Therefore,

$$K_Y = f^*(K_X + \lceil D\rceil) + \sum_i (a_i - b_i) - \lceil\widetilde{D}\rceil$$

and since $(X, \lceil D\rceil)$ is lc we have $a_i - b_1 \geq -1$. On the other hand

$$\widetilde{D} = f^*D - \sum_i t_i E_i$$

with $t_i < b_i$ because $d_i < 1$ for any i. This yields $a_i - t_i > a_i - b_i \geq -1$, and the pair (X, D) is klt. □

PROPOSITION 1.12. *Let X be a projective toric variety. Then X is log Fano.*

PROOF. Let $D_1^X, ..., D_r^X$ be the irreducible toric invariant divisors on X. Then we have $K_X = -\sum_i D_i^X$, see **[Ful]**. Now, let $A = \sum_i a_i D_i^X$ be an ample toric invariant divisor, and ϵ a rational number $0 < \epsilon \ll 1$. Therefore

$$-K_X - \epsilon A = \sum_i (1 - \epsilon a_i) D_i^X$$

with $1 - \epsilon a_i < 1$. The divisor $D = \sum_i (1 - \epsilon a_i) D_i^X$ is such that $\epsilon A = -K_X - D$ is ample. Note that $\lceil D\rceil \sim -K_X$. Let $f : Y \to X$ be a toric log resolution of $(X, \lceil D\rceil)$, and let $D_1^Y, ..., D_h^Y$ be the invariant toric divisors on Y. We have

$$K_Y = f^*(K_X + \lceil D\rceil) + \sum a_i E_i - \lceil\widetilde{D}\rceil = \sum a_i E_i - \lceil\widetilde{D}\rceil$$

because $\lceil D\rceil \sim -K_X$. On the other hand $K_Y = -\sum_i D_i^Y$ yields

$$K_Y = \sum a_i E_i - \lceil\widetilde{D}\rceil = -\sum_i D_i^Y.$$

This forces $a_i = -1$ for any i. Therefore, the pair $(X, \lceil D\rceil)$ is lc. To conclude it is enough to apply Lemma 1.11. □

It turns out that weak log Fano is equivalent to log Fano.

PROPOSITION 1.13. *Let X be a projective variety with at most klt singularities. Then X is log Fano if and only if X is weak log Fano.*

PROOF. Clearly X log Fano implies X weak log Fano. Now, let X be weak log Fano. Then there exists an effective divisor D such that $-K_X - D$ is big and nef and (X, D) is klt. By Lemma 1.8 there exists an effective divisor E such that $-K_X - D - \epsilon E = -K_X - (D + \epsilon E)$ is ample for $0 < \epsilon \ll 1$.
Let $D' = D + \epsilon E$. Therefore, D' is effective and $-K_X - D'$ is ample. Furthermore, since X has at most klt singularities and (X, D) is klt we get that (X, D') is klt for $0 < \epsilon \ll 1$. □

Finally, we have two important facts about log Fano varieties. We will prove just the latter, for the first one we refer to **[GOST]**.

LEMMA 1.14. **[GOST**, Corollary 1.3**]** *Let $f : X \to Y$ be a projective surjective morphism between normal projective varieties over an algebraically closed field of characteristic zero. If X is log Fano then Y is log Fano.*

The second result says that being log Fano is preserved under small transformations.

LEMMA 1.15. *Let X and Y be normal varieties over a field of characteristic zero that are isomorphic in codimension one. Then X is log Fano if and only if Y is so.*

PROOF. There exists a small transformation $f : X \dashrightarrow Y$. Such a small transformation can be factored as $f = f_k \circ ... \circ f_1$ where any $f_i : X_i \dashrightarrow X_{i+1}$ is small, and fits in a diagram of the following form

$$\begin{tikzcd} X_i \arrow[rr, dashed, "f_i"] \arrow[dr, "g_i"'] & & X_{i+1} \arrow[dl, "r_i"] \\ & Z_i & \end{tikzcd}$$

where f_i is a small projective birational contraction. To conclude, we have to prove that if X and Y are normal varieties over a field of characteristic zero and $f : X \to Y$ is a small birational morphism the X is log Fano if and only if Y is log Fano.
Assume that X is log Fano. Then there exists D effective such that $-K_X - D$ is ample and (X, D) is klt. Let us take an ample divisor H on Y such that $-K_X - D - \epsilon f^*H$ is ample and $(X, D + \epsilon f^*H)$ is klt. Note that since f is small $f_*(D + \epsilon f^*H)$ may not be $\mathbb{Q}$-Cartier. To deal with this we need the following trick. We take an ample divisor A on X such that $(X, D + \epsilon f^*H + A)$ is klt and

$$K_X + D + \epsilon f^*H + A \sim_{\mathbb{Q}} 0.$$

Therefore,

$$K_Y + f_*D + \epsilon H + f_*A = f_*(K_X + D + \epsilon f^*H + A) \sim_{\mathbb{Q}} 0.$$

Now, since f is small we have

$$f^*(K_Y + f^*D + \epsilon H + f_*A) = K_X + D + \epsilon f^*H + A.$$

We conclude that $(Y, f_*D + f_*A)$ is klt and $-(K_Y + f_*D + f_*A) \sim_{\mathbb{Q}} \epsilon H$ is ample.
Now, let us assume that Y is log Fano, and let D an effective divisor on Y such that $-K_Y - D$ is ample and (Y, D) is klt. Let $\widetilde{D}$ be the strict transform of D in X. Since f is small we have

$$K_X + \widetilde{D} = f^*(K_Y + D).$$

Therefore, $(X, \widetilde{D})$ is klt and $-K_X - \widetilde{D}$ is nef and big. This means that X is weak log Fano, and by Proposition 1.13 it is log Fano. □

CHAPTER 4

Blow-ups of $\mathbb{P}^n$ in k general points

The aim of this chapter is to prove the following result:

THEOREM 0.1. *Let X_k^n be a blow-up of $\mathbb{P}^n$ at k points in general position, with $n \geq 2$ and $k \geq 0$. Then X_k^n is log Fano if and only if one of the following holds:*
- *$n = 2$ and $k \leq 8$,*
- *$n = 3$ and $k \leq 7$,*
- *$n = 4$ and $k \leq 8$,*
- *$n > 4$ and $k \leq n + 3$.*

1. Root systems

Let V be an Euclidean space over a field k with inner product $\langle -, - \rangle : V \times V \to k$. For any non-zero vector $w \in V$ we may consider its orthogonal hyperplane

$$H_w = w^\perp = \{v \in V \mid \langle v, w \rangle = 0\}.$$

Let us consider the following map:

$$\begin{aligned} R_w : \quad & V \longrightarrow V \\ & v \mapsto v - 2\tfrac{\langle v,w \rangle}{\langle w,w \rangle} w \end{aligned} \tag{1.1}$$

Note that $R_w(w) = -w$ and $R_w(v) = v$ for any $v \in H_w$. Therefore, R_w is the reflection with respect to the hyperplane $H_w = w^\perp$.

DEFINITION 1.1. *A root system in V is a finite set $\mathfrak{R}$ of non-zero vectors of V such that:*
- *the vectors in $\mathfrak{R}$ generate V,*
- *if $v, \lambda v \in \mathfrak{R}$ then $\lambda = \pm 1$,*
- *for any $w \in \mathfrak{R}$ we have $R_w(\mathfrak{R}) \subseteq \mathfrak{R}$,*
- *for any $v, w \in \mathfrak{R}$ the projection of w onto the line generated by v is a half-integer multiple of v, that is $2\frac{\langle v,w \rangle}{\langle v,v \rangle} \in \mathbb{Z}$.*

The vectors in $\mathfrak{R}$ are the roots of the root system. The root lattice of a root system $\mathfrak{R}$ is the $\mathbb{Z}$-submodule of V generated by the roots of $\mathfrak{R}$.

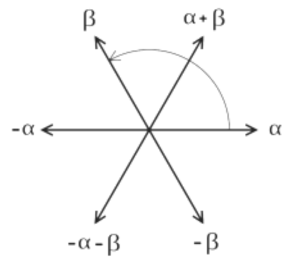
Let $\mathfrak{R} = \{r_1, ..., r_h\} \subset V$. The subgroup $\mathcal{W}_{\mathfrak{R}}$ of the group of isometries of V generate by $R_{r_1}, ..., R_{r_h}$ is the *Weyl group* of $\mathfrak{R}$.

DEFINITION 1.2. *Let $\mathfrak{R} \subset V$ be a root system. A subset $\mathfrak{S} \subseteq \mathfrak{R}$ is a set of simple roots in $\mathfrak{R}$ if*
- *the elements in $\mathfrak{S}$ form a basis of V,*
- *any $v \in \mathfrak{R}$ can be written as a linear combination of elements of $\mathfrak{S}$ with integer coefficients all of the same sing.*

The root $v \in \mathfrak{R}$ is positive if all the coefficients are nonnegative. The set of positive root is denoted by $\mathfrak{R}^+$. The vectors in $\mathfrak{R}^- = \mathfrak{R} \setminus \mathfrak{R}^+$ are called negative roots.

We can associate to a root system a graphs, called the *Dynkin diagram* of the root system. Given a root system $\mathfrak{R}$, we choose a set $\mathfrak{S}$ of simple roots. The vertices of the associated Dynkin diagram correspond to vectors in $\mathfrak{S}$. Any non-orthogonal pair of vectors is connected by an edge. This edge is an undirected single edge if they make an angle of $\frac{2}{3}\pi$ radians, a directed double edge if they make an angle of $\frac{3}{4}\pi$ radians, and a directed triple edge if they make an angle of $\frac{5}{6}\pi$ radians. Where "directed edge" means that double and triple edges are marked with an angle sign pointing toward the shorter vector.

EXAMPLE 1.3. The following is a representation of the rank two root system $A_2 = \{\alpha, \beta, \alpha + \beta, -\alpha, -\beta, -\alpha - \beta\}$



We may choose $A_2^+ = \{\alpha, \beta, \alpha + \beta\}$. Since α and β are simple the Dynkin diagram associated to A_2 is the following:

$$\circ \text{———} \circ$$

Note that the Dynking diagram A_2 is exactly the dual graph of the cyclic quotient singularity of type A_2 given by $\{y_0^2 + y_1^2 + y_2^3 = 0\}$.

1.1. Representations of semi-simple Lie algebras. A complex *Lie algebra* is a $\mathbb{C}$-vector space $\mathfrak{g}$ with a binary operation $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the *Lie bracket* such that:

- the Lie bracket is bilinear,
- $[g, g] = 0$ for any $g \in \mathfrak{g}$,
- the Lie bracket satisfies the *Jacobi identity*
$$[g_1, [g_2, g_3]] + [g_3, [g_1, g_2]] + [g_2, [g_3, g_1]] = 0$$
for any $g_1, g_2, g_3 \in \mathfrak{g}$.

A *simple* Lie algebra is a non-abelian Lie algebra that does not have non-trivial ideals. A direct sum of simple Lie algebras is called a *semi-simple* Lie algebra.
Now, let $\mathfrak{g}$ be a semi-simple Lie algebra and let $\mathfrak{h} \subseteq \mathfrak{g}$ be a *Cartan subalgebra* that is a subalebra which is maximal among abelian, diagonalizable subalgebras.
For any $g \in \mathfrak{g}$ we may consider the endomorphism

$$\begin{aligned} ad_g : \quad & \mathfrak{g} \longrightarrow \mathfrak{g} \\ & x \mapsto [g, x] \end{aligned}$$

The linear map

$$\begin{aligned} ad : \quad & \mathfrak{g} \longrightarrow \mathrm{End}(\mathfrak{g}) \\ & g \longmapsto ad_g \end{aligned}$$

is called the *adjoint representation* of $\mathfrak{g}$.
Now, we consider the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$. Since this action is diagonalizable we get a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_r \mathfrak{g}_r$$

called *Cartan decomposition*. The elements $r \in \mathfrak{h}^*$ are the eigenvalues of the action and for any $H \in \mathfrak{h}$, $X \in \mathfrak{g}_r$ we have $ad(H)(X) = r(H)X$. The eigenvalues $r \in \mathfrak{h}^*$ are called the roots of the Lie algebra $\mathfrak{g}$, we denote by $\mathfrak{R}(\mathfrak{g}) \subset \mathfrak{h}^*$ the set of all roots of $\mathfrak{g}$. In the following we concentrate on the adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$. For details on the general theory see [**FH**, Lecture 14].
Adjoint representation of $\mathfrak{sl}_3(\mathbb{C})$. We consider the Lie algebra $\mathfrak{sl}_3(\mathbb{C})$ of traceless 3×3 matrices:

$$sl_3(\mathbb{C}) = \{X \in M_3(\mathcal{C}) \mid \operatorname{tr}(X) = 0\}.$$

We have $\dim(sl_3(\mathbb{C})) = 8$, and we consider the bases of $sl_3(\mathbb{C})$ given by:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the matrices $E_{i,j}$ for $1 \leq i \neq j \leq 3$ having the entry (i,j) equal to 1 and all the other entries equal to zero. Now, let

$$\mathfrak{h} = \{X = \{x_{i,j}\} \in \mathfrak{sl}_3(\mathbb{C}) \mid x_{i,j} = 0, \forall\, i \neq j\}$$

be the Cartan subalgebra of diagonal traceless matrices. Note that $\dim(\mathfrak{h}) = 2$ and $\mathfrak{h} = \langle H_1, H_2 \rangle$. The linear functionals $L_i : \mathfrak{h} \to \mathbb{C}$ given by

$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i$$

for $i = 1, 2, 3$ form a basis of $\mathfrak{h}^*$. For any

$$H = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \in \mathfrak{h}$$

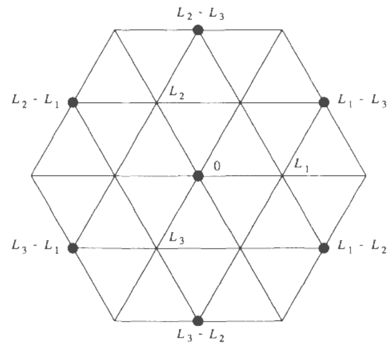
we have

$$ad(H)(E_{i,j}) = ad_H(E_{i,j}) = [H, E_{i,j}] = (a_i - a_j)E_{i,j}.$$

Therefore, we have the six eigenvalues $L_1 - L_2, L_1 - L_3, L_2 - L_3, L_3 - L_2, L_3 - L_1, L_2 - L_1$ for the adjoint action of $\mathfrak{h}$, and the eigenspace corresponding to $L_i - L_j$ is the 1-dimensional subspace $\mathfrak{sl}_3(\mathbb{C})_{L_i - L_j} = \langle E_{i,j} \rangle$. Since $\dim(\mathfrak{sl}_3(\mathbb{C}))$ we get

$$\mathfrak{sl}_3(\mathbb{C}) = \mathfrak{h} \oplus \bigoplus_{1 \leq i \neq j \leq 3} \mathfrak{sl}_3(\mathbb{C})_{L_i - L_j}.$$

The roots $L_i - L_j$ are represented as follows:



Note that the $L_i - L_j$ are exactly the roots of the root system A_2 in Example 1.3. The Weyl group of A_2 acts as the symmetric group S_3 on the generators L_1, L_2, L_3 of $\mathfrak{h}^*$. The example of $\mathfrak{sl}_3(\mathbb{C})$ reflects a general phenomenon. The classification of semi-simple Lie algebras proceeds by considering a Cartan subalgebra and the adjoint action of the Lie algebra on this subalgebra. The root system of the action determines the Lie algebra and the corresponding Dynkin diagram.

2. Intersection theory of a blow-up

Let X be a smooth projective variety, and let $i : Z \hookrightarrow X$ be a smooth subvariety. Let $\pi : Y = Bl_Z X \to X$ be the blow-up of X along Z with exceptional divisors $j : E \hookrightarrow Y$. We have the following commutative diagram:

$$\begin{tikzcd} E \arrow[r, "j"] \arrow[d, "\pi_E"'] & Y = Bl_Z X \arrow[d, "\pi"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

We have $E = \mathbb{P}(N_{Z/X})$, and let $\xi = c_1(\mathcal{O}_E(1)) \in A^1(E)$. Furthermore, $N_{E/Y} = \mathcal{O}_E(-1)$, so that $c_1(N_{E/Y}) = -c_1(\mathcal{O}_E(1)) = -\xi$.

PROPOSITION 2.1. [**EH**, Proposition 15.10] *The Chow ring $A(Y)$ of $Y = Bl_Z X$ is generated by $\pi^* A(X)$ and $j_* A(E)$ with the following multiplication rules:*

$$\begin{array}{rcll} \pi^*\alpha \cdot \pi^*\beta & = & \pi^*(\alpha \cdot \beta) & for\ \alpha, \beta \in A(X), \\ \pi^*\alpha \cdot j_*\gamma & = & j_*(\gamma \cdot \pi^*_{|E} i^*\alpha) & for\ \alpha \in A(X),\ \gamma \in A(E), \\ j_*\gamma \cdot j_*\delta & = & -j_*(\gamma \cdot \delta \cdot \xi) & for\ \gamma, \delta \in A(E). \end{array}$$

EXAMPLE 2.2. Let us take $X = \mathbb{P}^n$ and $Z = p \in \mathbb{P}^n$ a point. The groups $A^0(Y)$ and $A^n(Y)$ are both isomorphic to $\mathbb{Z}$, generated respectively by the fundamental class of Y and the class of a point. The group $A^1(X)$ is generated by $\widetilde{H}$ and E while $A^{n-1}(Y)$ is generated by the pull-back $\widetilde{L}$ of the class of a line of $\mathbb{P}^n$, and by the class of a line R in the exceptional divisor $E \cong \mathbb{P}^{n-1}$.
In this case $\mathcal{O}_E(1) = \mathcal{O}_{\mathbb{P}^{n-1}}(1)$ and $\xi = H$, where H is the hyperplane section of $E \cong \mathbb{P}^{n-1}$. For instance, we get $E^k = (-1)^{k-1} H^{k-1}$, and in particular:

$$E^n = (-1)^{n-1} H^{n-1} = (-1)^{n-1}.$$

EXAMPLE 2.3. Let us consider the case $X = \mathbb{P}^3$ and $Z = C \subset \mathbb{P}^3$ a smooth curve of degree d and genus g. Let $H \in A^1(\mathbb{P}^3)$ be the class of an hyperplanes, and $L = H^2 \in A^2(\mathbb{P}^3)$ the class of a line. We denote by $\widetilde{H}$ and $\widetilde{L}$ their pull-backs in Y. For any divisor $D \in Z^1(C)$ let F_D be the corresponding linear combination of the fibers of $\pi_{|E} : E \to C$.
Clearly, $A^0(Y) \cong \mathbb{Z}$, and $A^3(Y) \cong \mathbb{Z}$, generated by the fundamental class of Y and the class of point respectively.
Now, $A^1(Y)$ is generated by $\widetilde{H}$ and E. Furthermore, $A^2(Y)$ is generated by $\widetilde{L}$, $j_*\xi$, and j_*F_D for $D \in A^1(C)$. Note that geometrically the class $j_*\xi$ corresponds to a curve in $E \cong C \times \mathbb{P}^1$ that is mapped by π isomorphically to C. By Proposition 2.1 in $A^1(Y)$ we have:

$$\widetilde{H}^2 = \widetilde{L}^2, \quad ,\widetilde{H} \cdot E = j_*(E \cdot F_H) = j_*(F_H), \quad E^2 = -j_*(E \cdot E \cdot \xi) = -j_*\xi$$

where H is the the hyperplane section of C. The pairing between $A^1(Y)$ and $A^2(Y)$ is given by

$$\widetilde{H} \cdot \widetilde{L} = 1, \quad \widetilde{H}^2 \cdot j_*F_D = 0, \quad \widetilde{H} \cdot j_*\xi = H \cdot C = d.$$

Furthermore,

$$E \cdot \widetilde{L} = 0, \quad E \cdot j_*F_D = -j_*(E \cdot F_D \cdot \xi) = -\deg(D), \quad E \cdot j_*\xi = -j_*\xi = -c_1(N_{C/\mathbb{P}^3}).$$

Now, let us consider the exact sequence

$$0 \mapsto T_C \to T_{\mathbb{P}^3|C} \to N_{C/\mathbb{P}^3} \mapsto 0.$$

For the Chern polynomials we have $c_t(T_{\mathbb{P}^3|C}) = c_t(T_C) \cdot c_t(N_{C/\mathbb{P}^3})$. Now, by the Euler's sequence

$$0 \mapsto \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_{\mathbb{P}^3}(1)^{\oplus 4} \to T_{\mathbb{P}^3} \mapsto 0$$

we get

$$c_t(T_{\mathbb{P}^3}) = 1 + 4ht + 6h^2t^2 + 4h^3t^3 + h^4t^4.$$

Since $c_t(T_C) = 1 + c_1(T_C)t = 1 + (2-2g)t$ we have

$$(1 + (2-2g)t) \cdot (1 + c_1(N_{C/\mathbb{P}^3})t + c_2(N_{C/\mathbb{P}^3})t^2) = 1 + 4h_{|C}t + 6h_{|C}^2t^2 + 4h_{|C}^3t^3 + h_{|C}^4t^4 = 1 + 4h_{|C}t$$

and since $h_{|C} = \deg(C) = d$ we get

$$1 + (c_1(N_{C/\mathbb{P}^3}) + 2 - 2g)t + (c_2(N_{C/\mathbb{P}^3}) + c_1(N_{C/\mathbb{P}^3})(2-2g))t^2 + c_2(N_{C/\mathbb{P}^3})(2-2g)t^3 = 1 + 4dt.$$

This yields

$$E \cdot j_*\xi = -c_1(N_{C/\mathbb{P}^3}) = -4d - 2g + 2.$$

Finally,

$$\widetilde{H}^3 = 1, \quad \widetilde{H}^2 \cdot E = 0, \quad \widetilde{H} \cdot E^2 = -\widetilde{H} \cdot j_*\xi = -d, \quad E^3 = -j_*E \cdot j_*\xi = j_*(E \cdot \xi) = -4d - 2g + 2.$$

3. The standard Cremona transformation of $\mathbb{P}^n$

Let $p_1, ..., p_{n+1} \in \mathbb{P}^n$ be general points. We may assume

$$p_1 = [1 : 0 : ... : 0], ..., p_{n+1} = [0 : ... : 0 : 1].$$

We consider the standard Cremona transformation:

$$\begin{array}{rcl} \psi : \mathbb{P}^n & \dashrightarrow & \mathbb{P}^n \\ [x_0 : ... : x_n] & \longmapsto & [\frac{1}{x_0} : ... : \frac{1}{x_n}] \end{array}$$

Note that $\psi\circ\psi = Id_{\mathbb{P}^n}$, and $\psi^{-1}=\psi$. Let $H_1,...,H_{n+1}$ be the coordinate hyperplanes of $\mathbb{P}^n$. Then ψ is not defined on the locus

$$\bigcup_{1\leq i<j\leq n+1} H_i\cap H_j.$$

Furthermore, ψ is an isomorphism off of the union

$$\bigcup_{1\leq i\leq n+1} H_i.$$

Now, ψ induces a birational transformation $\widetilde{\psi}: X^n_{n+1} \dashrightarrow X^n_{n+1}$ and we have the following commutative diagram:

$$\begin{tikzcd} X^n_{n+1} \arrow[r, dashed, "\widetilde{\psi}"] \arrow[d] & X^n_{n+1} \arrow[d] \\ \mathbb{P}^n \arrow[r, dashed, "\psi"] & \mathbb{P}^n \end{tikzcd}$$

Note that, since ψ contracts the hyperplane H_i passing spanned by the n points $p_1,...,\hat{p}_i,...,p_{n+1}$ to the point p_i, the map $\widetilde{\psi}$ maps the strict transform of H_i onto the exceptional divisor E_i. Therefore $\widetilde{\psi}$ is an isomorphism in codimension one. Indeed, it is a composition of flops. In particular $\widetilde{\psi}$ induces an isomorphism $\mathrm{Pic}(X^n_{n+1})\to\mathrm{Pic}(X^n_{n+1})$.
Now, the linear system on $\mathbb{P}^n$ associated to the standard Cremona ψ is

$$\mathcal{H} = \mathcal{O}_{\mathbb{P}^n}(n)\otimes\mathcal{I}_{(n-1)(p_1+...+p_{n+1})},$$

that is $\mathcal{H}$ is the linear system of hypersurfaces in $\mathbb{P}^n$ of degree n having points of multiplicity at least $n-1$ in $p_1,...,p_{n+1}$. Therefore, the inverse image of a general hyperplane of $\mathbb{P}^n$ via ψ is an hypersurface of degree n with points of multiplicity $n-1$ in $p_1,...,p_{n+1}$, and

$$\widetilde{\psi}^*H = nH-(n-1)(E_1+...+E_{n+1}).$$

Furthermore, since ψ contracts the hyperplane H_i passing spanned by the n points $p_1,...,\hat{p}_i,...,p_{n+1}$ to the point p_i we have

$$\widetilde{\psi}^*E_i = H-E_1-...-\hat{E}_i-...-E_{n+1}.$$

We conclude that the simple reflection R_{α_k} with respect to $\alpha_k = H-E_1-...-E_{n+1}$ is realized by the small transformation $\widetilde{\psi}$.

PROPOSITION 3.1. *Let $D\subset\mathbb{P}^n$ be an hypersurface of degree d having points of multiplicities $m_1,...,m_{n+1}$ in $p_1,...,p_{n+1}$, and let $\psi:\mathbb{P}^n\dashrightarrow\mathbb{P}^n$ be the standard Cremona of $\mathbb{P}^n$. Then*

$$\deg(\psi(D)) = dn-\sum_{i=1}^{n+1} m_i$$

and

$$\mathrm{mult}_{p_i}\psi(D) = d(n-1)-\sum_{j\neq i} m_j$$

for any $i=1,...,n+1$.

PROOF. Let $X^n_{n+1} = Bl_{p_1,...,p_{n+1}}\mathbb{P}^n$, and $\widetilde{\psi} : X^n_{n+1} \dashrightarrow X^n_{n+1}$ be the birational map induced by ψ. The strict transform of D in $X^n_{n+1} \dashrightarrow X^n_{n+1}$ can be written as $\widetilde{D} \cong dH - \sum_{i=1}^{n+1} m_i E_i$.
Now, since $\widetilde{\psi}_* H = nH - \sum_{i=1}^{n+1}(n-1)E_i$, and $\widetilde{\psi}_* E_i = H - \sum_{j \neq i} E_i$ we get

$$\begin{aligned}\widetilde{\psi}_* D = \; & d(nH - \textstyle\sum_{i=1}^{n+1} E_i) - \sum_{i=1}^{n+1} m_i(H - \sum_{j\neq i} E_j) = \\ & dnH - d\textstyle\sum_{i=1}^{n+1}(n-1)E_i - \sum_{i=1}^{n+1} H + \sum_{i=1}^{n+1} m_i \sum_{j\neq i} E_j = \\ & (dn - \textstyle\sum_{i=1}^{n+1} m_i)H - \sum_{i=1}^{n+1}(d(n-1) - \sum_{j\neq i} m_j)E_j.\end{aligned}$$

□

Let $p_1, ..., p_k \in \mathbb{P}^n$ be general points with $k > n$, and let $X^n_k = Bl_{p_1,...,p_k}\mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ in $p_1, ..., p_k$. The Picard group $\mathrm{Pic}(X^n_k)$ is a free $\mathbb{Z}$-module of rank $k+1$. Let H be the pull-back of the hyperplane class of $\mathbb{P}^n$, and E_i be the class of the exceptional divisor over p_i. Then

$$H, E_1, ..., E_k,$$

is a basis of $\mathrm{Pic}(X^n_k)$. The anti-canonical class of X^n_k is given by

$$-K_{X^n_k} = (n+1)H - (n-1)(E_1 + ... + E_k).$$

In [**Mu1**] *S. Mukai* defines the following symmetric bilinear form on $\mathrm{Pic}(X^n_k)$:

$$\langle H, E_i\rangle = 0, \quad \langle H, H\rangle = n-1, \quad \left\langle E_i, E_j\right\rangle = -\delta_{i,j}. \tag{3.1}$$

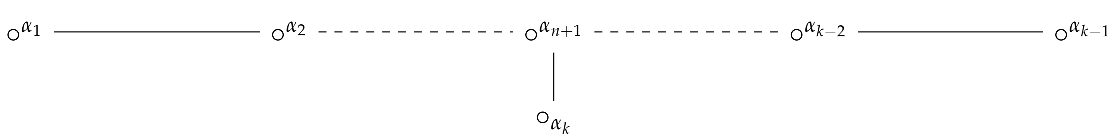
A straightforward computation, see [**Mu1**], shows that $\mathrm{Pic}(X^n_k)$ has another $\mathbb{Z}$-basis $\alpha_1, ..., \alpha_k, E_k$, where

$$\begin{aligned}&\alpha_1 = E_1 - E_2,\\ &\vdots\\ &\alpha_i = E_i - E_{i+1},\\ &\vdots\\ &\alpha_{k-1} = E_{k-1} - E_k,\\ &\alpha_k = H - E_1 - ... - E_{n+1}.\end{aligned}$$

Furthermore, $\alpha_1, ..., \alpha_k$ is a $\mathbb{Z}$-basis of the orthogonal complement $K^{\perp}_{X^n_k}$ of $K_{X^n_k}$ with respect to (3.1). For instance, we have

$$\left\langle K_{X^n_k}, \alpha_k \right\rangle = (n+1)\langle H, H\rangle - (n-1)\sum_{i=1}^{n+1}\langle E_i, E_i\rangle = (n+1)(n-1) - (n-1)(n+1) = 0.$$

Moreover, $\alpha_1, ..., \alpha_k$ is a system of simple roots of a finite root system with Dynkin diagram $T_{2,k-n-1,n+1}$:



Let $\mathcal{W}$ be the Weyl group of orthogonal reflection with respect to $\alpha_1, ..., \alpha_k$. Clearly $K_{X^n_k}$ is $\mathcal{W}$-invariant. Following Mukai [**Mu1**] we give the following definition.

DEFINITION 3.2. *Let X be normal $\mathbb{Q}$-factorial variety. A (-1)-divisor in X is a divisor $D \subset X$ such that there exists a small $\mathbb{Q}$-factorial transformation $f : X \dashrightarrow Y$ and a morphism $\pi : Y \to Z$ where π is the blow-up of a projective variety Z in a smooth point and D is the strict transform via f of the exceptional divisor of π.*

Now, our aim is to prove that any transformation $w \in \mathcal{W}$ is induced by a small $\mathbb{Q}$-factorial transformation of X_k^n.

THEOREM 3.3. [**Mu1**, Theorem 1] *For any transformation $w : \mathrm{Pic}(X_k^n) \to \mathrm{Pic}(X_k^n)$ in $\mathcal{W}$ there exists a small $\mathbb{Q}$-factorial transformation $f_w : X_k^n \dashrightarrow X$ such that X is also a blow-up of $\mathbb{P}^n$ in k general points, and the pull-back via f_w of the tautological basis of X coincides with the transformation of the tautological basis of X_k^n by w.*

PROOF. It is enough to prove the theorem for simple reflection. Note that a simple reflection with respect to $\alpha_i = E_i - E_{i+1}$ corresponds to a transposition of a pair of centers. Indeed by 1.1 we have

$$R_{\alpha_i}(H) = H - 2\frac{\langle H, E_i - E_{i+1}\rangle}{\langle E_i - E_{i+1}, E_i - E_{i+1}\rangle}(E_i - E_{i+1}) = H,$$

for any $k \neq i, i+1$ we have

$$R_{\alpha_i}(E_k) = E_k - 2\frac{\langle E_k, E_i - E_{i+1}\rangle}{\langle E_i - E_{i+1}, E_i - E_{i+1}\rangle}(E_i - E_{i+1}) = E_k,$$

furthermore

$$R_{\alpha_i}(E_i) = E_i - 2\frac{\langle E_i, E_i - E_{i+1}\rangle}{\langle E_i - E_{i+1}, E_i - E_{i+1}\rangle}(E_i - E_{i+1}) = E_i - (E_i - E_{i+1}) = E_{i+1},$$

and finally

$$R_{\alpha_i}(E_{i+1}) = E_{i+1} - 2\frac{\langle E_{i+1}, E_i - E_{i+1}\rangle}{\langle E_i - E_{i+1}, E_i - E_{i+1}\rangle}(E_i - E_{i+1}) = E_{i+1} - (-E_i + E_{i+1}) = E_i.$$

Therefore, the simple reflection with respect to α_i is realized by the lifting to X_k^n of an automorphism of $\mathbb{P}^n$ switching p_i and p_{i+1}, and fixing the p_j's with $j \neq i, i+1$.
For any $I \subset \{1, ..., k\}$ with $|I| = n+1$ we have that $\alpha_I = H - \sum_{i \in I} E_i$ is a root. The reflection R_I with respect to α_I is given by:

$$\left\{\begin{array}{lll} H \mapsto H + (n-1)\alpha_I = nH - (n-1)\sum_{i\in I} E_i, & & \\ E_i \mapsto E_i + \alpha_I & for & i \in I, \\ E_j \mapsto E_j & for & j \notin I. \end{array}\right. \tag{3.2}$$

In particular the simple reflection R_{α_k} with respect to $\alpha_k = H - E_1 - ... - E_{n+1}$ on the tautological basis of $\mathrm{Pic}(X_k^n)$ is given by:

$$\left\{\begin{array}{lll} H \mapsto H + (n-1)\alpha_I = nH - (n-1)\sum_{i=1}^{n+1} E_i, & & \\ E_i \mapsto H - E_1 - ... - \hat{E}_i - ... - E_{n+1} & for & 1 \leq i \leq n+1, \\ E_i \mapsto E_i & for & n+2 \leq i \leq k. \end{array}\right.$$

Let $p_1, ..., p_{n+1} \in \mathbb{P}^n$ be general points. We may assume

$$p_1 = [1 : 0 : ... : 0], ..., p_{n+1} = [0 : ... : 0 : 1].$$

We consider the standard Cremona transformation:

$$\begin{array}{rcl} \psi : \mathbb{P}^n & \dashrightarrow & \mathbb{P}^n \\ [x_0 : ... : x_n] & \longmapsto & [\frac{1}{x_0} : ... : \frac{1}{x_n}] \end{array}$$

Note that $\psi \circ \psi = Id_{\mathbb{P}^n}$, and $\psi^{-1} = \psi$. Let $H_1, ..., H_{n+1}$ be the coordinate hyperplanes of $\mathbb{P}^n$. Then ψ is not defined on the locus

$$\bigcup_{1 \leq i < j \leq n+1} H_i \cap H_j.$$

Furthermore, ψ is an isomorphism off of the union

$$\bigcup_{1 \leq i \leq n+1} H_i.$$

Now, ψ induces a birational transformation $\widetilde{\psi} : X^n_{n+1} \dashrightarrow X^n_{n+1}$ and we have the following commutative diagram:

$$\begin{tikzcd} X^n_{n+1} \arrow[r, dashed, "\widetilde{\psi}"] \arrow[d] & X^n_{n+1} \arrow[d] \\ \mathbb{P}^n \arrow[r, dashed, "\psi"] & \mathbb{P}^n \end{tikzcd}$$

Note that, since ψ contracts the hyperplane H_i passing spanned by the n points $p_1, ..., \hat{p}_i, ..., p_{n+1}$ to the point p_i, the map $\widetilde{\psi}$ maps the strict transform of H_i onto the exceptional divisor E_i. Therefore $\widetilde{\psi}$ is an isomorphism in codimension one. Indeed, it is a composition of flops. In particular $\widetilde{\psi}$ induces an isomorphism $\mathrm{Pic}(X^n_{n+1}) \rightarrow \mathrm{Pic}(X^n_{n+1})$.
Now, the linear system on $\mathbb{P}^n$ associated to the standard Cremona ψ is

$$\mathcal{H} = \mathcal{O}_{\mathbb{P}^n}(n) \otimes \mathcal{I}_{(n-1)(p_1 + ... + p_{n+1})},$$

that is $\mathcal{H}$ is the linear system of hypersurfaces in $\mathbb{P}^n$ of degree n having points of multiplicity at least $n-1$ in $p_1, ..., p_{n+1}$. Therefore, the inverse image of a general hyperplane of $\mathbb{P}^n$ via ψ is an hypersurface of degree n with points of multiplicity $n-1$ in $p_1, ..., p_{n+1}$, and

$$\widetilde{\psi}^* H = nH - (n-1)(E_1 + ... + E_{n+1}).$$

Furthermore, since ψ contracts the hyperplane H_i passing spanned by the n points $p_1, ..., \hat{p}_i, ..., p_{n+1}$ to the point p_i we have

$$\widetilde{\psi}^* E_i = H - E_1 - ... - \hat{E}_i - ... - E_{n+1}.$$

We conclude that the simple reflection R_{α_k} with respect to $\alpha_k = H - E_1 - ... - E_{n+1}$ is realized by the small transformation $\widetilde{\psi}$. □

4. Cox rings and the effective cone

Let X be a normal and $\mathbb{Q}$-factorial projective variety with finitely generated and free Picard group and Picard number ρ_X. Let $D_1, ..., D_{\rho_X}$ be a basis of Cartier divisors of $\mathrm{Pic}(X)$. Then

$$\mathrm{Cox}(X) = \bigoplus_{L \in \mathrm{Pic}(X)} H^0(X, L) = \bigoplus_{m_1, ..., m_{\rho_X} \in \mathbb{Z}} H^0(X, \sum_{i=1}^{\rho_X} m_i D_i).$$

Note that $\mathrm{Cox}(X)$ is an integral domain graded by the free abelian group $\mathrm{Pic}(X)$. Now, let us consider the following definition:

DEFINITION 4.1. *Let A be an integral domain graded by a free abelian group G,*

$$A = \bigoplus_{g \in G} A_g.$$

The support of A is the semi-group

$$\mathrm{Supp}(A) = \{g \in G \mid A_g \neq 0\}.$$

LEMMA 4.2. *Let $e \in G$ be the identity element. If $\mathrm{Supp}(A)$ is not finitely generated as a semi-group then A is not finitely generated as a ring over A_e.*

PROOF. Assume that A is finitely generated. Then, there exist finitely many non-zero homogeneous elements $a_i \in A_{g_i}$ for $i = 1, ..., h$ such that $a_1, ..., a_h$ generate A. Therefore, $g_1, ..., g_h$ generate $\mathrm{Supp}(A)$. □

For instance the support of $\mathrm{Cox}(X)$ is the semi-group of linear equivalence classes of effective divisors on X:

$$\mathrm{Eff}(X) = \{L \in \mathrm{Pic}(X) \mid H^0(X, L) \neq 0\}.$$

When $X = X_k^n = Bl_{p_1,...,p_k}\mathbb{P}^n$ is the blow-up of $\mathbb{P}^n$ in $p_1, ..., p_k$ we have

$$\mathrm{Cox}(X_k^n) = \bigoplus_{L \in \mathrm{Pic}(X_k^n)} H^0(X_k^n, L) = \bigoplus_{a, b_1, ..., b_k \in \mathbb{Z}} H^0(X_k^n, \mathcal{O}_{X_k^n}(aH - b_1E_1 - ... - b_kE_k)).$$

The following result is fundamental for our study of $\mathrm{Eff}(X_k^n)$.

LEMMA 4.3. *Let $\pi : X \to Y$ be the blow-up of a projective variety Y at a point $y \in Y$. Let E be exceptional divisor of π. Then E belongs to any system of generators of the semi-group $\mathrm{Eff}(X)$.*

PROOF. Let us assume that E is linearly equivalent to the sum $D_1 + D_2$ of two effective divisors. Let A be the pull-back of an ample divisor on Y. Then

$$E \cdot A^{\dim(Y)-1} = 0.$$

Therefore

$$D_1 \cdot A^{\dim(Y)-1} + D_2 \cdot A^{\dim(Y)-1} = E \cdot A^{\dim(Y)-1} = 0$$

yields $D_1 \cdot A^{\dim(Y)-1} = D_2 \cdot A^{\dim(Y)-1} = 0$. Hence, both $\mathrm{Supp}(D_1)$ and $\mathrm{Supp}(D_2)$ are contained in E. Then either $D_1 = 0$ or $D_2 = 0$. □

Now, let us consider the variety X_k^n.

DEFINITION 4.4. *We define the H-degree of $D = aH - \sum_{i=1}^k b_iE_i \in \mathrm{Pic}(X_k^n)$ as*

$$\deg(D) = a.$$

PROPOSITION 4.5. *If the following inequality holds*

$$\frac{1}{2} + \frac{1}{n+1} + \frac{1}{k-n-1} \leq 1$$

then the $\mathcal{W}$-orbit of E_k is infinite.

PROOF. Let $w \in \mathcal{W}$. Then there exists a subset $I \subset \{1,...,k\}$ with $|I| = n+1$ such that

$$\sum_{i\in I}\deg(w(E_i)) \leq \frac{n+1}{k}\sum_{i=1}^{k}\deg(w(E_i)).$$

Since $-K_{X_k^n}$ is $\mathcal{W}$-invariant we have

$$w(-K_{X_k^n}) = (n+1)w(H) - (n-1)\sum_{i=1}^{k} w(E_i) = -K_{X_k^n},$$

and in particular

$$(n+1)\deg(w(H)) - (n-1)\sum_{i=1}^{k}\deg(w(E_i)) = \deg(-K_{X_k^n}) = n+1.$$

Then

$$\deg(w(H)) - \sum_{i\in I}\deg(w(E_i)) \geq \deg(w(H)) - \frac{n+1}{k}\sum_{i=1}^{k}\deg(w(E_i)).$$

Now, $\sum_{i=1}^{k}\deg(w(E_i)) = \frac{1}{n-1}((n+1)\deg(w(H)) - n - 1)$ yields

$$\deg(w(H)) - \sum_{i\in I}\deg(w(E_i)) \geq \deg(w(H)) - \frac{(n+1)^2}{k(n-1)}(\deg(w(H)) - 1).$$

Now, note that

$$\frac{1}{2} + \frac{1}{n+1} + \frac{1}{k-n-1} \leq 1 \iff \frac{(n+1)^2}{k(n-1)} \leq 1.$$

In particular, $\frac{1}{2} + \frac{1}{n+1} + \frac{1}{k-n-1} \leq 1$ yields

$$\deg(w(H)) - \sum_{i\in I}\deg(w(E_i)) > 0.$$

Now, consider the reflection R_I with respect to α_I in (3.2). We have:

$$R_I(H) - H = nH - (n-1)\sum_{i\in I}E_i - H = (n-1)(H - \sum_{i\in I}E_i),$$

and

$$\deg(w(R_I(H))) - \deg(w(H)) = (n-1)(\deg(w(H)) - \sum_{i\in I}\deg(w(E_i))) > 0.$$

Therefore, the degree of H is increased by the reflection R_I, and the $\mathcal{W}$-orbit of H is infinite. Finally, since $(n+1)\deg(w(H)) - (n-1)\sum_{i=1}^{k}\deg(w(E_i)) = \deg(-K_{X_k^n}) = n+1$, we see that the degree of E_k is increased by R_i as well. Therefore, also the $\mathcal{W}$-orbit of E_k is infinite. □

THEOREM 4.6. *If the following inequality holds*

$$\frac{1}{2} + \frac{1}{n+1} + \frac{1}{k-n-1} \leq 1$$

then $\mathrm{Cox}(X_k^n)$ *is not finitely generated.*

PROOF. By Lemma 4.2 it is enough to prove that $\mathrm{Eff}(X_k^n)$ is not finitely generated. In order to to this, by Lemma 4.3 it is enough to prove that X_k^n contains infinitely many (-1)-divisors, in the sense of Definition 3.2.
Let us begin with the (-1)-divisor E_k. By Proposition 4.5 we have that if $\frac{1}{2}+\frac{1}{n+1}+\frac{1}{k-n-1}\leq 1$ the the orbit of E_k under the action of the Weyl group $\mathcal{W}$ is infinite. Let E be an element of this orbit. By Theorem 3.3 there exists a small transformation which is a lifting of a suitable standard Cremona centred in $n+1$ points among $p_1, ..., p_k$ such that D is linearly equivalent to the pull-back of an exceptional divisor via the lifting of the Cremona. Therefore, we produce infinitely (-1)-divisors in X_k^n and $\mathrm{Eff}(X_k^n)$ can not be finitely generated. $\square$

Explicitly, Theorem 4.6 can be rephrased as follows: if
- $n=2, k\geq 9$,
- $n=3, k\geq 8$,
- $n=4, k\geq 9$,
- $n\geq 5, k\geq n+4$

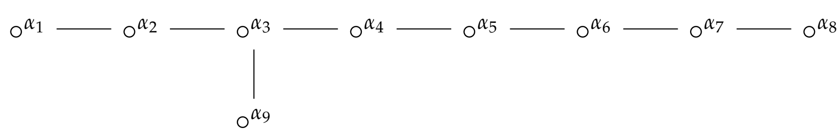
then $\mathrm{Cox}(X_k^n)$ is not finitely generated.

4.1. Blow-up of $\mathbb{P}^2$ in nine points. Let $p_1, ..., p_9$ be nine points in $\mathbb{P}^2$. First, let us assume that $\{p_1, ..., p_9\}$ is the complete intersection of two general cubics $C=Z(f), \Gamma=Z(g)$ in $\mathbb{P}^2$. Let $X=\{p_1, ..., p_8\}$. Since C and Γ are irreducible X does not have either four points on a line or seven points on a conic. Then the set X imposes independent conditions to the cubic, and C, Γ are a basis of the cubics through X. Therefore, any other cubic $D=Z(h)$ through X is such that $h=\alpha f+\beta g$. In particular $h(p_9)=0$. We conclude that the cubics through $p_1, ..., p_9$ are parametrized by $\mathbb{P}^1$. Therefore the linear system of cubics through $p_1, ..., p_9$ induces a rational map $\phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$, which in turns induces a morphism $\widetilde{\phi} : X_9^2 \to \mathbb{P}^1$:

$$\begin{tikzcd} X_9^2 \arrow[d] \arrow[dr, "\widetilde{\phi}"] & \\ \mathbb{P}^2 \arrow[r, dashed, "\phi"] & \mathbb{P}^1 \end{tikzcd}$$

Now, the general fiber of $\widetilde{\phi} : X_9^2 \to \mathbb{P}^1$ is an elliptic curve, and the nine exceptional divisors $E_1, ..., E_9$ are sections of the fibration $\widetilde{\phi}$. The generic fiber C of $\widetilde{\phi}$ is an elliptic curve over $\mathbb{C}(\mathbb{P}^1)=\mathbb{C}(t)$. Now, by considering the orbits of the nine sections $E_1, ..., E_9$ via the group law of C we produce infinitely many (-1)-curves in X_9^2.

Now, we consider the case of nine general points. In this case we follow the general philosophy of Theorem 3.3. Our aim is to produce infinitely many (-1)-curves in X_9^2 by applying iteratively the standard Cremona of $\mathbb{P}^2$, and by replacing the symmetries of the elliptic fibration $\widetilde{\phi} : X_9^2 \to \mathbb{P}^1$ with the symmetries of $\mathrm{Pic}(X_9^2)$ with respect to the Weyl group of the Dynkin diagram $T_{2,6,3}$:



We begin with the line $C_2=\langle p_1, p_2\rangle$, and consider the standard Cremona $f_2 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ centred in p_7, p_8, p_9. By Proposition 3.1 the curve $C_3=f_2(C_2)$ is a conic through $p_1, ..., p_5$. We proceed recursively by taking at the step i the standard Cremona $f_{i-1} : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ centred at the three points among $p_1, ..., p_9$ of lowest multiplicity for the curve $C_i = f_{i-1}(C_{i-1})$. We denote by d the degree and by m_i the degree and the multiplicity in p_i of these curves. The following table displays the step from $i = 2$ to $i = 16$ of the iteration.

i	d	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
2	1	1	1	0	0	0	0	0	0	0
3	2	1	1	1	1	1	0	0	0	0
4	4	2	2	2	1	1	1	1	1	0
5	6	3	3	2	2	2	2	1	1	1
6	9	4	4	4	3	3	2	2	2	2
7	12	5	5	5	4	4	4	3	3	2
8	16	7	7	6	5	5	5	4	4	4
9	20	8	8	8	7	7	6	5	5	5
10	25	10	10	10	8	8	8	7	7	6
11	30	12	12	11	10	10	10	8	8	8
12	36	14	14	14	12	12	11	10	10	10
13	42	16	16	16	14	14	14	12	12	11
14	49	19	19	18	16	16	16	14	14	14
15	56	21	21	21	19	19	18	16	16	16
16	64	24	24	24	21	21	21	19	19	18

Now, we want to prove that:

$$deg(C_i) \sim \frac{i^2}{4}$$

and $\mathrm{mult}_{p_j} C_i \sim \frac{(i+2)^2-2(i+2)-1}{12}$ for $j = 1,2,3$, $\mathrm{mult}_{p_j} C_i \sim \frac{(i+1)^2-2(i+1)-1}{12}$ for $j =, 4, 5, 6$, $\mathrm{mult}_{p_j} C_i \sim \frac{i^2-2i-1}{12}$ for $j = 7, 8, 9$, where $\sim$ means that the values differs at most by a rational number $-1 < \epsilon < 1$. This is verified for all the steps in the table. Let us assume it is true at the step i. Then

$$\deg(C_{i+1}) = 2\deg(C_i) - m_7(C_i) - m_8(C_i) - m_9(C_i) \sim 2\frac{i^2}{4} - 3\frac{i^2-2i-1}{12} = \frac{(i+1)^2}{4}.$$

From the table we see that

$$m_j(C_{i+1}) = m_{j-3}(C_i) \sim \frac{(i+2)^2 - 2(i+2) - 1}{12}$$

for $j = 4, 5, 6$. Furthermore

$$m_j(C_{i+1}) = m_{j-3}(C_i) \sim \frac{(i+1)^2 - 2(i+1) - 1}{12}$$

for $j = 7, 8, 9$. Finally

$$m_j(C_{i+1}) \sim \frac{i^2}{4} - 2\frac{i^2-2i-1}{12} = \frac{i^2+4i+2}{12} = \frac{(i+3)^2 - 2(i+3) - 1}{12}.$$

The line $C_2 = \langle p_1, p_2 \rangle$ is a (-1)-curve in X_2^9. Therefore, C_i is a (-1)-curve as well. Finally, since

$$deg(C_i) \sim \tfrac{i^2}{4} \xrightarrow[i\mapsto\infty]{} \infty$$

we get infinitely many (-1)-curves in X_9^2.

4.2. Blow-up of $\mathbb{P}^2$ in eight points. Now, let us apply the same procedure to X_8^2. In this case we get the following table:

i	d	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8
2	1	1	1	0	0	0	0	0	0
3	2	1	1	1	1	1	0	0	0
4	4	2	2	2	1	1	1	1	1
5	5	3	3	3	2	2	2	1	1
6	6	3	3	3	3	2	2	2	2
7	6	3	3	3	3	2	2	2	2

We see that $\deg(C_i) = 6$ for any $i \geq 6$.

4.3. The Mori cone of X_k^n. In this section we determine the cone of curves of X_k^n.

LEMMA 4.7. *Let $p_1, ..., p_8 \in \mathbb{P}^3$ be general points, and $C \subset \mathbb{P}^3$ an irreducible curve of degree d having multiplicity $m_i = \operatorname{mult}_{p_i}(C)$ at p_i, $1 \leq i \leq 8$. Then $m_1 + ... + m_8 \leq 2d$.*

PROOF. If C is degenerate, then $m_i \neq 0$ for at most three points p_i, and the conclusion follows easily from Bézout. So from now on we assume that C is non degenerate. Let Λ be the pencil of irreducible quadric surfaces passing through $p_1, \dots, p_8$. Suppose that $m_1 + ... + m_8 > 2d$. It follows from Bézout that C is contained in every member of Λ. In particular, C is a non degenerate irreducible curve contained in the intersection of two irreducible quadric surfaces. So $d \in \{3, 4\}$. Suppose that $d = 3$. Then C must be a twisted cubic through at most 6 of the p_i's, and thus $m_1 + ... + m_8 \leq 2d = 6$, contradicting our assumptions. We conclude that $d = 4$, $m_i \geq 1$ for every i, and $m_j \geq 2$ for some j. If follows from Bézout that $m_j = 2$, and $m_i = 1$ for $i \neq j$. Consider the projection from p_j

$$\pi_{p_j} : C \dashrightarrow \mathbb{P}^2.$$

The image $\overline{\pi_{p_1}(C)}$ is a conic though the seven general points $\pi_{p_j}(p_i)$, $i \neq j$, which is impossible. This shows that $m_1 + ... + m_8 \leq 2d$. □

PROPOSITION 4.8. *Let X_k^n be the blow-up of $\mathbb{P}^n$ at points in general position $p_1, \dots, p_k$, $n \geq 2$. Denote by R_i a line in the exceptional divisors over p_i, and by $L_{i,j}$ the strict transforms of the line through $p_i \neq p_j$. Suppose that either of the following holds:*

- *$k \leq 2n$.*
- *$n = 3$ and $k \leq 8$.*

Then the Mori cone $\mathrm{NE}(X_k^n)$ is generated by the classes of the R_i's and $L_{i,j}$'s.

PROOF. Let X_k^n be the blow-up of $\mathbb{P}^n$, $n \geq 2$, at points in general position $p_1, \dots, p_k$. First of all, note that

$$L \equiv L_{i,j} + R_i + R_j \text{ and } L_i \equiv L - R_i \equiv L_{i,j} + R_j. \tag{4.1}$$

Let $\widetilde{C} \subset X_k^n$ be an irreducible curve not contained in any exceptional divisor E_i, and denote by C the image of $\widetilde{C}$ in $\mathbb{P}^n$. It is an irreducible curve of degree $d > 0$ and multiplicity $m_i = \operatorname{mult}_{p_i} C \geq 0$ at p_i, $\widetilde{C}$ is the strict transform of C, and

$$\widetilde{C} \equiv dL - m_1 R_1 - ... - m_k R_k. \tag{4.2}$$

We must show that the class of $\widetilde{C}$ in $N_1(X_k^n)$ lies in the cone generated by the R_i's and $L_{i,j}$'s. We may assume that $m_1 \leq m_2 \leq \cdots \leq m_k$.
First let us assume that k is even. We write

$$\begin{aligned}\widetilde{C} \equiv \quad & dL - m_1(R_1+R_2) - (m_2-m_1)R_2 - m_3(R_3+R_4) - (m_4-m_3)R_4 - \\ & ... - m_{k-1}(R_{k-1}+R_k) - (m_k - m_{k-1})R_k.\end{aligned} \tag{4.3}$$

Note that $m_1 + (m_2 - m_1) + m_3 + (m_4 - m_3) + ... + m_{k-1} + (m_k - m_{k-1}) = m_2 + m_4 + ... + m_k$. We claim that $m_2 + m_4 + ... + m_k \leq d$. Indeed, since $k \leq 2n$, the set $\{p_2, p_4, ..., p_k\}$ has cardinality at most n. Consider the linear space $P = \langle p_2, p_4, ..., p_k \rangle \subsetneqq \mathbb{P}^n$. If $m_2 + m_4 + ... + m_k > d$, then $C \subset P$ by Bézout . Since the p_i's are general, $p_1, p_3, ..., p_{k-1} \notin P$, and so $m_1 = m_3 = ... = m_{k-1} = 0$. But this implies that $m_i = 0$ for $i \leq k-1$ and $m_k > d$, which is impossible. This proves the claim. So we can rewrite (4.3) as

$$\begin{aligned}\widetilde{C} \equiv \quad & m_1 L_{1,2} + (m_2 - m_1)L_2 + m_3 L_{3,4} + (m_4 - m_3)L_4 - \\ & ... + m_{k-1}L_{k-1,k} + (m_k - m_{k-1})L_k + (d - m_2 - m_4 - ... - m_k)L.\end{aligned}$$

It follows from (4.1) that the class of $\widetilde{C}$ in $N_1(X_k^n)$ lies in the cone generated by the R_i's and $L_{i,j}$'s.
Now suppose that k is odd, and write

$$\begin{aligned}\widetilde{C} \equiv \quad & dL - m_1(R_1+R_2) - (m_2-m_1)R_2 - m_3(R_3+R_4) - (m_4-m_3)R_4 - \\ & ... - m_{k-2}(R_{k-2}+R_{k-1}) - (m_{k-1} - m_{k-2})R_{k-1} - m_k R_k.\end{aligned} \tag{4.4}$$

In this case $m_1 + (m_2 - m_1) + m_3 + (m_4 - m_3) + ... + m_{k-1} + (m_k - m_{k-1}) = m_2 + m_4 + ... + m_{k-1} + m_k$. Just as in the even case, one shows that $m_2 + m_4 + ... + m_{k-1} + m_k \leq d$ and rewrite (4.4) as an effective linear combination of the R_i's and $L_{i,j}$'s.
From now on we suppose that $n = 3$ and $k \leq 8$. Then $m_i \leq d$ and $m_1 + ... + m_k \leq 2d$by Lemma 4.7. If $m_{k-1} = 0$, then $\widetilde{C} \equiv m_k L_k + (d - m_k)L$. It follows from (4.1) that the class of $\widetilde{C}$ in $N_1(X_k^n)$ lies in the cone generated by the R_i's and $L_{i,j}$'s. If $m_{k-1} \neq 0$, then rewrite (4.2) as

$$\widetilde{C} \equiv (L_{k-1,k}) - d'L - m'_1 R_1 - ... - m'_k R_k,$$

where $d' = d - 1$, $m'_i = m_i$ for $i \leq k-2$, and $m'_i = m_i - 1$ for $i = k-1$ or k. Note that $m'_i \leq d'$. This is clear for $i = k-1$ or k. For $i \leq k-2$ it follows from the assumptions that $m_1 \leq m_2 \leq \cdots \leq m_k \leq d$ and $m_1 + ... + m_k \leq 2d$. We also have $m'_1 + ... + m'_k \leq 2d'$. So we can repeat the process and conclude by induction that the class of $\widetilde{C}$ in $N_1(X_k^n)$ lies in the cone generated by the R_i's and $L_{i,j}$'s. □

5. Proof of Theorem 0.1

By Theorem 4.6 we know that if the hypothesis of Theorem 0.1 are not satisfied then X_k^n is not a Mori Dream Space. In particular by Proposition 1.2 it is not log Fano. In this section our aim it to prove the other implication by producing explicitly an effective divisor D such that $-K_{X_k^n} - D$ is ample and (X_k^n, D) is klt.
In order to clarify ideas let us consider the following example.

5.1. Blow-ups of $\mathbb{P}^3$. A version of the following result with more details on the postulation of the points $p_1, ..., p_k \in \mathbb{P}^3$ has been proven in [**BL**, Proposition 2.9].

PROPOSITION 5.1. *Let X_k^3 be the blow-up of $\mathbb{P}^3$ at k general points $p_1, ..., p_k$. Then X_k^3 is weak Fano if and only if $k \leq 7$.*

PROOF. The anti-canonical divisor of X_k^3 is given by
$$-K_{X_k^3} = 4H - 2E_1 - ... - 2E_k = 2(2H - E_1 - ... - E_k).$$
If $k > 7$ then $(-K_{X_k^3})^3 \leq 0$ and X_k^3 can not be weak Fano. We have that $-K_{X_k^3} \cdot R_i = 2$, and $-K_{X_k^3} \cdot L_i = 0$. By Proposition 4.8 $-K_{X_k^3}$ is nef if $k \leq 7$. Furthermore $(-K_{X_k})^3 > 0$ for $k \leq 7$, and by [**La**, Theorem 2.2.14] $-K_{X_k}$ is big. ☐

REMARK 5.2. Let X_8^3 be the blow-up of $\mathbb{P}^3$ at eight general points $p_1, ..., p_8$. By Proposition 4.8, $-K_{X_8}$ is nef. On the other hand $(-K_{X_8})^3 = 0$, and so $-K_{X_8}$ is not big by [**La**, Theorem 2.2.14].

By Proposition 1.9 we have that X_k^3 is log Fano for any $k \leq 7$. In the following we show how to produce an explicit $\mathbb{Q}$-divisor D such that $-(K_{X_k} + D)$ is ample and the pair (X_k, D) is klt. First of let us observe that if $k \leq 4$ then X_k^3 is a toric variety. In this case it is enough to take D as a suitable combination of toric invariant divisors. However, in the case $k = 4$ we may choose an irreducible divisor D. For instance we may consider the cubic surface
$$\Delta = \{x_0x_1x_2 + x_0x_1x_3 + x_0x_2x_3 + x_1x_2x_3 = 0\} \subset \mathbb{P}^3,$$
that is the Cayley's nodal cubic surface. Note that Δ is an element of the linear system of the standard Cremona transformation of $\mathbb{P}^3$. The surface Δ has exactly four singularities in the fundamental points of $\mathbb{P}^3$ that are ordinary double points. We may write the strict transform D of Δ as
$$D = 3H - 2(E_1 + ... + E_4).$$
Then
$$-K_{X_4^3} - \epsilon D = (4 - 3\epsilon)H - (2 - 2\epsilon)(E_1 + ... + E_4).$$
We have $(-K_{X_4^3} - \epsilon D) \cdot R_i = 2 - 2\epsilon$ and $(-K_{X_4^3} - \epsilon D) \cdot L_{i,j} = 4 - 3\epsilon - 2(2 - 2\epsilon) = 4\epsilon$. By Proposition 4.8 we conclude that $-(K_{X_4^3} + \epsilon D)$ is ample for any $0 < \epsilon < 1$.

Let us consider the other three cases. If $k = 5$ we consider all the planes $H_{i,j,k}$ spanned by three of the p_i's. We get a divisor $\Delta = \sum_{i,j,k} H_{i,j,k}$, and any of the p_i is a of multiplicity six for Δ. Therefore, we may write the strict transform D of Δ, through the blow-up morphism, as
$$D = 10H - 6(E_1 + ... + E_5).$$
Then $-K_{X_5^3} - \epsilon D = (4 - 10\epsilon)H - (2 - 6\epsilon)(E_1 + ... + E_5)$. We have $-(K_{X_5^3} - \epsilon D) \cdot R_i = 2 - 6\epsilon$ and $-(K_{X_5^3} - \epsilon D) \cdot L_{i,j} = 2\epsilon$. Then, by Proposition 4.8 we have that $-(K_{X_5^3} + \epsilon D)$ is ample for any $0 < \epsilon < \frac{1}{3}$. Furthermore we can take $\epsilon > 0$ arbitrarily small in order to have have the pair $(X_5, \epsilon D)$ klt.

If $k = 6$ we have to take care of the twisted cubic through the p_i's. Therefore linear subspaces are not enough. Let Q_i be the unique quadric cone with vertex p_i and having a simple point in p_j for any $j \neq i$. We consider the divisor $\Delta = Q_1 + Q_2 + Q_3 + H_{4,5,6}$, where $H_{4,5,6}$ is the plane spanned by p_4, p_5, p_6. Since any of the p_i is a points of multiplicity four for Δ we can write
$$D = 7H - 4(E_1 + ... + E_6).$$
Then $-K_{X_6^3} - \epsilon D = (4 - 7\epsilon)H - (2 - 4\epsilon)(E_1 + ... + E_6)$. Therefore, $-(K_{X_6^3} + \epsilon D) \cdot R_i = 2 - 4\epsilon$ and $-(K_{X_6^3} + \epsilon D) \cdot R_i = \epsilon$. By Proposition 4.8 we have that $-(K_{X_5^3} + \epsilon D)$ is ample for any $0 < \epsilon < \frac{1}{2}$.

If $k = 7$ we have to consider cubic surfaces. First, we claim that there exists an irreducible cubic surface having nodes at $p_1, ..., p_4$ and simple points at p_5, p_6, p_7. Cubic surfaces are parametrized by $\mathbb{P}(k[x_0, ..., x_3]_3) \cong \mathbb{P}^{19}$. Furthermore, any node imposes at most four conditions, and simple point at most one condition. We have exactly at most $4 \cdot 4 + 3 = 19$ conditions. Therefore, there exists a cubic surface S having nodes at $p_1, ..., p_4$ and simple points at p_5, p_6, p_7. Now, we want to prove that S is irreducible. Since the p_i's are general and S passes through all of them with three planes we can construct a cubic having nodes at most at two of the p_i's. Similarly with an irreducible quadric and a plane we can construct a cubic having nodes at most at three of the p_i's. We conclude that S is irreducible. Let $S_{i,j,k}$ be a cubic surface having simple points at p_i, p_j, p_k and nodes at p_h for any $h \neq, i, j, k$. Any of the p_i is of multiplicity fifty-five for the surface $\Delta = \sum_{i,j,k} S_{i,j,k}$. Since Δ has $\binom{7}{4} = 35$ components of degree three we can write its strict transform as

$$D = 105H - 55(E_1 + ... + E_7).$$

Then $-K_{X_7^3} - \epsilon D = (4 - 105\epsilon)H - (2 - 55\epsilon)(E_1 + ... + E_7)$, $-(K_{X_7^3} + \epsilon D) \cdot R_i = 2 - 55\epsilon$ and $-(K_{X_7^3} + \epsilon D) \cdot L_{i,j} = 5\epsilon$. By Proposition 4.8 $-(K_{X_5^3} + \epsilon D)$ is ample for any $0 < \epsilon < \frac{2}{55}$.

6. Blow-ups of $\mathbb{P}^n$, $n \geq 4$

6.1. Blow-ups of $\mathbb{P}^n$ in $n+1$ points. The variety X^n_{n+1} is toric. Therefore it is log Fano. In this section we show that for X^n_{n+1} the divisor D in the definition of log Fano variety can be chosen irreducible.

Let $|\mathcal{H}| \subset |\mathcal{O}_{\mathbb{P}^n}(n)|$ be the linear system of hypersurfaces of degree n and having multiplicity at least $n-1$ in $p_1, ..., p_{n+1}$. This linear system induces the standard Cremona transformation of $\mathbb{P}^n$ given by:

$$\begin{array}{cccc} \phi_{|\mathcal{H}|}: & \mathbb{P}^n & \dashrightarrow & \mathbb{P}^n \\ & [x_0 : ... : x_n] & \longmapsto & [\frac{1}{x_0} : ... : \frac{1}{x_n}] \end{array}$$

LEMMA 6.1. *Let $\Delta \in |\mathcal{H}|$ be a general element, $L_{i_1,...,i_h} = \langle p_{i_1}, ..., p_{i_h} \rangle$, and $\pi : Y \to \mathbb{P}^n$ be the blow-up of all the strict transforms of the linear subspaces $L_{i_1,...,i_h}$ for $h = 1, ..., n-1$ in order of increasing dimension. Then the strict transform $\widetilde{D}$ of Δ in Y is smooth and transversal to all the exceptional divisors of π. Furthermore*

$$\mathrm{mult}_{L_{i_1,...,i_h}} \Delta = n - h$$

for any $h = 1, ..., n-1$.

PROOF. By [**MM**, Theorem 1] the Cremona $\phi_{|\mathcal{H}|}$ lifts to an automorphism of Y. In particular this implies that the strict transform $\widetilde{D}$ of Δ via π is smooth and transversal to all the exceptional divisors of π. Therefore, Δ is smooth out of the union of the codimension two linear subspaces $L_{i_1,...,i_n}$.

Now, let us consider the element of the linear system $|\mathcal{H}|$ given by:

$$\Delta_0 := \{x_0x_1...x_{n-1} + x_0x_1...x_{n-2}x_n + ... + x_1x_2...x_n = 0\}.$$

We may assume that the p_i's are the fundamental points of $\mathbb{P}^n$, and consider $L_{1,...,h} = \langle p_1, ..., p_h \rangle$. Let $x \in L_{1,...,h}$ be a general point. Then $x_0 \neq 0, ..., x_{h-1} \neq 0$. Substituting $x_0 = 1, ..., x_{h-1} = 1$ in the equation of Δ_0 and taking the monomials of lowest degree we get that the projective tangent cone of Δ_0 in x is the hypersurface $T \subset \mathbb{P}^{n-1}$ given by:

$$T = \{x_h x_{h+1} ... x_{n-2} x_{n-1} + x_h x_{h+1} ... x_{n-2} x_n + ... + x_{h+1} x_{h+2} ... x_{n-1} x_n = 0\}.$$

Then $\operatorname{mult}_{L_{1,...,h}} \Delta_0 = \deg(T) = n - h$. To conclude it is enough to observe that for any $\Delta \in |\mathcal{H}|$ we have $\operatorname{mult}_{L_{i_1,...,i_h}} \Delta \geq n - h$. ☐

PROPOSITION 6.2. *Let Δ be a general element in the linear system of the standard Cremona of $\mathbb{P}^n$ and let D be its strict transform in X^n_{n+1}. For any $\frac{n-3}{n-2} < \epsilon < 1$ the divisor $-(K_{X^n_{n+1}} + \epsilon D)$ is ample, and the pair $(X^n_{n+1}, \epsilon D)$ is klt.*

PROOF. We have

$$D = nH - (n-1)(E_1 + ... + E_{n-1})$$

and

$$-K_{X^n_{n+1}} - \epsilon D = (n + 1 - \epsilon n)H - (n - 1 - \epsilon(n-1))(E_1 + ... + E_{n+1}).$$

Therefore, $-(K_{X^n_{n+1}} + \epsilon D) \cdot R_i = n - 1 - \epsilon(n-1) > 0$ if and only if $\epsilon < 1$. Furthermore, $-(K_{X^n_{n+1}} + \epsilon D) \cdot L_{i,j} = (n + 1 - \epsilon n) - 2(n - 1 - \epsilon(n-1)) = (n-1)\epsilon - n + 3 > 0$ if and only if $\epsilon < \frac{n-3}{n-2}$. By Proposition 4.8 we conclude that for any $\frac{n-3}{n-2} < \epsilon < 1$ the divisor $-K_{X^n_{n+1}} - \epsilon D$ is ample. Now, by Lemma 6.1 we have that the blow-up $\pi : Y \to \mathbb{P}^n$ of all the strict transforms of the linear subspaces $L_{i_1,...,i_h}$ for $h = 1, ..., n-1$ in order of increasing dimension is a log resolution of the pair $(X^n_{n+1}, \epsilon D)$. Now, let ρ_h be the number of linear subspaces of dimension $h - 1$ that have been blown-up, and let $E_1^{h-1}, ..., E_{\rho_h}^{h-1}$ be the exceptional divisors over such linear subspaces. Then, we may write

$$K_Y = \pi^* K_{X^n_{n+1}} + \sum_{h=2}^{n-1} (n-h)(E_1^{h-1} + ... + E_{\rho_h}^{h-1}).$$

Furthermore, by Lemma 6.1 we have

$$\pi^*(\epsilon D) = \epsilon \sum_{h=2}^{n-1} (n-h)(E_1^{h-1} + ... + E_{\rho_h}^{h-1}) + \epsilon \widetilde{D},$$

where we denote by $\widetilde{D}$ the strict transform of D in Y. Therefore, we get

$$K_Y = \pi^*(K^n_{n+1} + \epsilon D) + \sum_{h=2}^{n-1} (n - h - \epsilon(n-h))(E_1^{h-1} + ... + E_{\rho_h}^{h-1}) - \epsilon \widetilde{D}.$$

We see that for $\epsilon < 1$ all the discrepancies are greater than zero. Therefore the pair $(X^n_{n+1}, \epsilon D)$ is terminal and hence klt. ☐

REMARK 6.3. The toric variety Y used as a log resolution in the proof of Proposition 6.2, that is is $\mathbb{P}^n$ blown-up at all the linear spaces of codimension at least two spanned by subsets of $n+1$ points in linear general position, is the Losev-Manin's moduli space $\overline{L}_{n+1}$ introduced by *A. Losev* and *Y. Manin* in **[LM]**, see **[Ha**, Section 6.4]. The space $\overline{L}_{n+1}$ parametrizes $(n+1)$-pointed chains of projective lines $(C, x_0, x_\infty, x_1, ..., x_{n+1})$ where:

- C is a chain of smooth rational curves with two fixed points x_0, x_∞ on the extremal components,
- $x_1, ..., x_{n+1}$ are smooth marked points different from x_0, x_∞ but non necessarily distinct,
- there is at least one marked point on each component.

6.2. Hyperplane arrangements and the blow-ups of $\mathbb{P}^n$ in $n+2$ points. Let $p_1, ..., p_{n+2}$ be general points in $\mathbb{P}^n$. We consider the hyperplane arrangement

$$\mathcal{H} = \{\langle p_{i_1}, ..., p_{i_n}\rangle \mid i_j \in \{1, ..., n+2\}\}.$$

Note that $\mathcal{H}$ is supported on a reducible divisor $H = \sum_{i=1}^{\rho_{n-1}} H_i$, where H_i is an hyperplane and $\rho_{n-1} = \binom{n+2}{n} = \frac{(n+2)(n+1)}{2}$. We keep denoting by $\mathcal{H}$ the strict transform of $\mathcal{H}$ in X^n_{n+2}. Although any H_i is smooth the H_i's intersects with hight multiplicity along the strict transforms of the linear subspaces of $\mathbb{P}^n$ determined by the intersections of the H_i's themselves. Therefore $(X^n_{n+2}, \mathcal{H})$ is not a log resolution of $(\mathbb{P}^n, \mathcal{H})$. Let us consider the set $\mathcal{H}^h = \{\langle p_{i_1}, ..., p_{i_{h+1}}\rangle \mid i_j \in \{1, ..., n+2\}\}$ of all the h-planes spanned by the p_i's.

PROPOSITION 6.4. *Let $\pi : Y \to X^n_{n+2}$ be the blow-up of all the lines in $\mathcal{H}^1$, all the planes in $\mathcal{H}^2$,..., all the $(n-2)$-planes in $\mathcal{H}^{n-2}$, in order of increasing dimension. Let us consider the pair $(X^n_{n+2}, \epsilon\mathcal{H})$ where $\epsilon \in \mathbb{Q}$ is a rational number, and let $\widetilde{\mathcal{H}}$ be the strict transform of $\mathcal{H}$ through π. Then $(Y, \epsilon\widetilde{\mathcal{H}})$ is a log resolution of $(X^n_{n+2}, \epsilon\mathcal{H})$. Let E^h_j be the exceptional divisor over an h-plane, and let us write*

$$K_Y = \pi^*(K_{X^n_{n+2}} + \epsilon\mathcal{H}) + d_{n-1}(H_1 + ... + H_{\rho_{n-1}}) + \sum_{h=1}^{n-2} d_h(E^h_1 + ... + E^h_{\rho_h}).$$

Then $\rho_{n-1} = \binom{n+2}{n}$, $d_{n-1} = -\epsilon$, and

$$\rho_h = \binom{n+2}{h+1},\ d_h = (n-h-1) - \epsilon\binom{n-h+1}{n-h-1}.$$

PROOF. Clearly $\rho_{n-1} = \binom{n+2}{n} = \frac{(n+2)(n+1)}{2}$. At each step, the spaces to be blown-up do not intersect because their intersections have been blown-up at an earlier step. Clearly the divisor $\mathrm{Exc}(\pi) \cup H_1 \cup ... \cup H_{\rho_{n-1}}$ is simple normal crossing. Therefore $(Y, \epsilon\widetilde{\mathcal{H}})$ is a log resolution of $(X^n_{n+2}, \epsilon\mathcal{H})$. Clearly, any element of $\mathcal{H}^h$ is determined by $h+1$ points. Therefore $\rho_h = \binom{n+2}{h+1}$ for $h = 1, ..., n-1$. Now, let us compute the discrepancies. First of all we have

$$K_Y = \pi^* K_{X^n_{n+2}} + \sum_{h=1}^{n-2}(n-h-1)(E^h_1 + ... + E^h_{\rho_h}). \tag{6.1}$$

Now, fix an h-plane, let us say $\langle p_1, ..., p_{h+1}\rangle$. In order to construct an hyperplane in $\mathcal{H}$ containing $\langle p_1, ..., p_{h+1}\rangle$ we have to choose $n-h-1$ points out of $p_{h+2}, ..., p_{n+2}$. Therefore we have $\binom{n-h+1}{n-h-1}$ of them, and

$$\pi^*(\epsilon\mathcal{H}) = \epsilon\sum_{h=1}^{n-2}\binom{n-h+1}{n-h-1}(E^h_1 + ... + E^h_{\rho_h}) + \epsilon(H_1 + ... + H_{\rho_{n-1}}). \tag{6.2}$$

Finally, subtracting 6.2 from 6.1 we get

$$K_Y = \pi^*(K_{X^n_{n+2}} + \epsilon\mathcal{H}) - \epsilon\sum_{i=1}^{\rho_{n-1}} H_i + \sum_{h=1}^{n-2}\left((n-h-1) - \epsilon\binom{n-h+1}{n-h-1}\right)(E^h_1 + ... + E^h_{\rho_h}).$$

□

THEOREM 6.5. *If $p_1, ..., p_k \in \mathbb{P}^n$ are general points then X^n_k is log Fano for any $k \leq n+2$.*

PROOF. If $k \leq n+1$ then X^n_k is toric and hence log Fano. Let us consider X^n_{n+2} and the divisor $\mathcal{H}$ of Proposition 6.4. Our aim is to prove that there exits a rational number ϵ such that $-(K_{X^n_{n+2}} + \epsilon\mathcal{H})$ is ample and $(X^n_k, \epsilon\mathcal{H})$ is klt. Let $E_1, ..., E_{n+2}$ be the exceptional divisors over $p_1, ..., p_{n+2}$. Then

$$-K_{X^n_{n+2}} = (n+1)H - (n-1)E_1 - ... - (n-1)E_{n+2}.$$

Furthermore, since we have $\binom{n+2}{n} = \frac{1}{2}(n+2)(n+1)$ hyperplanes in $\mathcal{H}$, and thorough any p_i there are $\binom{n+1}{n-1} = \frac{1}{2}(n+1)n$ hyperplanes we have

$$\mathcal{H} = \frac{1}{2}(n+2)(n+1)H - \frac{1}{2}(n+1)nE_1 - ... - \frac{1}{2}(n+1)nE_{n+2}.$$

Therefore

$$-(K_{X^n_{n+2}} + \epsilon\mathcal{H}) = (n+1-\frac{\epsilon}{2}(n+2)(n+1))H - (n-1-\frac{\epsilon}{2}(n+1)n)\sum_{i=1}^{n+2} E_i.$$

Now, we have:

$$-(K_{X^n_{n+2}} + \epsilon\mathcal{H}) \cdot R_i = (n-1-\frac{\epsilon}{2}(n+1)n)$$

and

$$-(K_{X^n_{n+2}} + \epsilon\mathcal{H}) \cdot L_{i,j} = (n+1-\frac{\epsilon}{2}(n+2)(n+1)) - 2(n-1-\frac{\epsilon}{2}(n+1)n).$$

By Proposition 4.8 we have that for any rational number ϵ such that

$$\frac{2(n-3)}{(n+1)(n-2)} < \epsilon < \frac{2(n-1)}{n(n+1)}$$

the divisor $-(K_{X^n_{n+2}} + \epsilon\mathcal{H})$ is ample.
To conclude we have to show that $(X^n_{n+2}, \epsilon\mathcal{H})$ is klt. By Proposition 6.4 the discrepancy of the log resolution $\pi : Y \to X^n_{n+2}$ with respect to the exceptional divisor E^h_j over an h-plane is given by $d_h = (n-h-1) - \epsilon\binom{n-h+1}{n-h-1}$. Now, $(n-h-1) - \epsilon\binom{n-h+1}{n-h-1} > -1$ if and only if $\epsilon < \frac{2}{n-h+1}$. Therefore, if $\epsilon < \frac{2}{n}$ then $d_h > -1$ for any $h = 1, ..., n-2$. To conclude, it is enough to observe that $\frac{2(n-1)}{n(n+1)} < \frac{2}{n}$. $\square$

6.3. Blow-ups of $\mathbb{P}^n$ in $n+3$ points.

6.6 (The effective cone of the blow-up of $\mathbb{P}^n$ at $n+3$ points). *Let X be the blow-up of $\mathbb{P}^n$ at $n+3$ points p_i in general position. By* [**CT2**, Theorems 1.3], *X is a Mori dream space. Next we describe the* 1*-dimensional faces of* $\mathrm{Eff}(X)$ *(*[**CT2**, Theorem 1.2]*). We denote by H the pullback to X of a hyperplane in $\mathbb{P}^n$, and by E_i the exceptional divisor over the point p_i. For each subset $I \subset \{1, \cdots, n+3\}$ whose complement has odd cardinality $|I^c| = 2k+1$, consider the divisor*

$$E_I := kH - k\sum_{i\in I} E_i - (k-1)\sum_{i\in I^c} E_i.$$

There is a unique divisor in the linear system $|E_I|$, which we also denote by E_I. When $k=0$ we have $E_{\{i\}^c} = E_i$ When $k \geq 1$, E_I can be described as follows. Let $\pi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{2k-2}$ be the projection from the linear space $\langle p_i \rangle_{i\in I}$. Let $C_I \subset \mathbb{P}^{2k-2}$ be the image of the unique rational normal curve through all the p_i's. The divisor E_I is the cone with vertex $\langle p_i \rangle_{i\in I}$ over $\mathrm{Sec}_{k-1}C_I$. Each E_I generates a 1*-dimensional face of* $\mathrm{Eff}(X)$*, and all* 1*-dimensional faces are of this form.*

In this section we exhibit integral divisors $D \subset X^n_{n+3}$ and rational numbers $\epsilon > 0$ such that $\Delta = \epsilon D$ makes X^n_{n+3} log Fano. In the previous cases, D was taken as sum of strict transforms of hyperplanes through n of the $n+3$ points. For X^n_{n+3}, we will also need to add other extremal divisors $E_I \subset X^n_{n+3}$ introduced in Paragraph 6.6. This will make the log resolution of (X, Δ) more complicated, and we will need to understand well how the divisors E_I's intersect. For this purpose, we start this section with some preliminaries on secant varieties of rational normal curves. Then we will consider separately the cases $n = 2h+1$ odd, and $n = 2h$ even.

6.4. Preliminaries on secant varieties of rational normal curves. Given an irreducible and reduced non-degenerate variety $X \subset \mathbb{P}^n$, and a positive integer $h \leq n$ we denote by $\mathbb{S}ec_k(X)$ the *k-secant variety* of X. This is the subvariety of $\mathbb{P}^n$ obtained as the closure of the union of all $(k-1)$-planes $\langle x_1, ..., x_k \rangle$ spanned by k general points of X. We will be concerned with the case when $X = C$ is a rational normal curve of degree n in $\mathbb{P}^n$. The following proposition gathers some of the basic properties of the secant varieties $\mathbb{S}ec_k(C)$ in this case.

PROPOSITION 6.7. *Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree n, and let k be an integer such that $1 \leq k \leq \frac{n}{2}$. Then the following hold.*

(1) $\dim(\mathbb{S}ec_k(C)) = 2k-1$ *(see for instance* [**Har**, Proposition 11.32]*).*
(2) $\deg(\mathbb{S}ec_k(C)) = \binom{n-k+1}{k}$ *(see for instance* [**EH**, Theorem 12.16]*).*
(3) $\mathbb{S}ec_k(C)$ *is normal and* $\operatorname{Sing}(\mathbb{S}ec_k(C)) = \mathbb{S}ec_{k-1}(C)$ *(see for instance* [**Ve1**, Theorem 1.1]*).*
(4) *If $n = 2h$ is even, then for any $1 \leq t < h$ we have*

$$\operatorname{mult}_{\mathbb{S}ec_{h-t}(C)} \mathbb{S}ec_h(C) = t+1.$$

PROOF OF (4). Suppose that $n = 2h$ is even, and consider the $(h+1) \times (h+1)$ matrix

$$M_h = \begin{pmatrix} x_0 & x_1 & \dots & x_h \\ x_1 & x_2 & \dots & x_{h+1} \\ \vdots & \vdots & \ddots & \vdots \\ x_h & x_{h+1} & \dots & x_{2h} \end{pmatrix}. \tag{6.3}$$

For any $1 \leq k \leq h$, the secant variety $\mathbb{S}ec_k(C)$ can be described as the determinantal variety:

$$\mathbb{S}ec_k(C) \;=\; \{\operatorname{rank}(M_h) \leq k\}.$$

(See for instance [**Har**, Proposition 9.7]). In particular, $\mathbb{S}ec_h(C) \subset \mathbb{P}^{2h}$ is the degree $h+1$ hypersurface defined by the polynomial $F := \det(M_h)$. For each $j \in \{0, ..., 2h\}$, let $\{M^j_i\}$ be the set of $h \times h$ minors of M_h produced by erasing in M_h a row and a column meeting in an entry of type x_j Denote by ρ_j be the number of such minors. Then

$$\frac{\partial F}{\partial x_j} = \sum_{i=1}^{\rho_j} \alpha^j_i \det(M^j_i),$$

for suitable $\alpha^j_i > 0$. Inductively, we see that for any $1 \leq t < h$ the partial derivatives of order t of F are linear combinations of determinants of $(h+1-t) \times (h+1-t)$ minors of M_h. The vanishing of such determinants defines $\mathbb{S}ec_{h-t}(C)$, while the vanishing of the of determinants of the $(h-t) \times (h-t)$ minors of M_h defines $\mathbb{S}ec_{h-t-1}(C) \subsetneq \mathbb{S}ec_{h-t}(C)$. Therefore, there is at least one partial derivative of order $t+1$ of F not vanishing on $\mathbb{S}ec_{h-t}(C)$. This means that $\operatorname{mult}_{\mathbb{S}ec_{h-t}(C)} \mathbb{S}ec_h(C) = t+1$ for any $1 \leq t < h$. $\square$

The following proposition is just a particular instance of [**Be**, Theorem 1]. The general statement for smooth curves embedded via a $2h$-very ample line bundle can be found in [**Ve**, Theorem 3.1] as well.

PROPOSITION 6.8. *Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree n, and set $h := \lfloor \frac{n}{2} \rfloor$. Consider the following sequence of blow-ups:*

- $\pi_1 : X_1 \to \mathbb{P}^n$ *the blow-up of C,*
- $\pi_2 : X_2 \to X_1$ *the blow-up of the strict transform of $\mathrm{Sec}_2(C)$,*

$\vdots$

- $\pi_h : X_h \to X_{h-1}$ *the blow-up of the strict transform of $\mathrm{Sec}_h(C)$.*

Let $\pi : X \to \mathbb{P}^n$ be the composition of these blow-ups. Then, for any $k \leq h$ the strict transform of $\mathrm{Sec}_k(C)$ in X_{k-1} is smooth and transverse to all exceptional divisors. In particular X is smooth and the exceptional locus of π is a simple normal crossing divisor.

NOTATION 6.9. Let $p_1, ..., p_{n+3} \in \mathbb{P}^n$ be general points, and let $C \subset \mathbb{P}^n$ be the unique rational normal curve of degree n through these points. Given $1 \leq m \leq n$, $I = \{i_1 < \cdots < i_m\} \subset \{1, \ldots, n+3\}$, and a positive integer k such that $0 \leq k \leq \frac{n-m}{2}$, we consider the following variety of dimension $d = 2k - 1 + m$:

$$Y_I^d := \mathrm{Join}\left(\langle p_{i_1}, \ldots, p_{i_m} \rangle , \mathrm{Sec}_k(C) \right).$$

Alternatively, Y_I^d can be defined as follows. Let $\pi_I : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-m}$ be the projection from the linear space $\langle p_{i_1}, \ldots, p_{i_m} \rangle$. Let $C_I \subset \mathbb{P}^{n-m}$ be the image of C under π_I. It is the the unique rational normal curve of degree $n-m$ through the points $\pi(p_j)$, $j \notin I$. Then Y_I^d is the cone with vertex $\langle p_{i_1}, \ldots, p_{i_m} \rangle$ over $\mathrm{Sec}_k(C_I)$.

By convent, when $k = 0$, we set $Y_I^{m-1} := \langle p_{i_1}, \ldots, p_{i_m} \rangle$.

Fix $I = \{i_1 < \cdots < i_m\} \subset \{1, \ldots, n+3\}$, with $m \leq n$. Given k such that $0 \leq k \leq \frac{n-m}{2}$, set $d := 2k - 1 + m$. By Proposition 6.7, we have

$$\deg(Y_I^d) = \binom{n-m-k+1}{k} \quad \text{and} \quad \mathrm{Sing}(Y_I^d) = Y_I^{d-2} \tag{6.4}$$

Moreover, if $n - m$ is even and $d_1 = 2k_1 - 1 + m > 2k_2 - 1 + m = d_2$, then $Y_I^{d_2} \subset Y_I^{d_1}$ and

$$\mathrm{mult}_{Y_I^{d_2}} Y_I^{d_1} = \frac{d_1 - d_2}{2} + 1. \tag{6.5}$$

We also have analogs of Proposition 6.8 for sequences of blow-ups of Y_I^d, for $|I| - 1 \leq d \leq n - 1$. More precisely:

PROPOSITION 6.10. *Let $C \subset \mathbb{P}^n$ be a rational normal curve of degree n, $p_1, \ldots, p_m \in C$ distinct points, with $1 \leq m \leq n$, and set $h := \lfloor \frac{n-m}{2} \rfloor$. Consider the following sequence of blow-ups:*

- $\pi_1 : X_1 \to \mathbb{P}^n$ *the blow-up of $Y_I^{m-1} := \langle p_1, \ldots, p_m \rangle$,*
- $\pi_2 : X_2 \to X_1$ *the blow-up of the strict transform of Y_I^{m+1},*

$\vdots$

- $\pi_h : X_h \to X_{h-1}$ *the blow-up of the strict transform of Y_I^{m+2h-1}.*

Let $\pi : X \to \mathbb{P}^n$ be the composition of these blow-ups. Then, for any $k \leq h$ the strict transform of Y_I^{m+2k-1} in X_{k-1} is smooth and transverse to all exceptional divisors.

Proposition 6.10 follows easily from Proposition 6.8. In the next sections, we will blow-up varieties of type Y_I^d for several subsets $I \subset \{1, \ldots, n+3\}$, in a suitable order. In order to show the smoothness and transversality of the strict transforms of the Y_I^d 's in the intermediate blow-ups, we will need the following result.

PROPOSITION 6.11. *Let $W \subset Z \subset X$ be smooth projective varieties, and let $Y \subset X$ be a projective variety such that* $\mathrm{Sing}(Y) = Z$ *and Y has ordinary singularities along Z. Let $\pi_W : X_W \to X$ be the blow-up of W, and denote by Z_W and Y_W the strict transforms of Z and Y, respectively. Then* $\mathrm{Sing}(Y_W) = Z_W$ *and Y_W has ordinary singularities along Z_W.*

PROOF. Denote by E_W the exceptional divisor of π_W. Then $\pi_W^{-1}(Z) = Z_W \cup E_W$. Let $\pi_{Z_W} : X_{Z_W} \to X_W$ be the blow-up of X_W along Z_W, with exceptional divisor E_{Z_W}.

We claim that the composite morphism $\pi_W \circ \pi_{Z_W} : X_{Z_W} \to X$ is isomorphic to the blow-up $\pi_Z : X_Z \to X$ of X along Z, followed by the blow-up of X_Z along $\pi_Z^{-1}(W)$. Indeed, by the universal property of the blow-up ([**Har**, Proposition 7.14]), there exits a unique morphism $f : X_{Z_W} \to X_Z$ making the following diagram commute.

$$\begin{tikzcd} X_{Z_W} \arrow[r, "f"] \arrow[d, "\pi_{Z_W}"'] & X_Z \arrow[d, "\pi_Z"] \\ X_W \arrow[r, "\pi_W"] & X \end{tikzcd}$$

Note that all varieties in this diagram are smooth. Since Z and W are smooth, the intersection $Z_W \cap E_W \subset X_W$ is smooth. Thus, any normal direction of Z_W in X_W at a point $p \in Z_W \cap E_W$ is the image of a normal direction at p of $Z_W \cap E_W$ in E_W. In other words, the inverse image of W in X_{Z_W} consists of the strict transform $\widetilde{E}_W$ of E_W in X_{Z_W}. Therefore, the inverse image of the smooth variety $\pi_Z^{-1}(W)$ in X_W is precisely $\widetilde{E}_W$. Using the the universal property of the blow-up, and comparing the Picard number of these smooth varieties, we conclude that $f : X_{Z_W} \to X_Z$ is the blow-up of X_Z along $\pi_Z^{-1}(W)$, proving the claim.

Next we prove that $\mathrm{Sing}(Y_W) = Z_W$. Clearly $Z_W \subset \mathrm{Sing}(Y_W)$. Suppose that this inclusion is strict. Then the strict transform Y_{Z_W} of Y_W in X_{Z_W} is singular. Since $f : X_{Z_W} \to X_Z$ is a smooth blow-up, $f(Y_{Z_W}) \subset X_Z$ is singular as well. But notice that $f(Y_{Z_W}) \subset X_Z$ is the strict transform of $Y \subset X$ via π_Z. Since $\mathrm{Sing}(Y) = Z$ and Y has ordinary singularities along Z, the blow-up π_Z resolves the singularities of Y. This contradiction shows that $\mathrm{Sing}(Y_W) = Z_W$. Moreover, since Y has ordinary singularities along Z, the intersection of its strict transform Y_Z with the exceptional divisor E_Z of π_Z is transverse. This implies that the intersection $Y_{Z_W} \cap E_{Z_W}$ is also transverse, i.e., Y_W has ordinary singularities along Z_W. □

We end this section by describing the intersection of some of the Y_I^d's. This can be computed using elementary projective geometry. In what follows we adopt the following notation. Given two finite sets I and J, we define their distance to be

$$d(I, J) \; := \; \big|(I \cup J) \setminus (I \cap J)\big|.$$

We start by intersecting varieties Y_I^d's with the same dimension.

PROPOSITION 6.12. *Let the assumptions and notation be as in Notation 6.9. Let $I_1, I_2 \subset \{1, \ldots, n+3\}$ be subsets with cardinality m_1 and m_2, respectively, and suppose that $I_1 \cap I_2 = \emptyset$. Let k_1 and k_2 be* *integers such that* $0 \leq k_i \leq \frac{n-m_i}{2}$, $i = 1, 2$, *and* $m_1 + 2k_1 - 1 = m_2 + 2k_2 - 1 =: d$. *Set* $s = \frac{m_1+m_2}{2}$ *and suppose that* $d \leq n - s$. *Then*

$$Y^d_{I_1} \cap Y^d_{I_2} = \bigcup_J Y^{d-s}_J,$$

where the union is taken over all subsets $J \subset I_1 \cup I_2$ *satisfying* $d(I_i, J) = s$ *for* $i = 1, 2$.

Moreover, for a general point in any irreducible component of the above intersections, the intersection is transverse.

PROOF. We note that the assumptions of the theorem imply that $d = k_1 + k_2 + s - 1$ and $m_1 - m_2 = 2(k_2 - k_1)$.

Let $J \subset I_1 \cup I_2$ be such that $d(I_i, J) = s$ for $i = 1, 2$. We shall prove that $Y^{d-s}_J \subset Y^d_{I_1} \cap Y^d_{I_2}$. Write $J = J_1 \cup J_2$, where $J_i \subset I_i$, $i = 1, 2$, set $\ell_i := |J_i|$, $i = 1, 2$, and $\ell = |J| = \ell_1 + \ell_2$. The assumption that $d(I_i, J) = s$ for $i = 1, 2$ implies that $k_2 - k_1 = \ell_1 - \ell_2$. We set $k := k_2 - \ell_1 = k_1 - \ell_2$, and note that $d - s = \ell + 2k - 1$.

Let $x \in Y^{d-s}_J$. Then there exists a point $q \in \mathbb{S}ec_k(C)$ such that $x \in \langle\, q, p_i \mid i \in J \,\rangle \cong \mathbb{P}^\ell$. The following two linear subspaces of this $\mathbb{P}^\ell$

$$\langle\, x, p_i \mid i \in I_1 \,\rangle \cong \mathbb{P}^{\ell_1} \text{ and } \langle\, q, p_i \mid i \in I_2 \,\rangle \cong \mathbb{P}^{\ell_2}$$

have complementary dimensions. Hence there exists a point

$$z \in \langle\, x, p_i \mid i \in J_1 \,\rangle \cap \langle\, q, p_i \mid i \in J_2 \,\rangle.$$

In particular, $z \in \mathbb{S}ec_{k+\ell_2}(C)$. Since $k + \ell_2 = k_1$, we conclude that $x \in Y^d_{I_1}$. Similarly we show that $x \in Y^d_{I_2}$.

Now assume that x is a general point of Y^{d-s}_J. Keeping the same notation as above, we will prove now that $Y^d_{I_1}$ and $Y^d_{I_2}$ intersect transversely at x. This amounts to proving that $T_x(Y^d_{I_1}) \cap T_x(Y^d_{I_2}) = T_x(Y^{d-s}_J)$. By Terracini's Lemma **[Te]**, we have

$$\begin{aligned}
T_x(Y^d_{I_1}) &= \langle\, \langle p_i \mid i \in I_1 \,\rangle, \langle\, T_{q_i}C \mid 1 \leq i \leq k \,\rangle, \langle\, T_{p_i}C \mid i \in J_2 \,\rangle \,\rangle,\\
T_x(Y^d_{I_2}) &= \langle\, \langle p_i \mid i \in I_2 \,\rangle, \langle\, T_{q_i}C \mid 1 \leq i \leq k \,\rangle, \langle\, T_{p_i}C \mid i \in J_1 \,\rangle \,\rangle,\\
T_x(Y^{d-s}_J) &= \langle\, \langle p_i \mid i \in J \,\rangle, \langle\, T_{q_i}C \mid 1 \leq i \leq k \,\rangle \,\rangle,
\end{aligned}$$

where $q_1, \ldots, q_k \in C$ are such that $q \in \langle\, q_i \mid 1 \leq i \leq k \,\rangle$.

Consider the linear subspaces:

$$\begin{aligned}
L_1 &:= \langle\, \langle p_i \mid i \in I_1 \,\rangle, \langle\, T_{p_i}C \mid i \in J_2 \,\rangle \,\rangle,\\
L_2 &:= \langle\, \langle p_i \mid i \in I_2 \,\rangle, \langle\, T_{p_i}C \mid i \in J_1 \,\rangle \,\rangle,\\
L &:= \langle\, \langle p_i \mid i \in J \,\rangle \,\rangle \subset L_1 \cap L_2.
\end{aligned}$$

We have that $\dim(\langle L_1, L_2 \rangle) \leq m_1 + m_2 + \ell - 1$, and equality holds if and only if $L_1 \cap L_2 = L$. On the other hand, note that L intersects C in at least $m_1 + m_2 + \ell$ points, counted with multiplicity. Therefore we must have $\dim(\langle L_1, L_2 \rangle) = m_1 + m_2 + \ell - 1$, and $L_1 \cap L_2 = L$. It follows from the description of the tangent spaces above that $T_x(Y^d_{I_1}) \cap T_x(Y^d_{I_2}) = T_x(Y^{d-s}_J)$.

It remains to prove that $Y^d_{I_1} \cap Y^d_{I_2} \subset \bigcup_J Y_J^{d-s}$. Write $\{p_i \mid i \in I_1\} = \{x_1, \ldots, x_{m_1}\}$ and $\{p_i \mid i \in I_2\} = \{y_1, \ldots, y_{m_2}\}$. Suppose that $x \in Y^d_{I_1} \cap Y^d_{I_2}$. This means that there exist points $z_1, \ldots, z_{k_1}, w_1, \ldots, w_{k_2} \in C$ such that:

$$\langle x_1, \ldots, x_{m_1}\rangle \cap \langle z_1, \ldots, z_{k_1}\rangle = \emptyset = \langle y_1, \ldots, y_{m_2}\rangle \cap \langle w_1, \ldots, w_{k_2}\rangle, \text{ and}$$
$$x \in \langle x_1, \ldots, x_{m_1}, z_1, \ldots, z_{k_1}\rangle \cap \langle y_1, \ldots, y_{m_2}, w_1, \ldots, w_{k_2}\rangle.$$

The assumption that $d \leq n - s$ implies that $m_1 + m_2 + k_1 + k_2 \leq n + 1$, and thus

$$\langle x_1, \ldots, x_{m_1}, z_1, \ldots, z_{k_1}\rangle \cap \langle y_1, \ldots, y_{m_2}, w_1, \ldots, w_{k_2}\rangle =$$
$$\langle \{x_1, \ldots, x_{m_1}, z_1, \ldots, z_{k_1}\} \cap \{y_1, \ldots, y_{m_2}, w_1, \ldots, w_{k_2}\} \rangle.$$

By relabeling the points if necessary, we may write, for suitable integers s_1, s_2 and r:

$$\{x_1, \ldots, x_{s_1}\} = \{x_1, \ldots, x_{m_1}\} \cap \{w_1, \ldots, w_{k_2}\}$$
$$\{y_1, \ldots, y_{s_2}\} = \{y_1, \ldots, y_{m_2}\} \cap \{z_1, \ldots, z_{k_1}\}$$
$$\{z_1 = w_1, \ldots, z_r = w_r\} = \{z_1, \ldots, z_{k_1}\} \cap \{w_1, \ldots, w_{k_2}\}.$$

Note that $s_i + r \leq k_j$, $\{i, j\} = \{1, 2\}$, and we have

$$x \in \langle x_1, \ldots, x_{s_1}, y_1, \ldots, y_{s_2}, z_1, \ldots, z_r \rangle. \tag{6.6}$$

Let $J_0 \subset I_1 \cup I_2$ be the subset corresponding to the points $\{x_1, \ldots, x_{s_1}, y_1, \ldots, y_{s_2}\} \subset \{p_1, ..., p_{n+3}\}$. Note that $d(J_0, I_i) = m_i - s_i + s_j$, for $\{i, j\} = \{1, 2\}$. In particular we have

$$d(J_0, I_1) + d(J_0, I_2) = 2s.$$

Suppose first that $d(J_0, I_1) = d(J_0, I_2) = s$. It follows from (6.6) that

$$x \in \text{Join}\left(\langle p_i \mid i \in J_0 \rangle , \mathcal{S}ec_r(C) \right).$$

Since $s_i + r \leq k_j$, $\{i, j\} = \{1, 2\}$, we get that

$$|J_0| + 2r - 1 = s_1 + s_2 + 2r - 1 \leq k_1 + k_2 - 1 = d - s.$$

Hence $x \in Y^{d-s}_{J_0}$.

From now on we consider the case when $d(J_0, I_1) \neq d(J_0, I_2)$. Without lost of generality, we assume that

$$d(J_0, I_1) - d(J_0, I_2) = m_1 - m_2 + 2s_2 - 2s_1 > 0.$$

We will modify the subset $J_0 \subset I_1 \cup I_2$ by adding points of $I_1 \setminus J_0$ or removing points of $I_2 \cap J_0$ to obtain another subset $J \subset I_1 \cup I_2$ satisfying $d(I_i, J) = s$ for $i = 1, 2$. Note that if $i \in I_1 \setminus J_0$, then $d(J_0 \cup \{i\}, I_1) = d(J_0, I_1) - 1$ and $d(J_0 \cup \{i\}, I_2) = d(J_0, I_2) + 1$. Similarly, if $i \in I_2 \cap J_0$, then $d(J_0 \setminus \{i\}, I_1) = d(J_0, I_1) - 1$ and $d(J_0 \setminus \{i\}, I_2) = d(J_0, I_2) + 1$. So we have to modify J_0 by adding or removing exactly $\frac{m_1 - m_2}{2} + s_2 - s_1$ points of the appropriate I_i.

Suppose first that $|I_1 \setminus J_0| = m_1 - s_1 \geq \frac{m_1 - m_2}{2} + s_2 - s_1$. This is equivalent to the inequality $s \geq s_2$. We construct $J_1 \subset I_1 \cup I_2$ by adding to J_0 $\frac{m_1 - m_2}{2} + s_2 - s_1$ points of $I_1 \setminus J_0$. Then $d(I_i, J_1) = s$ for $i = 1, 2$, and it follows from (6.6) that

$$x \in \text{Join}\left(\langle p_i \mid i \in J_1 \rangle , \mathcal{S}ec_r(C) \right).$$

Since $s_2 + r \leq k_1$, we get that

$$|J_1| + 2r - 1 = (k_2 - k_1 + 2s_2) + 2r - 1 \leq k_1 + k_2 - 1 = d - s.$$

Hence $x \in Y^{d-s}_{J_1}$.

Next we suppose that $s < s_2$. Let $I_2' \subset I_2$ be the subset corresponding to the points $\{y_1, \dots, y_s\}$, and set $J_2 := I_1 \cup I_2'$. Then $d(I_i, J_2) = s$ for $i = 1, 2$, and it follows from (6.6) that

$$x \in \mathrm{Join}\left(\langle p_i \mid i \in J_2 \rangle , \mathcal{S}ec_{r+s_2-s}(C) \right).$$

Since $s_2 + r \leq k_1$, we get that

$$|J_2| + 2(r + s_2 - s) - 1 \;=\; m_1 + 2(r + s_2) - s - 1 \;\leq\; m_1 + 2k_1 - 1 - s \;=\; d - s.$$

Hence $x \in Y^{d-s}_{J_2}$.

□

6.5. The odd case $n = 2h + 1$. In this subsection we construct divisors Δ making X^n_{n+3} log Fano when $n = 2h + 1$ is odd. In order to clarify the ideas we begin by developing an example. The case $n = 3$ is in Section 5.1. Therefore, the first non trivial case is $n = 5$. Let $\pi_i : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ be the projection from p_i, and let $C_i \subset \mathbb{P}^4$ be the unique rational normal curve of degree four passing through $\pi_i(p_j)$ for $j \neq i$. By Proposition 6.7 the secant variety $\mathcal{S}ec_2(C_i) \subset \mathbb{P}^4$ is an hypersurface of degree three, $\mathrm{Sing}(\mathcal{S}ec_2(C_i)) = C_i$ and $\mathrm{mult}_{C_i}(\mathcal{S}ec_2(C_i)) = 2$. Let Δ_i be the cone over $\mathcal{S}ec_2(C_i)$, and Γ^2_i be the cone over C_i with vertex p_i, that is

$$\Delta_i = \mathrm{Join}(p_i, \mathcal{S}ec_2(C)), \quad \Gamma^2_i = \mathrm{Join}(p_i, C).$$

We denote by D_i the strict transform of Δ_i in X^5_8.

LEMMA 6.13. *For any $i = 1, ..., 8$ we have*

$$\deg(\Delta_i) = 3, \quad \mathrm{mult}_{p_i} \Delta_i = 3, \quad \mathrm{mult}_{\Gamma^2_i} \Delta_i = 2.$$

Furthermore, let $\pi : Y \to X^5_8$ be the blow-up of X^5_8 along the strict transform of Γ^2_i. Then the strict transform of D_i in Y is smooth and transversal to the exceptional divisor of π over the strict transform of Γ^2_i.

PROOF. Since Δ_i is a cone over $\mathcal{S}ec_2(C_i)$ by Proposition 6.7 we have $\deg(\Delta_i) = \deg(\mathcal{S}ec_2(C_i)) = 3$, $\mathrm{mult}_{p_i} \Delta_i = \deg(\mathcal{S}ec_2(C_i)) = 3$, $\mathrm{mult}_{\Gamma^2_i} \Delta_i = \mathrm{mult}_{C_i} \mathcal{S}ec_2(C_i) = 2$.
Now, the projection $\pi_i : \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ from p_i lifts to a morphism $Bl_{p_i}\mathbb{P}^5 \to \mathbb{P}^4$ and therefore induces a morphism $\widetilde{\pi}_i : X^5_8 \to \mathbb{P}^4$. Let Z be the blow-up of $\mathbb{P}^4$ along C_i. Since $\widetilde{\pi}_i^{-1}(C_i)$ is the strict transform of Γ^2_i, by [**Har**, Corollary 7.15] there exists a unique morphism $f_i : Y \to Z$ such that the following diagram

$$\begin{tikzcd} Y \arrow[r, "f_i"] \arrow[d] & Z \arrow[d] \\ X^5_8 \arrow[r, "\widetilde{\pi}_i"] & \mathbb{P}^4 \end{tikzcd}$$

is commutative. Therefore, if $\widetilde{\mathcal{S}ec_2(C_i)}$ is the strict transform of $\mathcal{S}ec_2(C_i)$ in Z we have $f_i^{-1}(\widetilde{\mathcal{S}ec_2(C_i)}) = D_i$. Now, to conclude it is enough to observe that by Proposition 6.8 $\widetilde{\mathcal{S}ec_2(C_i)}$ is smooth and transversal to the exceptional divisor over C_i. □

Now, let $H_{4,...,8} \subset \mathbb{P}^5$ be the hyperplane spanned by $p_4, ..., p_8$, and consider the divisor

$$\Delta = \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup H_{4,...,8}.$$

Note that $\Delta_i \cap \Delta_j$ is a 3-fold of degree nine. Let $C \subset \mathbb{P}^5$ be the rational normal curve of degree five through $p_1, ..., p_8$, $L_{i,j}$ the line spanned by p_i and p_j, and $C_{i,j} = \pi_{i,j}(C)$, where $\pi_{i,j} : \mathbb{P}^5 \dashrightarrow \mathbb{P}^3$ is the projection from $L_{i,j}$. Let $Y^3_{i,j} = \mathrm{Join}(L_{i,j}, C)$ be the cone over $C_{i,j}$ with vertex $L_{i,j}$. Then $\mathbb{S}ec_2(C)$ and $Y^3_{i,j}$ are both contained in $\Delta_i \cap \Delta_j$. Furthermore $\deg(\mathbb{S}ec_2(C)) = 6$ and $\deg(Y^3_{i,j}) = 3$ yield

$$\Delta_i \cap \Delta_j = \mathbb{S}ec_2(C) \cup Y^3_{i,j} \tag{6.7}$$

scheme-theoretically. We denote by D the strict transform of Δ in X^5_8.

PROPOSITION 6.14. *Let us consider the following chain of blow-ups:*

- *blow-up the strict transforms in X^5_8 of the lines $L_{1,2}, L_{1,3}, L_{2,3}$ and of the rational normal curve C,*
- *blow-up the strict transforms of $\Gamma^2_1, \Gamma^2_2, \Gamma^2_3$,*
- *blow-up the strict transform of $\mathbb{S}ec_2(C)$,*

and let $\pi : Y \to X^5_8$ be the composition of these blow-ups. Then π is a log resolution of the pair (X^5_8, D).

PROOF. Let as assume that there exists a point $p \in \Gamma^2_i \cap \Gamma^2_j$ with $p \notin C \cup L_{i,j}$. This means that there are two lines $L_{p_i,q_1} = \langle p_i, q_1 \rangle$, $L_{p_j,q_2} = \langle p_j, q_2 \rangle$ such that $q_1, q_2 \in C$ and $p \in L_{p_i,q_1} \cap L_{p_j,q_2}$. Therefore, the plane spanned by L_{p_i,q_1} and L_{p_j,q_2} intersects C in at least four points. A contradiction because $\deg(C) = 5$. Now, let $p \in C$ be a general point and assume that $\mathbb{T}_p\Gamma^2_i = \mathbb{T}_p\Gamma^2_j$. Then, $p_i, p_j \in \mathbb{T}_p\Gamma^2_i \cap C$ and $\mathbb{T}_p\Gamma^2_i$ intersects C in p with multiplicity at least two. This means that $\mathbb{T}_p\Gamma^2_i$ is a plane intersecting C in at least four points counted with multiplicity. Again we find a contradiction. Now, since the strict transforms $\widetilde{C}$ and $\widetilde{L}_{i,j}$ of C and $L_{i,j}$ in X^5_8 are smooth and disjoint we conclude that the strict transforms of Γ^2_i and Γ^2_j in X^5_8 are smooth and intersects transversally along the disjoint union $\widetilde{C} \cup \widetilde{L}_{i,j}$.
Next, we want to prove that $\mathbb{S}ec_2(C) \cap Y^3_{i,j} = \Gamma^2_i \cup \Gamma^2_j$. Assume that there is a point $p \in \mathbb{S}ec_2(C) \cap Y^3_{i,j}$ with $p \notin \Gamma^2_i \cup \Gamma^2_j$. Then, there is are a secant line $L_{q,t} = \langle q, t \rangle$ with $q, t \in C$ and a line $L_{r,s} = \langle r, s \rangle$ with $r \in L_{i,j}$ and $s \in C$ such that $p \in L_{q,t} \cap L_{r,s}$. The 3-plane spanned by $L_{q,t}, L_{r,s}$ and $L_{i,j}$ intersects C in at least five points. A contradiction. Now, let $p \in \Gamma^2_i \cup \Gamma^2_j$ be a general point and assume that $\mathbb{T}_p\mathbb{S}ec_2(C) = \mathbb{T}_pY^3_{i,j}$. We may assume that p lies on a secant line $L_{p_i,q} = \langle p_i, q \rangle$ with $q \in C$. Note that $p_j \in L_{i,j} \subset \mathbb{T}_pY^3_{i,j}$. By Terracini's Lemma 1.1 we have that $\mathbb{T}_{p_i}C, \mathbb{T}_qC \subset \mathbb{T}_p\mathbb{S}ec_2(C)$. This means that $\mathbb{T}_p\mathbb{S}ec_2(C) = \mathbb{T}_pY^3_{i,j}$ is a 3-plane intersecting C in at least five points counted with multiplicity. A contradiction.
Now, after blowing-up $L_{i,j}$ and C we know that the strict transforms $\widetilde{\Gamma}^2_i, \widetilde{\Gamma}^2_j$ of Γ^2_i and Γ^2_j are smooth and do not intersect. Furthermore,

$$\widetilde{\mathbb{S}ec_2(C)} \cap \widetilde{Y}_{i,j} = \widetilde{\Gamma}^2_i \cup \widetilde{\Gamma}^2_j.$$

Now, we blow-up the $\widetilde{\Gamma}^2_i$'s. By the previous part of the proof, Proposition 6.8 and Lemma 6.13 we know that now $\widetilde{\mathbb{S}ec_2(C)}$ and $\widetilde{Y}_{i,j}$, where we keep the same notations for the strict transforms on this further blow-up, are smooth, disjoint and intersects transversally along all the exceptional divisors. By equation 6.7 we get that $\widetilde{\Delta}_i$ and $\widetilde{\Delta}_j$ are smooth and intersects transversally along the disjoint union $\widetilde{\mathbb{S}ec_2(C)} \cup \widetilde{Y}_{i,j}$.
Let us consider a third cone Δ_k. Since the p_i's are general Δ_k does not contain $Y^3_{i,j}$. On the other

hand $\mathcal{S}ec_2(C) \subset \Delta_k$. At the last step we blow-up $\mathcal{S}ec_2(C)$. Clearly the hyperplane $H_{4,\dots,8}$ intersects transversally C in $p_4, \dots, p_8$ and intersects transversally all the subvarieties that have been blown-up. Finally, by Proposition 6.8, and Lemma 6.13 we conclude that the divisor

$$\widetilde{\Delta}_1 \cup \widetilde{\Delta}_2 \cup \widetilde{\Delta}_3 \cup \widetilde{H}_{4,\dots,8} \cup \mathrm{Exc}(\pi)$$

in Y is simple normal crossing. □

PROPOSITION 6.15. *The variety X^5_8 is log Fano.*

PROOF. Let us consider the divisor $D \subset X^5_8$ that is the strict transform of $\Delta = \Delta_1 + \Delta_2 + \Delta_3 + H_{4,\dots,8}$ in X^5_8. By Lemma 6.13 we have

$$\deg(\Delta) = 10, \quad \mathrm{mult}_{p_i} \Delta = 7.$$

Therefore, we have

$$D = 10H - 7(E_1 + \dots + E_8)$$

and

$$-(K_{X^5_8} + \epsilon D) = (6 - 10\epsilon)H - (4 - 7\epsilon)(E_1 + \dots + E_8).$$

By Proposition 4.8 in order two find a rational number ϵ such that $-(K_{X^5_8} + \epsilon D)$ is ample we have two consider its intersection with the curves of type $L_{i,j}$ and R_i. We have

$$-(K_{X^5_8} + \epsilon D) \cdot R_i = 4 - 7\epsilon$$

and

$$-(K_{X^5_8} + \epsilon D) \cdot L_{i,j} = 6 - 10\epsilon - 2(4 - 7\epsilon) = 4\epsilon - 2.$$

Therefore, for any $\frac{1}{2} < \epsilon < \frac{4}{7}$ the divisor $-(K_{X^5_8} + \epsilon D)$ is ample.
Now, by Proposition 6.14 we know that $\pi : Y \to X^5_8$ is a log resolution of the pair $(X^5_8, \epsilon D)$. We denote by $E_{L_{i,j}}$, E_C, $E_{\Gamma^2_i}$, $E_{\mathcal{S}ec_2(C)}$ the exceptional divisors over $L_{i,j}$, C, Γ^2_i and $\mathcal{S}ec_2(C)$ respectively. The canonical divisor of Y is given by

$$K_Y = \pi^* K_{X^5_8} + 3(E_{L_{1,2}} + E_{L_{1,3}} + E_{L_{2,3}}) + 3E_C + 2(E_{\Gamma^2_1} + E_{\Gamma^2_2} + E_{\Gamma^2_3}) + E_{\mathcal{S}ec_2(C)}.$$

Now, $L_{i,j}$ has multiplicity two for Δ_i, Δ_j and one for Δ_k with $k \neq i, j$. The curve C has multiplicity two for any Δ_i. The cone Γ^2_i has multiplicity two for Δ_i and one for Δ_j with $j \neq i$. Finally $\mathcal{S}ec_2(C)$ has multiplicity one for any Δ_i. Then we may write

$$\pi^*(\epsilon D) = \epsilon \widetilde{D} + 5\epsilon(E_{L_{1,2}} + E_{L_{1,3}} + E_{L_{2,3}}) + 6\epsilon E_C + 4\epsilon(E_{\Gamma^2_1} + E_{\Gamma^2_2} + E_{\Gamma^2_3}) + 3\epsilon E_{\mathcal{S}ec_2(C)}$$

and

$$K_Y = \pi^*(K_{X^5_8} + \epsilon D) + (3 - 5\epsilon)\sum_{i,j} E_{L_{i,j}} + (3 - 6\epsilon)E_C + (2 - 4\epsilon)\sum_i E_{\Gamma^2_i} + (1 - 3\epsilon)E_{\mathcal{S}ec_2(C)} - \epsilon \widetilde{D}.$$

Therefore, for any $\epsilon < \frac{2}{3}$ all the discrepancies are greater than -1. We conclude that for any $\frac{1}{2} < \epsilon < \frac{4}{7}$ the divisor $-(K_{X^5_8} + \epsilon D)$ is ample and the pair $(X^5_8, \epsilon D)$ is klt. □

Let us move to the general case. We follow Notation 6.9. For each $1 \leq i \leq 3$, let $\Delta_i \subset X^n_{n+3}$ be the strict transform of the divisor $Y^{2h}_i \subset \mathbb{P}^n$, and denote by $H_{4,\dots,n+3} \subset X^n_{n+3}$ the strict transform of the hyperplane $\langle p_4, \dots, p_{n+3} \rangle \subset \mathbb{P}^{n+3}$.

THEOREM 6.16. *Let $n = 2h + 1 \geq 5$ be an odd integer. Set*

$$D := \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup H_{4,\dots,n+3} \subset X^n_{n+3}.$$

For any $\frac{2h-2}{3h-2} < \epsilon < \frac{2h}{3h+1}$ the divisor $-(K_{X^n_{n+3}} + \epsilon D)$ is ample, and the pair $(X^n_{n+3}, \epsilon D)$ is klt.

For the proof of Theorem 6.16, we will need the following.

PROPOSITION 6.17. *Let the assumptions be as in Theorem 6.16, and follow Notation 6.9. For $0 \leq m \leq n-3$, we define a modification X_m of X^n_{n+3} recursively as follows:*

- *$X_0 = X^n_{n+3}$,*
- *X_{2k+1} is the blow-up of X_{2k} along the strict transforms of $\mathcal{S}ec_{k+1}(C)$, and of the $Y^{2k+1}_{i,j}$'s ($0 \leq k \leq h-2$),*
- *X_{2k} is the blow-up of X_{2k-1} along the strict transforms of the Y^{2k}_i's, and of $Y^{2k}_{1,2,3}$ ($1 \leq k \leq h-1$),*
- *X_{n-2} is the blow-up of X_{n-3} along the strict transform of $\mathcal{S}ec_h(C)$.*

Then for any k the strict transforms of $\mathcal{S}ec_{k+1}(C)$ and of the $Y^{2k+1}_{i,j}$'s in X_{2k}, and of the Y^{2k}_i's and $Y^{2k}_{1,2,3}$ in X_{2k-1} are smooth, disjoint and intersect transversally all the exceptional divisors.
In particular, let $\pi : X_{n-2} \to X^n_{n+3}$ be the composition of these blow-ups. Then π is a log resolution of the pair (X^n_{n+3}, D).

PROOF. We proceed by induction on X_m. The center of the blow-up $X_{m+1} \to X_m$ is a disjoint union of smooth subvarieties, all transverse to the exceptional divisors of $X_m \to X_0$. For simplicity of notation we will denote by $\widetilde{Z}$ the strict transform of a subvariety $Z \subset X^n_{n+3}$ in any X_m.
First of all note that by Proposition 6.11 the blow-up $p_1, ..., p_{n+3}$ resolves the vertex singularity of the Y^d_i and does not produce any effect on the singularities of the other Y^d_l's involved in the resolution. Since the p_i's have been blown-up the strict transforms of C and of the lines $Y^1_{i,j}$ do not intersect in X_0 and the statement is verified for $m = 0$. Now, by Proposition 6.12 in X_0 we have

$$\widetilde{Y}^2_i \cap \widetilde{Y}^2_j = \widetilde{C} \cup \widetilde{Y}^1_{i,j}, \quad \widetilde{Y}^2_i \cap \widetilde{Y}^2_{1,2,3} = \widetilde{Y}^1_{i,r} \cup \widetilde{Y}^1_{i,s}.$$

By blowing-up the $\widetilde{Y}^1_{i,j}$'s and $\widetilde{C}$ we separate the $\widetilde{Y}^2_i$'s and $\widetilde{Y}^2_{1,2,3}$. In X_1 the $\widetilde{Y}^2_i$'s and $\widetilde{Y}^2_{1,2,3}$ are smooth and disjoint. So the statement is verified for $m = 1$
Now, suppose that the statement is true $m = 2k$. We will show that it holds for X_{2k+1} and X_{2k+2}. By Propositions 6.8 and 6.10, we know that the subvarieties $\widetilde{Y}^{2k+2}_i \subset X_{2k+1}$, $1 \leq i \leq 3$, and $\widetilde{\mathcal{S}ec_{k+2}}(C)$, $\widetilde{Y}^{2k+3}_{i,j} \subset X_{2k+2}$, $1 \leq i < j \leq 3$, are all smooth and transverse to the exceptional divisors over X_0.
It remains to show that the the $\widetilde{Y}^{2k+2}_i$'s and $\widetilde{Y}^{2k+2}_{1,2,3}$ are pairwise disjoint in X_{2k+1}, and similarly for $\widetilde{\mathcal{S}ec_{k+2}}(C)$ and the $\widetilde{Y}^{2k+3}_{i,j}$'s in X_{2k+2}.
Consider the blow-up $X_{2k+1} \to X_{2k}$. By Proposition 6.12, on X_{2k} we have

$$\widetilde{Y}^{2k+2}_i \cap \widetilde{Y}^{2k+2}_j = \widetilde{\mathcal{S}ec_{k+1}}(C) \cup \widetilde{Y}^{2k+1}_{i,j}, \quad \widetilde{Y}^{2k+2}_i \cap \widetilde{Y}^{2k+2}_{i,r,s} = \widetilde{Y}^{2k+1}_{i,r} \cup \widetilde{Y}^{2k+1}_{i,s}.$$

By the induction hypothesis, $\widetilde{\mathcal{S}ec_{k+1}}(C)$ and $\widetilde{Y}^{2k+1}_{i,j}$ are smooth and disjoint. So the intersection is everywhere transverse. We conclude that on X_{2k+1}, which is obtained from X_{2k} by blowing-up $\widetilde{\mathcal{S}ec_{k+1}}(C)$ and $\widetilde{Y}^{2k+1}_{i,j}$, the $\widetilde{Y}^{2k+2}_i$'s and $\widetilde{Y}^{2k+2}_{1,2,3}$ are pairwise disjoint. Note that the $\widetilde{Y}^{2k+2}_i$'s and

$\widetilde{Y}^{2k+2}_{1,2,3}$ are smooth because their singular loci have been blown-up in the preceding step. Furthermore, keeping in mind that $\operatorname{Sing}(\mathbb{S}ec_t(C)) = \mathbb{S}ec_{t-1}(C)$, $\operatorname{Sing}(Y_i^{2t+2}) = Y_i^{2t}$, $\operatorname{Sing}(Y_{i,j}^{2t+1}) = Y_{i,j}^{2t-1}$, $\operatorname{Sing}(Y_{1,2,3}^{2t+2}) = Y_{1,2,3}^{2t}$, we have the following.

CLAIM 6.18. We have $\mathbb{S}ec_{k+1}(C) \subset \mathbb{S}ec_{t-1}(C)$ for $t \geq k+2$, $\mathbb{S}ec_{k+1}(C) \subset Y_i^{2t+2}$ for $t \geq k$, $\mathbb{S}ec_{k+1}(C) \subset Y_{i,j}^{2t+1}$ for $t \geq k+1$, $\mathbb{S}ec_{k+1}(C) \subset Y_{1,2,3}^{2t+2}$ for $t \geq k+1$, and $\mathbb{S}ec_{k+1} \cap Y_{1,2,3}^{2k+2} = Y_1^{2k} \cup Y_2^{2k} \cup Y_3^{2k}$.
For any $i,r,s \in \{1,2,3\}$ and $t \geq k+1$ we have $Y_{i,s}^{2k+1} \subset Y_{r,s}^{2t+1}$ while $Y_{i,s}^{2k+1} \cap Y_{r,s}^{2k+1} = Y_s^{2k} \cup Y_{i,r,s}^{2k}$. Furthermore, $Y_{i,j}^{2k+1} \subset \operatorname{Sing}(\mathbb{S}ec_t(C)) = \mathbb{S}ec_{t-1}(C)$ for any $t \geq k+3$, while $Y_{i,j}^{2k+1} \cap \operatorname{Sing}(\mathbb{S}ec_{k+2}(C)) = Y_i^{2k} \cup Y_j^{2k}$, and $Y_{i,j}^{2k+1} \subset Y_{1,2,3}^{2t+2}$ for any $t \geq k$. Finally, $Y_{i,j}^{2k+1} \subset Y_r^{2t+2}$ for any $i,j,r \in \{1,2,3\}$ and $t > k+1$, and for $t = k+1$ we have

$$Y_{i,j}^{2k+1} \cap \operatorname{Sing}(Y_r^{2k+4}) = \begin{cases} Y_{i,j}^{2k+1} & \text{if } r \in \{i,j\}, \\ Y_i^{2k} \cup Y_j^{2k} \cup Y_{i,j,r}^{2k} & \text{if } r \notin \{i,j\}. \end{cases}$$

Moreover, for a general point in any irreducible component of the above intersections, the intersection is transverse.

PROOF. Let us prove the last equality. The others can be proved by similar arguments. We have $Y_{i,j}^{2k+1} \cap Y_r^{2k+2} = (Y_{i,j,r}^{2k+2} \cap Y_r^{2k+2}) \cap Y_{i,j}^{2k+1}$. Now $Y_{i,j,r}^{2k+2} \cap Y_r^{2k+2}$ is the cone with vertex p_r over $Y_{i,j}^{2k+1} \cap \mathbb{S}ec_{k+1}(C)$, and by Proposition 6.12 the last intersection is given by $Y_i^{2k} \cup Y_j^{2k}$. Therefore, $Y_{i,j,r}^{2k+2} \cap Y_r^{2k+2} = Y_{i,r}^{2k+1} \cup Y_{j,r}^{2k+1}$ and by Proposition 6.12 $(Y_{i,r}^{2k+1} \cup Y_{j,r}^{2k+1}) \cap Y_{i,j}^{2k+1} = Y_i^{2k} \cup Y_j^{2k} \cup Y_{i,j,r}^{2k}$. □

This means that we blow-up either a smooth variety contained in in the singular loci of the strict transforms of the cones that have not yet been blown-up or a smooth variety not intersecting these strict transforms. Therefore, by Proposition 6.8 $\widetilde{\mathbb{S}ec_{k+2}(C)}$ is smooth and transversal to all the exceptional divisors. By Proposition 6.10 the same is true for the $\widetilde{Y}_{i,j}^{2k+3}$'s. On the other hand, by Proposition 6.11 the singularities of $\widetilde{\mathbb{S}ec_t(C)}$ for $t \geq k+3$, of the $\widetilde{Y}_{i,j}^{2t+1}$'s for $t \geq k+2$, and of the $\widetilde{Y}_i^{2t+2}$'s and $\widetilde{Y}_{1,2,3}^{2t+2}$ for $t \geq k$ are not affected by these blow-ups, so that we can proceed with the induction.
Now consider the blow-up $X_{2k+2} \to X_{2k+1}$. By Proposition 6.12, on X_{2k+1} we have

$$\widetilde{\mathbb{S}ec_{k+2}(C)} \cap \widetilde{Y}_{i,j}^{2k+3} = \widetilde{Y}_i^{2k+2} \cup \widetilde{Y}_j^{2k+2}, \quad \widetilde{Y}_{i,j}^{2k+3} \cap \widetilde{Y}_{i,r}^{2k+3} = \widetilde{Y}_i^{2k+2} \cup \widetilde{Y}_{i,j,r}^{2k+2}.$$

By the induction hypothesis, the $\widetilde{Y}_i^{2k+2}$'s and $\widetilde{Y}_{1,2,3}^{2k+2}$ are smooth and pairwise disjoint. So the intersection is everywhere transverse. We conclude that on X_{2k+2}, which is obtained from X_{2k+1} by blowing-up the $\widetilde{Y}_i^{2k+2}$'s and $\widetilde{Y}_{1,2,3}^{2k+2}$, the varieties $\widetilde{\mathbb{S}ec_{k+2}(C)}$ and the $\widetilde{Y}_{i,j}^{2k+3}$'s are pairwise disjoint. Furthermore, arguing as in the proof of Claim 6.18 we have the following.

CLAIM 6.19. For any $t \geq k+1$ and $i,j \in \{1,2,3\}$ we have $Y_{1,2,3}^{2k+2} \subset Y_{i,j}^{2t+1}$ and $Y_{1,2,3}^{2k+2} \subset Y_i^{2t+2}$. Moreover

$$Y_{1,2,3}^{2k+2} \cap \mathbb{S}ec_t(C) = \begin{cases} Y_{1,2,3}^{2k+2} & \text{if } t \geq k+3, \\ Y_{1,2}^{2k+1} \cup Y_{1,3}^{2k+1} \cup Y_{2,3}^{2k+1} & \text{if } t \geq k+2. \end{cases}$$

For the Y_i^{2k+2}'s we have $Y_i \subset \mathcal{S}ec_t(C)$ for $t \geq k+1$, $Y_i^{2k+1} \subset Y_{1,2,3}^{2t+2}$ for $t \geq k+1$. Moreover, $Y_i^{2k+2} \subset Y_j^{2t+2}$ for $t \geq k+1$, $Y_i^{2k+2} \subset Y_{r,s}^{2t+1}$ for $t \geq k+2$ and $i,j,r,s \in \{1,2,3\}$. Finally

$$Y_i^{2k+2} \cap Y_{r,s}^{2k+3} = \begin{cases} Y_i^{2k+2} & \text{if } i \in \{r,s\}, \\ \mathcal{S}ec_{k+1}(C) \cup Y_{i,r}^{2k+1} \cup Y_{i,s}^{2k+1} & \text{if } i \notin \{r,s\}. \end{cases}$$

Moreover, for a general point in any irreducible component of the above intersections, the intersection is transverse.

By Proposition 6.10 the $\widetilde{Y}_i^{2k+4}$'s and $\widetilde{Y}_{1,2,3}^{2k+4}$ are smooth and transverse to all the exceptional divisors. By Claim 6.19 we blow-up either a smooth variety contained in in the singular loci of the strict transforms of the cones that have not yet been blown-up or a smooth variety not intersecting these strict transforms. Therefore, by Proposition 6.11 the singularity of the $\widetilde{Y}_i^{2t+2}$'s, $\widetilde{Y}_{1,2,3}^{2t+2}$ and $\widetilde{\mathcal{S}ec_t(C)}$ for $t \geq k+2$, and of the $\widetilde{Y}_{i,j}^{2t+1}$'s for $t \geq k+1$ are not affected by these blow-ups.
On X_{n+3}, the divisors $\widetilde{\Delta}_1$, $\widetilde{\Delta}_2$ and $\widetilde{\Delta}_3$ are smooth and transverse to the exceptional divisors over X_0 by Propositions 6.10 and 6.11.
The same is clearly true for $\widetilde{H}_{4,\dots,n+3}$. Moreover, the same argument used above shows that their intersection are pairwise smooth and everywhere transverse. At the last step we blow-up $\widetilde{\mathcal{S}ec_h(C)}$. By Proposition 6.12 we have

$$\widetilde{\Delta}_1 \cap \widetilde{\Delta}_2 \cap \widetilde{\Delta}_3 = \widetilde{\mathcal{S}ec_h(C)}.$$

So, after the blow-up of $\widetilde{\mathcal{S}ec_h(C)}$ in the last step, we get a log resolution of (X_{n+3}^n, D). □

PROOF OF THEOREM 6.16. We have

$$D = \Delta_1 + \Delta_2 + \Delta_3 + H_{4,\dots,n+3} \sim (3h+4)H - (3h+1)(E_1 + \dots + E_{n+3}).$$

Recall from Proposition 4.8 that the Mori cone of X_{n+3}^n is generated by the classes R_i's and $L_{i,j}$'s. One computes

$$-(K_{X_{n+3}^n} + \epsilon D) \cdot R_i = 2h - \epsilon(3h+1) \text{ and } -(K_{X_{n+3}^n} + \epsilon D) \cdot L_{i,j} = \epsilon(3h-2) - 2h + 2.$$

Therefore $-K_{X_{n+3}^n} - \epsilon D$ is ample provided that $\frac{2h-2}{3h-2} < \epsilon < \frac{2h}{3h+1}$.

Next we check when the pair $(X_{n+3}^n, \epsilon D)$ is klt. Let $\pi : \widetilde{X} := X_{n-2} \to X_{n+3}^n$ be the log resolution of $(X_{n+3}^n, \epsilon D)$ introduced in Proposition 6.17 above. We have

$$K_{\widetilde{X}} = \pi^* K_{X_{n+3}^n} + \sum_{k=1}^{h} (n-2k) E_{\mathcal{S}ec_k(C)} + \sum_{k=1}^{h-1} (n-2k) \sum_{i,j} E_{Y_{i,j}^{2k-1}} + \sum_{k=1}^{h-1} (n-2k-1) (\sum_i E_{Y_i^{2k}} + E_{Y_{1,2,3}^{2k}}).$$

Here we denote by E_Y the exceptional divisor with center $Y \subset \mathbb{P}^n$. In order to compute discrepancies, we will compute the the multiplicities of the Y_i^{2h}'s along the images in $\mathbb{P}^n$ of the subvarieties blown-up by π. By Proposition 6.7 we have $\operatorname{mult}_{\mathcal{S}ec_k(C)} \mathcal{S}ec_h(C) = h-k+1$. Moreover, $\operatorname{mult}_{\mathcal{S}ec_k(C)} Y_r^{2h} = h-k+1$,

$$\operatorname{mult}_{Y_{i,j}^{2k-1}} Y_r^{2h} = \begin{cases} \operatorname{mult}_{\mathcal{S}ec_k(C)} \mathcal{S}ec_h(C) = h-k+1 & \text{if } r \in \{i,j\}, \\ \operatorname{mult}_{\mathcal{S}ec_{k+1}(C)} \mathcal{S}ec_h(C) = h-k & \text{if } r \notin \{i,j\}, \end{cases}$$

$$\operatorname{mult}_{Y_i^{2k}} Y_r^{2h} = \begin{cases} \operatorname{mult}_{\mathcal{S}ec_k(C)} \mathcal{S}ec_h(C) = h-k+1 & \text{if } r = i, \\ \operatorname{mult}_{\mathcal{S}ec_{k+1}(C)} \mathcal{S}ec_h(C) = h-k & \text{if } r \neq i, \end{cases}$$

and $\operatorname{mult}_{Y^{2k}_{1,2,3}} Y^{2h}_r = \operatorname{mult}_{\mathbb{S}ec_{k+1}(C)} \mathbb{S}ec_h(C) = h-k$ for for $k = 1, ..., h-1$. Let $\Delta \subset \mathbb{P}^n$ be the divisor whose strict transform is D. We have

$$\begin{array}{lll} \operatorname{mult}_{\mathbb{S}ec_k(C)} \Delta & = & 3(h-k+1), \\ \operatorname{mult}_{Y^{2k-1}_{i,j}} \Delta & = & 2(h-k+1)+h-k = 3(h-k)+2, \\ \operatorname{mult}_{Y^{2k}_{i,j,r}} \Delta & = & 3(h-k), \\ \operatorname{mult}_{Y^{2k}_{i}} \Delta & = & h-k+1+2(h-k) = 3h-3k+1. \end{array} \tag{6.8}$$

Now, equalities 6.8 yield:

$$\begin{aligned} \pi^*(D) = \quad & \widetilde{D} + \textstyle\sum_{k=1}^{h} 3(h-k+1) E_{\mathbb{S}ec_k(C)} + \sum_{k=1}^{h-1}(3(h-k)+2)\sum_{i,j} E_{Y^{2k-1}_{i,j}} \\ & + \textstyle\sum_{k=1}^{h-1}(3h-3k+1)\sum_i E_{Y^{2k}_i} + \sum_{k=1}^{h-1} 3(h-k) E_{Y^{2k}_{1,2,3}}, \end{aligned}$$

and hence

$$\begin{aligned} K_{\widetilde{X}} = \quad \pi^*(K_{X^n_{n+3}} + \epsilon D) \quad & + \textstyle\sum_{k=1}^{h}(2h-2k+1-3\epsilon(h-k+1)) E_{\mathbb{S}ec_k(C)} \\ & + \textstyle\sum_{k=1}^{h-1}(2h-2k+1-\epsilon(3(h-k)+2))\sum_{i,j} E_{Y^{2k-1}_{i,j}} \\ & + \textstyle\sum_{k=1}^{h-1}(2(h-k)-\epsilon(3h-3k+1))\sum_i E_{Y^{2k}_i} \\ & + \textstyle\sum_{k=1}^{h-1}(2(h-k)-\epsilon(3h-3k)) E_{Y^{2k}_{1,2,3}} - \epsilon\widetilde{D}. \end{aligned}$$

Therefore the pair $(X^n_{n+3}, \epsilon D)$ is klt for any $0 \leq \epsilon < \frac{2}{3}$. □

6.6. The even case $n = 2h$. Let us begin with the case $n = 4$. For any $i, j = 1, ..., 7$ we consider the projection $\pi_{i,j} : \mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ from the line $L_{i,j} = \langle p_i, p_j \rangle$. Let $C_{i,j}$ be the unique conic through the points $\pi_{i,j}(p_k)$ for $k \neq i, j$, and $\Delta_{i,j} = \operatorname{Join}(L_{i,j}, C)$ the cone over $C_{i,j}$ with vertex $L_{i,j}$. By Proposition 6.7 the secant variety $\mathbb{S}ec_2(C) \subset \mathbb{P}^4$ of the rational normal curve C through the p_i's is an hypersurface of degree three, $\operatorname{Sing}(\mathbb{S}ec_2(C)) = C$ and $\operatorname{mult}_C(\mathbb{S}ec_2(C)) = 2$. Finally let $H_{5,6,7}$ be a general hyperplane through p_5, p_6, p_7, and consider the divisor

$$\Delta = \Delta_{1,2} \cup \Delta_{3,4} \cup \mathbb{S}ec_2(C) \cup H_{5,6,7}.$$

PROPOSITION 6.20. *Let us consider the following chain of blow-ups:*

- *blow-up the strict transforms of the lines $L_{1,2}$ and $L_{3,4}$,*
- *blow-up the strict transform of the rational normal curve C,*

and let $\pi : Y \to X^4_7$ be the composition of these blow-ups. Then π is a log resolution of the pair (X^4_7, D).

PROOF. First of all we want to prove that $\Delta_{1,2} \cap \Delta_{3,4} \cap \mathbb{S}ec_2(C) = C \cup L_{1,3} \cup L_{1,4} \cup L_{2,3} \cup L_{2,4}$. Assume that there is a point $p \in \Delta_{1,2} \cap \Delta_{3,4} \cap \mathbb{S}ec_2(C)$ such that $p \notin C \cup L_{1,3} \cup L_{1,4} \cup L_{2,3} \cup L_{2,4}$. Since $p \in \Delta_{1,2} \cap \mathbb{S}ec_2(C)$ there is secant line $L_{q,r} = \langle q, r \rangle$ with $q, r \in C$, and a line $L_{s,t} = \langle s, t \rangle$ with $s \in L_{1,2}$ and $t \in C$ such that $p \in L_{q,r} \cap L_{s,t}$. The lines $L_{1,2}, L_{q,r}, L_{s,t}$ generate an hyperplane intersecting C in at least five points. On the other hand, $\deg(C) = 4$ forces $q = p_1 = s$ and $r = t$. That is $p \in (\Delta_{1,2} \cap \mathbb{S}ec_2(C)) \setminus (C \cup L_{1,3} \cup L_{1,4} \cup L_{2,3} \cup L_{2,4})$ implies that there exists a point $r \in C$ such that $p \in L_{1,r} = \langle p_1, r \rangle$.
The same argument shows that $p \in (\Delta_{3,4} \cap \mathbb{S}ec_2(C)) \setminus (C \cup L_{1,3} \cup L_{1,4} \cup L_{2,3} \cup L_{2,4})$ implies that there is a point $u \in C$ such that $p \in L_{3,u} = \langle p_3, u \rangle$. Since $p \notin C \cup L_{1,3} \cup L_{1,4} \cup L_{2,3} \cup L_{2,4}$ the lines $L_{1,r}, L_{3,u}, L_{1,2}$ span an hyperplane intersecting C in at least five points. A contradiction.
Now, note that $\deg(\Delta_{1,2} \cap \Delta_{3,4} \cap \mathbb{S}ec_2(C)) = 12$. Since $\operatorname{mult}_C(\mathbb{S}ec_2(C)) = 2$ we get that

$$\Delta_{1,2} \cap \Delta_{3,4} \cap \mathbb{S}ec_2(C) = C \cup L_{1,3} \cup L_{1,4} \cup L_{2,3} \cup L_{2,4}$$

scheme-theoretically. After blowing-up the $L_{i,j}$'s the strict transform of the $\Delta_{i,j}$ are smooth. Similarly, by Proposition 6.8 blowing-up C we have that the strict transform of $\mathcal{S}ec_2(C)$ is smooth. Clearly the hyperplane $H_{5,6,7}$ intersects transversally C in p_5, p_6, p_7 and intersects transversally all the subvarieties that have been blown-up. Again by Proposition 6.8 we conclude that the divisor

$$\widetilde{\Delta}_{1,2} \cup \widetilde{\Delta}_{3,4} \cup \widetilde{\mathcal{S}ec_2(C)} \cup \widetilde{H}_{5,6,7} \cup \mathrm{Exc}(\pi)$$

in Y is simple normal crossing. □

PROPOSITION 6.21. *The variety X^4_7 is log Fano.*

PROOF. We consider the strict transform $D \subset X^5_8$ of $\Delta = \Delta_1 + \Delta_2 + \Delta_3 + H_{4,...,8}$ in X^5_8. By Lemma 6.13 we have

$$\deg(\Delta) = 8, \quad \mathrm{mult}_{p_i} \Delta = 5.$$

Therefore, we have

$$D = 8H - 5(E_1 + ... + E_8)$$

and

$$-(K_{X^4_7} + \epsilon D) = (5 - 8\epsilon)H - (3 - 5\epsilon)(E_1 + ... + E_7).$$

Intersecting with the curves of type $L_{i,j}$ and R_i, by Proposition 4.8 we get that $-(K_{X^4_7} + \epsilon D)$ is ample for any $\frac{1}{2} < \epsilon < \frac{3}{5}$.
Now, by Proposition 6.20 we have that $\pi : Y \to X^4_7$ is a log resolution. Furthermore we have

$$\pi^*(\epsilon D) = \epsilon \widetilde{D} + 3\epsilon(E_{L_{1,2}} + E_{L_{3,4}}) + 4\epsilon E_C$$

and

$$K_Y = \pi^*(K_{X^4_7} + \epsilon D) + (2 - 3\epsilon) \sum_{i,j} E_{L_{i,j}} + (2 - 4\epsilon) E_C - \epsilon \widetilde{D}.$$

Then for any $\epsilon < \frac{3}{4}$ the pair $(X^4_7, \epsilon D)$ is klt. We conclude that for any $\frac{1}{2} < \epsilon < \frac{3}{5}$ the divisor $-(K_{X^4_7} + \epsilon D)$ is ample and the pair $(X^4_7, \epsilon D)$ is klt. □

Now, let $H_{5,...,2h+3}$ be the strict transform in X^n_{n+3} of a general hyperplane through $p_5, ..., p_{2h+3}$, and $\Delta_{i,j}$ the strict transform of $Y^{2h-1}_{i,j}$ according to Notation 6.9.

THEOREM 6.22. *Let $n = 2h \geq 4$ be an even integer. Set*

$$D := \Delta_{1,2} \cup \Delta_{3,4} \cup \widetilde{\mathcal{S}ec_h(C)} \cup H_{5,...,2h+3} \subset X^n_{n+3}.$$

For any $\frac{2h-3}{3h-4} < \epsilon < \frac{2h-1}{3h-1}$ the divisor $-(K_{X^n_{n+3}} + \epsilon D)$ is ample, and the pair $(X^n_{n+3}, \epsilon D)$ is klt.

For the proof of Theorem 6.22, we will need the following results.

LEMMA 6.23. *Any point of $Y^{2h-1}_{1,2} \cap Y^{2h-1}_{3,4}$ which is smooth for both $Y^{2h-1}_{1,2}$ and $Y^{2h-1}_{3,4}$ is a smooth point of $Y^{2h-1}_{1,2} \cap Y^{2h-1}_{3,4}$ as well.*

PROOF. Let $x \in Y^{2h-1}_{1,2} \cap Y^{2h-1}_{3,4}$ be a point such that $x \notin \mathrm{Sing}(Y^{2h-1}_{1,2}) \cup \mathrm{Sing}(Y^{2h-1}_{3,4})$. It is enough to prove that the intersection of $Y^{2h-1}_{1,2}$ and $Y^{2h-1}_{3,4}$ in x is transverse, that is $T_x Y^{2h-1}_{1,2} \neq T_x Y^{2h-1}_{3,4}$.
Assume by contradiction that $T_x Y^{2h-1}_{1,2} = T_x Y^{2h-1}_{3,4} = H^{2h-1}$. Since $x \in Y^{2h-1}_{1,2}$ by Terracini's Lemma **[Te]** we have

$$H^{2h-1} = \langle p_1, p_2, T_{z_1}C, ..., T_{z_{h-1}}C \rangle = \langle p_2, p_3, T_{w_1}C, ..., T_{w_{h-1}}C \rangle$$

for $z_i, w_i \in C$. Now, let $s = |\{z_1, ..., z_{h-1}\} \cap \{w_1, ..., w_{h-1}\}|$, $r = |\{z_1, ..., z_{h-1}\} \cap \{p_3, p_4\}|$, and $r = |\{w_1, ..., w_{h-1}\} \cap \{p_1, p_2\}|$. Note that since $z_i \notin \{p_1, p_2\}$ and $w_i \notin \{p_3, p_4\}$ for any i, we have $s \leq h - 1 - \max(r, t)$. We may assume $r \geq t$ so that $s \leq h - 1 - r$. Therefore H^{2h-1} intersects C in at least

$$2(2(h-1) - s) + 2 - r + 2 - t \geq 2h + 2 + r - t \geq 2h + 2$$

points counted with multiplicity. A contradiction because the rational normal curve C has degree $2h$. $\square$

PROPOSITION 6.24. *Let the assumptions be as in Theorem 6.22, and follow Notation 6.9. For $0 \leq m \leq n-3$, we define a modification X_m of X^n_{n+3} recursively as follows:*

- *$X_0 = X^n_{n+3}$,*
- *X_{2k+1} is the blow-up of X_{2k} along the strict transforms of $\mathcal{S}ec_{k+1}(C)$, of the $Y^{2k+1}_{i,j}$'s, and of $Y^{2k+1}_{1,2,3,4}$ $(0 \leq k \leq h-3)$, (note that for $k = 0$ we do not have $Y^{2k+1}_{1,2,3,4}$),*
- *X_{2k} is the blow-up of X_{2k-1} along the strict transforms of the Y^{2k}_i's, and of the $Y^{2k}_{i,j,r}$'s $(1 \leq k \leq h-2)$.*
- *X_{n-3} is the blow-up of X_{n-4} along the strict transforms of $\mathcal{S}ec_{h-1}(C)$, of $Y^{2h-3}_{1,2}$ and of $Y^{2h-3}_{3,4}$.*

Then for any k the strict transforms of $\mathcal{S}ec_{k+1}(C)$, of the $Y^{2k+1}_{i,j}$'s, and of $Y^{2k+1}_{1,2,3,4}$ in X_{2k}; of the Y^{2k}_i's and of the $Y^{2k}_{i,j,r}$'s in X_{2k-1} are smooth, disjoint and intersect transversally all the exceptional divisors.
In particular, let $\pi : X_{n-3} \to X^n_{n+3}$ be the composition of these blow-ups. Then π is a log resolution of the pair (X^n_{n+3}, D).

PROOF. Following the same notation of the proof of Proposition 6.17 we proceed by induction on m. By Proposition 6.11 the blow-up of $p_1, ..., p_{n+3}$ does not affect the singularities of the Y^d_I's involved in the resolution. Since the p_i's have been blown-up the strict transforms of C and of the lines $Y^1_{i,j}$ do not intersect in X_0 and the statement is verified for $m = 0$. Now, by Proposition 6.12 in X_0 we have: $\widetilde{Y}^2_i \cap \widetilde{Y}^2_j = \widetilde{C} \cup \widetilde{Y}^1_{i,j}$, $Y^2_i \cap Y^2_{j,r,s} = \{p_j, p_r, p_s\}$ if $i \notin \{j, r, s\}$, $Y^2_i \cap Y^2_{i,j,r} = Y^1_{i,j} \cup Y^1_{i,r}$, and $Y^2_{i,j,r} \cap Y^2_{i,j,s} = Y^1_{i,j}$. By blowing-up the $\widetilde{Y}^1_{i,j}$'s and $\widetilde{C}$ we separate the $\widetilde{Y}^2_i$'s and the $\widetilde{Y}^2_{i,j,r}$'s which are smooth and disjoint. So the statement is verified for $m = 1$.
Now, suppose that the statement is true $m \leq 2k-1$. We will show that it holds for X_{2k} and X_{2k+1}. By Proposition 6.12 we have:

$$\begin{array}{lll} Y^{2k+1}_{i,j} \cap Y^{2k+1}_{i,j,r,s} & = & Y^{2k}_{i,j,r} \cup Y^{2k}_{i,j,s}, \\ \mathcal{S}ec_{k+1}(C) \cap Y^{2k+1}_{i,j} & = & Y^{2k}_i \cup Y^{2k}_j, \\ Y^{2k+1}_{i,j} \cap Y^{2k+1}_{i,r} & = & Y^{2k}_i \cup Y^{2k}_{i,j,r}, \\ Y^{2k+1}_{i,j} \cap Y^{2k+1}_{r,s} & = & \mathcal{S}ec_k(C) \cup Y^{2k-1}_{i,r} \cup Y^{2k-1}_{i,s} \cup Y^{2k-1}_{j,r} \cup Y^{2k-1}_{j,s} \cup Y^{2k-1}_{i,j,r,s}, \end{array}$$

and finally

$$\mathcal{S}ec_{k+1}(C) \cap Y^{2k+1}_{1,2,3,4} = \bigcup_{\{i,j\} \subset \{1,2,3,4\}} Y^{2k-1}_{i,j}.$$

Since all the irreducible components of these intersections have been blown-up either at the step $m = 2k-2$ or at the step $m = 2k-1$ we see that the strict transforms of $\mathcal{S}ec_{k+1}(C)$, of the $Y^{2k+1}_{i,j}$'s, and of $Y^{2k+1}_{1,2,3,4}$ in X_{2k} are disjoint. Furthermore, by Propositions 6.8 and 6.10 these strict transforms are smooth and transversal to all the exceptional divisors. Furthermore, as in Claims 6.18 and 6.19

it is easy to check that the intersections of $Y^{2k+1}_{1,2,3,4}$ with $\mathcal{S}ec_{t+1}(C)$, $Y^{2t+1}_{i,j}$, and $Y^{2t}_{i,j,r}$ for $t \geq k$ are union of cones of these four types and of dimension $d < 2k+1$. This fact together with Claims 6.18 and 6.19 implies that we blow-up either a smooth variety contained in in the singular loci of the strict transforms of the cones that have not yet been blown-up or a smooth variety not intersecting these strict transforms. By Proposition 6.11 these blow-ups do not modify the singularities of the strict transforms of the cones that we are going to blow-up in the following steps.

Now, let us consider X_{2k+1}. By Proposition 6.12 we have:

$$\begin{array}{rcl} Y^{2k+2}_{i,j,r} \cap Y^{2k+2}_{i,j,s} & = & Y^{2k+1}_{i,j} \cup Y^{2k+1}_{i,j,r,s}, \\ Y^{2k+2}_{i} \cap Y^{2k+2}_{j} & = & \mathcal{S}ec_{k+1}(C) \cup Y^{2k+1}_{i,j}, \\ Y^{2k+2}_{i} \cap Y^{2k+2}_{i,j,r} & = & Y^{2k+1}_{i,j} \cup Y^{2k+1}_{i,r}, \\ Y^{2k+2}_{i} \cap Y^{2k+2}_{j,r,s} & = & Y^{2k}_{i,j,r} \cup Y^{2k}_{i,j,s} \cup Y^{2k}_{i,r,s} \cup Y^{2k}_{j} \cup Y^{2k}_{r} \cup Y^{2k}_{s}. \end{array}$$

As before, all the irreducible components of these intersection have been blown-up either at the step $m = 2k$ or at the step $m = 2k-1$. Therefore, by Proposition 6.10 the strict transforms of the Y^{2k+2}_{i}'s, and of the $Y^{2k+2}_{i,j,r}$'s in X_{2k+1} are pairwise disjoint, smooth and transversal to all the exceptional divisors. As in the previous step it is easy to check that the intersections of the $Y^{2k+2}_{i,j,r}$ with $\mathcal{S}ec_{t+1}(C)$, $Y^{2t+1}_{i,j}$, Y^{2t+2}_{i} and $Y^{2t+2}_{i,j,s}$ for $t \geq k+1$ are union of cones of these four types and of dimension $d < 2k+2$. As before, this fact together with Claim 6.19 and Proposition 6.11 implies that these blow-ups do not affect the singularities of the strict transforms of the cones that will be blown-up in the next steps.

Now we have to take care of the last step. First of all we need to understand the intersection of $Y^{2h-1}_{i,j}$ and $\mathrm{Sing}(Y^{2h-1}_{r,s}) = Y^{2h-3}_{r,s}$.

CLAIM 6.25. We have

$$Y^{2h-1}_{i,j} \cap Y^{2h-3}_{r,s} = Y^{2h-4}_{r} \cup Y^{2h-4}_{s} \cup Y^{2h-4}_{i,r,s} \cup Y^{2h-4}_{j,r,s}.$$

Moreover, for a general point in any irreducible component of the above intersections, the intersection is transverse.

PROOF. We write $Y^{2h-1}_{i,j} \cap Y^{2h-3}_{r,s} = (Y^{2h-1}_{i,j} \cap Y^{2h-1}_{i,j,r,s}) \cap Y^{2h-3}_{r,s}$. By Proposition 6.12 we get $Y^{2h-1}_{i,j} \cap Y^{2h-1}_{i,j,r,s} = Y^{2h-2}_{i,j,r} \cup Y^{2h-2}_{i,j,s}$. Now, $Y^{2h-2}_{i,j,r} \cap Y^{2h-3}_{r,s} = (Y^{2h-2}_{i,j,r} \cap Y^{2h-2}_{i,r,s}) \cap Y^{2h-2}_{r,s}$ which by Proposition 6.12 is equal to $(Y^{2h-3}_{i,r} \cup Y^{2h-3}_{i,j,r,s}) \cap Y^{2h-3}_{r,s}$ which in turn by Proposition 6.12 is equal to $Y^{2h-4}_{r} \cup Y^{2h-4}_{i,r,s} \cup Y^{2h-4}_{j,r,s}$. In the same way we have $Y^{2h-2}_{i,j,s} \cap Y^{2h-3}_{r,s} = Y^{2h-4}_{s} \cup Y^{2h-4}_{i,r,s} \cup Y^{2h-4}_{j,r,s}$. □

The strict transforms $\widetilde{Y}^{2h-1}_{1,2}$ and $\widetilde{Y}^{2h-1}_{3,4}$ in X_{n-4} are still singular along $\widetilde{Y}^{2h-3}_{1,2}$ and $\widetilde{Y}^{2h-3}_{3,4}$ respectively. However, by Claim 6.25 we have

$$\widetilde{Y}^{2h-1}_{1,2} \cap \mathrm{Sing}(\widetilde{Y}^{2h-1}_{3,4}) = \widetilde{Y}^{2h-1}_{3,4} \cap \mathrm{Sing}(\widetilde{Y}^{2h-1}_{1,2}) = \emptyset.$$

Hence, Lemma 6.23 yields that $\widetilde{Y}^{2h-1}_{1,2} \cap \widetilde{Y}^{2h-1}_{3,4}$ is smooth. Therefore, after blowing-up the strict transforms of $\mathcal{S}ec_{h-1}(C)$, of $Y^{2h-3}_{1,2}$ and of $Y^{2h-3}_{3,4}$, by Proposition 6.10 in X_{n-3} the strict transforms $\widetilde{\mathcal{S}ec_h(C)}$, $\widetilde{Y}^{2h-1}_{1,2}$ $\widetilde{Y}^{2h-1}_{3,4}$ are smooth and intersect transversally all the exceptional divisors. We already know that the intersection $\widetilde{Y}^{2h-1}_{1,2} \cap \widetilde{Y}^{2h-1}_{3,4}$ is transversal. On the other hand, by Proposition 6.12 in X_{n-3} the intersection $\widetilde{\mathcal{S}ec_h(C)} \cap \widetilde{Y}^{2h-1}_{i,j} = \widetilde{Y}^{2h-2}_{i} \cup \widetilde{Y}^{2h-2}_{j}$ is a union of two smooth, disjoint subvarieties. Therefore $\widetilde{\mathcal{S}ec_h(C)} \cap \widetilde{Y}^{2h-1}_{i,j}$ is transversal as well. To conclude it is enough to observe that clearly in X_{n-3} the strict transform $\widetilde{H}_{5,\dots,2h+3}$ is transversal to $\widetilde{\mathcal{S}ec_h(C)}$, $\widetilde{Y}^{2h-1}_{1,2}$, $\widetilde{Y}^{2h-1}_{3,4}$ and to all the exceptional divisors. $\square$

PROOF OF THEOREM 6.22. We may write:

$$D \sim (3h+2)H - (3h-1)(E_1 + \dots + E_{2h+3})$$

and

$$-K_{X^n_{n+3}} - \epsilon D \sim (2h+1-\epsilon(3h+2))H - (2h-1-\epsilon(3h-1))(E_1 + \dots + E_{2h+3}).$$

Now, we have

$$(-K_{X^n_{n+3}} - \epsilon D) \cdot R_i = 2h-1-\epsilon(3h-1) \text{ and } (-K_{X^n_{n+3}} - \epsilon D) \cdot L_{i,j} = \epsilon(3h-4) - 2h + 3.$$

By Proposition 4.8 the divisor $-K_{X^n_{n+3}} - \epsilon D$ is ample for $\frac{2h-3}{3h-4} < \epsilon < \frac{2h-1}{3h-1}$.
Now, our aim is to compute the multiplicities of $\mathcal{S}ec_h(C)$, $Y^{2h-1}_{1,2}$, and $Y^{2h-1}_{3,4}$ along the subvarieties blown-up in the resolution $\pi : \widetilde{X} := X_{n-1} \to X^n_{n+3}$ of Proposition 6.24. First of all, we have:

$$\operatorname{mult}_{Y^{2k-1}_{r,s}} Y^{2h-1}_{i,j} = \begin{cases} \operatorname{mult}_{\mathcal{S}ec_{k-1}(C)} \mathcal{S}ec_{h-1}(C) = h-k+1 & \text{if } i,j \in \{r,s\}, \\ \operatorname{mult}_{\mathcal{S}ec_{k}(C)} \mathcal{S}ec_{h-1}(C) = h-k & \text{if either } i \in \{r,s\} \text{ or } j \in \{r,s\}, \\ \operatorname{mult}_{\mathcal{S}ec_{k+1}(C)} \mathcal{S}ec_{h-1}(C) = h-k-1 & \text{if } i,j \notin \{r,s\}, \end{cases}$$

$$\operatorname{mult}_{Y^{2k}_{r,s,t}} Y^{2h-1}_{i,j} = \begin{cases} \operatorname{mult}_{\mathcal{S}ec_{k+2}(C)} \mathcal{S}ec_{h-1}(C) = h-k-2 & \text{if } i,j \notin \{r,s,t\}, \\ \operatorname{mult}_{\mathcal{S}ec_{k+1}(C)} \mathcal{S}ec_{h-1}(C) = h-k-1 & \text{if either } i \in \{r,s,t\} \text{ or } j \in \{r,s,t\}, \\ \operatorname{mult}_{\mathcal{S}ec_{k}(C)} \mathcal{S}ec_{h-1}(C) = h-k & \text{if } i,j \in \{r,s,t\}, \end{cases}$$

$$\operatorname{mult}_{Y^{2k}_{r}} Y^{2h-1}_{i,j} = \begin{cases} \operatorname{mult}_{\mathcal{S}ec_{k+1}(C)} \mathcal{S}ec_{h-1}(C) = h-k-1 & \text{if } r \notin \{i,j\}, \\ \operatorname{mult}_{\mathcal{S}ec_{k}(C)} \mathcal{S}ec_{h-1}(C) = h-k & \text{if } r \in \{i,j\}, \end{cases}$$

for $k = 1, \dots, h-1$. Finally,

$$\operatorname{mult}_{Y^{2k-1}_{1,2,3,4}} Y^{2h-1}_{i,j} = \operatorname{mult}_{\mathcal{S}ec_{k}(C)} Y^{2h-1}_{i,j} = \operatorname{mult}_{\mathcal{S}ec_{k}(C)} \mathcal{S}ec_{h-1}(C) = h-k,$$

for $k = 1, \dots, h-1$. Now, let us consider the component $\mathcal{S}ec_h(C)$. We have:

$$\begin{array}{lcl} \operatorname{mult}_{\mathcal{S}ec_{k}(C)} \mathcal{S}ec_{h}(C) & = & h-k+1, \\ \operatorname{mult}_{Y^{2k-1}_{i,j}} \mathcal{S}ec_{h}(C) & = & \operatorname{mult}_{\mathcal{S}ec_{k+1}(C)} \mathcal{S}ec_{h}(C) = h-k, \\ \operatorname{mult}_{Y^{2k}_{i,j,r}} \mathcal{S}ec_{h}(C) & = & \operatorname{mult}_{\mathcal{S}ec_{k+2}(C)} \mathcal{S}ec_{h}(C) = h-k-1, \\ \operatorname{mult}_{Y^{2k}_{i}} \mathcal{S}ec_{h}(C) & = & \operatorname{mult}_{\mathcal{S}ec_{k+1}(C)} \mathcal{S}ec_{h}(C) = h-k, \\ \operatorname{mult}_{Y^{2k-1}_{1,2,3,4}} \mathcal{S}ec_{h}(C) & = & \operatorname{mult}_{\mathcal{S}ec_{k+2}(C)} \mathcal{S}ec_{h}(C) = h-k-1. \end{array}$$

Let $\Delta \subset \mathbb{P}^n$ be the divisor whose strict transform is D. We have:

$$\begin{array}{lcl} \operatorname{mult}_{\mathcal{S}ec_{k}(C)} \Delta & = & 2(h-k) + (h-k+1) = 3h-3k+1, \\ \operatorname{mult}_{Y^{2k-1}_{i,j}} \Delta & = & 2(h-k) + (h-k) = (h-k+1) + (h-k-1) + (h-k) = 3h-3k, \\ \operatorname{mult}_{Y^{2k}_{i,j,r}} \Delta & = & (h-k-1) + (h-k) + (h-k-2) = 3h-3k-3, \\ \operatorname{mult}_{Y^{2k}_{i}} \Delta & = & (h-k-1) + (h-k) + (h-k) = 3h-3k-1, \\ \operatorname{mult}_{Y^{2k-1}_{1,2,3,4}} \Delta & = & 2(h-k) + h-k-1 = 3h-3k-1. \end{array} \tag{6.9}$$

Let $\pi : \widetilde{X} := X_{n-3} \to X^n_{n+3}$ be the log resolution of the pair (X^n_{n+3}, D) in Proposition 6.24. The canonical divisor of $\widetilde{X}$ is given by:

$$\begin{aligned} K_{\widetilde{X}} = \quad & \pi^* K_{X^n_{n+3}} + \textstyle\sum_{k=1}^{h-1}(n-2k)E_{\mathbb{S}ec_k(C)} + \sum_{k=1}^{h-1}(n-2k)\sum_{i,j} E_{Y^{2k-1}_{i,j}} \\ & + \textstyle\sum_{k=1}^{h-2}(n-2k-1)(\sum_i E_{Y^{2k}_i} + \sum_{i,j,r} E_{Y^{2k}_{i,j,r}}) + \sum_{k=2}^{h-2}(n-2k)E_{Y^{2k-1}_{1,2,3,4}}. \end{aligned}$$

The equalities (6.9) yield:

$$\begin{aligned} \pi^*(D) = \quad & \widetilde{D} + \textstyle\sum_{k=1}^{h-1}(3h-3k+1)E_{\mathbb{S}ec_k(C)} + \sum_{k=1}^{h-1}(3h-3k)\sum_{i,j} E_{Y^{2k-1}_{i,j}} \\ & + \textstyle\sum_{k=1}^{h-2}(3h-3k-1)\sum_i E_{Y^{2k}_i} + \sum_{k=1}^{h-2}(3h-3k-3)\sum_{i,j,r} E_{Y^{2k}_{i,j,r}} \\ & + \textstyle\sum_{k=2}^{h-2}(3h-3k-1)E_{Y^{2k-1}_{1,2,3,4}}. \end{aligned}$$

and

$$\begin{aligned} K_{\widetilde{X}} = \quad \pi^*(K_{X^n_{n+3}} + \epsilon D) \quad & + \textstyle\sum_{k=1}^{h-1}(2h-2k-\epsilon(3h-3k+1))E_{\mathbb{S}ec_k(C)} \\ & + \textstyle\sum_{k=1}^{h-1}(2h-2k-\epsilon(3h-3k))\sum_{i,j} E_{Y^{2k-1}_{i,j}} \\ & + \textstyle\sum_{k=1}^{h-2}(2h-2k-1-\epsilon(3h-3k-1))\sum_i E_{Y^{2k}_i} \\ & + \textstyle\sum_{k=1}^{h-2}(2h-2k-1-\epsilon(3h-3k-3))\sum_{i,j,r} E_{Y^{2k}_{i,j,r}} \\ & + \textstyle\sum_{k=2}^{h-2}(2h-2k-\epsilon(3h-3k-1))E_{Y^{2k-1}_{1,2,3,4}} - \epsilon\widetilde{D}. \end{aligned}$$

For $\epsilon < \frac{2h-1}{3h-2}$ all the discrepancies are greater than -1. Finally, for $\frac{2h-3}{3h-4} < \epsilon < \frac{2h-1}{3h-1}$ the divisor $-K_{X^n_{n+3}} - \epsilon D$ is ample and the pair $(X^n_{n+3}, \epsilon D)$ is klt. $\square$

CHAPTER 5

Moduli of Curves

To fix the ideas, we work over an algebraically closed field k. Consider a class of objects $\mathcal{M}$ over k, for instance the class of closed subschemes of $\mathbb{P}^n$ with fixed Hilbert Polynomial, the class of curves of genus g over k, the class of vector bundles of given rank and Chern classes over a fixed scheme, and so on. We wish to classify the objects in $\mathcal{M}$.
The first step is to give a rule to determine when two objects of $\mathcal{M}$ are the same (usually isomorphic) and then to give the elements of $\mathcal{M}$ up to isomorphism. This determines $\mathcal{M}$ as a set. Now we want to put a natural structure of variety or scheme on $\mathcal{M}$. In other words we are looking for a scheme M whose closed points are in a one-to-one correspondence with the elements of $\mathcal{M}$, and whose scheme structure describes the variations of elements in $\mathcal{M}$, more precisely how they behave in families.

DEFINITION 0.1. *A family of elements of $\mathcal{M}$, over the parameter scheme S of finite type over k, is a scheme $X \to S$ flat over S, whose fibers at closed points are elements of $\mathcal{M}$.*

The first request on M, to be a Moduli Space for the class $\mathcal{M}$, is that for any family $f : X \to S$ of objects of $\mathcal{M}$ there exists a morphism $\phi : S \to M$ such that for any closed point $s \in S$, the image $f(s) \in \mathcal{M}$ corresponds to the isomorphism class of the fiber $X_s = f^{-1}(s)$ in $\mathcal{M}$.
Furthermore we want the assignment of the morphism ϕ to be functorial. To explain the last sentence consider the functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$, that assigns to S the set $\mathcal{F}(S)$ of families $X \to S$ of elements of $\mathcal{M}$ parametrized by S. If $S^{'} \to S$ is a morphism, for any family $X \to S$ we can consider the fiber product $X \times_S S^{'} \to S^{'}$, that is a family over $S^{'}$. In this way the morphism $S^{'} \to S$ gives rise to a map of set $\mathcal{F}(S) \to \mathcal{F}(S^{'})$, and $\mathcal{F}$ becomes a controvariant functor.
In this language to assign a morphism $\phi : S \to M$ to any family $X \to S$ with the required properties, means to give a functorial morphism $\alpha : \mathcal{F} \to Hom(-, M)$.
Finally we want to make M unique with the above properties. So we require that if N is any other scheme, and $\beta : \mathcal{F} \to Hom(-, N)$ is a functorial morphism, then there exists a unique morphism $e : M \to N$ such that $\beta = h_e \circ \alpha$, where $h_e : Hom(-, M) \to Hom(-, N)$ is the induced map on associated functors.

DEFINITION 0.2. *We define a coarse moduli space for the family $\mathcal{M}$ to be a scheme M over k, with a morphism of functors $\alpha : \mathcal{F} \to Hom(-, M)$ such that*

- *the induced map $\mathcal{F}(Spec(k)) \to Hom(Spec(k), M)$ is bijective i.e. there is a one-to-one correspondence with isomorphism classes of elements of $\mathcal{M}$ and closed points of M,*
- *α is universal in the sense explained above.*

We define a tautological family for $\mathcal{M}$ to be a family $X \to M$ such that for each closed point $m \in M$, the fiber X_m is the element of $\mathcal{M}$ corresponding to m by the bijection $\mathcal{F}(Spec(k)) \to Hom(Spec(k), M)$ above.

A *jump phenomenon* for $\mathcal{M}$ is a family $X \to S$, where S is an integral scheme of dimension at least one, such that all fibers X_s for $s \in S$ are isomorphic except for one X_{s_0} that is different. In this case the corresponding morphism $S \to M$ have to map s_0 to a point and all other closed points of S to another point, but this is not possible for a morphism of schemes, so a coarse moduli space for $\mathcal{M}$ fails to exist.

EXAMPLE 0.3. Consider the family $y^2 = x^3 + t^2x + t^3$ over the t-line. Then for any $t \neq 0$ we get smooth elliptic curves all with the same j-invariant

$$j = 12^3 \cdot \frac{4t^6}{4t^6 + 27t^6} = 12^3 \cdot \frac{4}{31},$$

and hence all isomorphic. But for $t = 0$ we get the cusp $y^2 = x^3$. This is a jump phenomenon, so the cuspidal curve cannot belong to a class having a coarse moduli space.

DEFINITION 0.4. *Let $\mathcal{F}$ be the functor associated to the moduli problem $\mathcal{M}$. If $\mathcal{F}$ is isomorphic to a functor of the form $Hom(-, M)$, then we say that $\mathcal{F}$ is representable, and we call M a fine moduli space for $\mathcal{M}$.*

Let $\alpha : \mathcal{F} \to Hom(-, M)$ be an isomorphism. In particular $\mathcal{F}(M) \to Hom(M, M)$ is an isomorphism, and there is a unique family $X_{\mathcal{U}} \to M$ corresponding to the identity map $Id_M \in Hom(M, M)$. The family $X_{\mathcal{U}}$ is called the *universal family* of the fine moduli space M. Note that for any family $X \to S$ there exists an unique morphism $S \to M$, such that $X \to S$ is obtained by base extension from the universal family. Conversely, if there is a scheme M and a family $X_{\mathcal{U}}$ with the above properties then $\mathcal{F}$ is represented by M.

REMARK 0.5. If M is a fine moduli space for $\mathcal{M}$ then it is also a coarse moduli space, furthermore the universal family $X_{\mathcal{U}} \to M$ is a tautological family.

A benefit of having a fine moduli space is that we can study it using infinitesimal methods.

PROPOSITION 0.6. *Let M be a fine moduli space for the moduli problem $\mathcal{M}$, and let $X_0 \in \mathcal{M}$ be an element corresponding to a point $x_0 \in M$. The Zariski tangent space $T_{x_0}M$ is in one-to-one correspondence with the set of families $X \to D$ over the dual numbers $D = k[\epsilon]/(\epsilon^2)$, whose closed fibers are isomorphic to X_0.*

PROOF. We know that to give a morphism $f : Spec(D) \to M$ is equivalent to give a closed point $x_0 \in M$ and a tangent direction $v \in T_{x_0}M$. But a morphism $f : Spec(D) \to M$ corresponds to a unique family $X \to Spec(D)$ whose closed fibers are isomorphic to $X_0 \in \mathcal{M}$ corresponding to the point $x_0 \in M$, where $x_0 = f((Spec(D))_{red})$. $\square$

Let $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ be the functor associated to the moduli problem $\mathcal{M}$. Suppose that $\mathcal{F}$ is representable, and let M be the corresponding fine moduli space. For any local Artin k-algebra A we have that $Spec(A)$ is a fat point and $(Spec(A))_{red}$ is a single point. For any $x_0 \in M$ we can define the infinitesimal deformation functor of $\mathcal{F}$ as the functor $\mathfrak{Art} \to \mathfrak{Sets}$ that sends A in the set of morphisms $f : Spec(A) \to M$ such that $f((Spec(A))_{red}) = x_0$. Clearly studying this functor we get information on the geometry of M in a neighborhood of x_0.
Recall that a pro-object is an inverse limit of objects in $\mathfrak{Art}$, the category of Artin local algebras over a field k. If $\mathcal{F} : \mathfrak{Art} \to \mathfrak{Sets}$ is a deformation functor we say that $\mathcal{F}$ is pro-representable if it is isomorphic to $Hom(-, R)$ for some pro-object R.

PROPOSITION 0.7. *Let $\mathcal{F}$ be the functor associated to the moduli problem $\mathcal{M}$, and $X_0 \in \mathcal{M}$. Consider the functor $\mathcal{F}_0$ that to each local Artin ring A over k assigns the set of families of $\mathcal{M}$ over $Spec(A)$ whose closed fiber is isomorphic to X_0. If $\mathcal{M}$ has a fine moduli space, then the functor $\mathcal{F}_0$ is pro-representable.*

PROOF. Let M be a fine moduli scheme for $\mathcal{M}$, and let $x_0 \in M$ corresponds to $X_0 \in \mathcal{M}$. Let $\mathcal{O}_{M,x_0}$ be the local ring of M at x_0 and $\mathfrak{M}_{x_0}$ its maximal ideal. The natural homomorphisms

$$... \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0}^3 \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0}^2 \to \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0},$$

make $(\mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0}^n)$ into an inverse system of rings. The inverse limit $\varprojlim \mathcal{O}_{M,x_0}/\mathfrak{M}_{x_0}^n$ is denoted by $\hat{\mathcal{O}}_{M,x_0}$, and is called the completion of $\mathcal{O}_{M,x_0}$ with respect to $\mathfrak{M}_{x_0}$ or the $\mathfrak{M}_{x_0}$-adic completion of $\mathcal{O}_{M,x_0}$.
Since M is a fine moduli space, each element of $\mathcal{F}_0(A)$ corresponds to a unique morphism $Spec(A) \to M$ that maps $(Spec(A)_{red}) = Spec(k)$ at x_0. Such morphism corresponds to a ring homomorphism $\hat{\mathcal{O}}_{M,x_0} \to A$. We conclude that the functor $\mathcal{F}_0$ is pro-representable and that it is represented by the pro-object $\hat{\mathcal{O}}_{M,x_0}$, $\mathfrak{M}_{x_0}$-adic completion of $\mathcal{O}_{M,x_0}$. $\square$

DEFINITION 0.8. *A controvariant functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ is a sheaf for the Zariski topology, if for every scheme S and every $\{\mathcal{U}_i\}$ open covering of S, the diagram*

$$\mathcal{F}(S) \to \prod \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$$

is exact. This means that:

- *given $x, y \in \mathcal{F}(S)$ whose restriction to $\mathcal{F}(\mathcal{U}_i)$ are equal for all i, then $x = y$,*
- *given a collection of elements $x_i \in \mathcal{F}(\mathcal{U}_i)$ for each i, such that for each i, j, the restrictions of x_i, x_j to $\mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$ are equal, then there exists an element $x \in \mathcal{F}(S)$ whose restriction to each $\mathcal{F}(\mathcal{U}_i)$ is x_i.*

PROPOSITION 0.9. *If the moduli problem $\mathcal{M}$ has a fine moduli space, then the associated functor $\mathcal{F}$ is a sheaf in the Zariski topology.*

PROOF. Since $\mathcal{M}$ has a fine moduli space, for any scheme S we have $\mathcal{F}(S) = Hom(S, M)$. Furthermore morphisms of schemes are determined locally, and can be glued if they are given locally and are compatible on overlaps. $\square$

REMARK 0.10. Using Grothendieck's theory of *descent* one can show that a representable functor is a sheaf for the faithfully flat quasi-compact topology, and hence also for the étale topology.

Examples of Moduli Spaces. We will give some examples of representable functors.

EXAMPLE 0.11. (Grassmannians) Let V be a k-vector space of dimension n, and let $r \leq n$ be a fixed integer. Consider the controvariant functor $Gr : \mathfrak{Sch} \to \mathfrak{Sets}$ defined as follows

- For any scheme S, $Gr(S)$ is the set of rank r vector subbundle of the trivial bundle $S \times V$.
- If $f : S \to S'$ is a morphism of schemes, and $E_{S'}$ is a rank r subbundle of $S' \times V$, we define

$$Gr(f)(E_{S'}) = f^*(E_{S'}) = (f \times Id_V)^{-1}(E_{S'}).$$

Note that for $S = Spec(k)$ we have that $Gr(Spec(k))$ is the set of rank r subbundle of $Spec(k) \times V = V$ i.e. the set of r-dimensional subspace of V, that is the Grassmannian $Gr(r, V)$.
If $E \in Gr(S)$ is a rank r subbundle of $S \times V$, we can construct a morphism $f_E : S \to Gr(r, V)$ defined by $s \mapsto E_s$, where E_s is the fiber of E over $s \in S$. In this way we get a map

$$\phi(S) : Gr(S) \to Hom(S, Gr(r, V)),\ E \mapsto f_E.$$

The collection $\{\phi(S)\}$ gives a functorial isomorphism between Gr and $Hom(-, Gr(r,V))$. Then the functor Gr is representable and the Grassmannian $Gr(r,V)$ is the corresponding fine moduli space. The universal family corresponding to the identity map $Id_{Gr(r,V)} \in Hom(Gr(r,V), Gr(r,V))$ is clearly the universal bundle on $Gr(r,V)$ given by $\{(W,v) \mid v \in W\} \subseteq Gr(r,V) \times V$.

EXAMPLE 0.12. (Hilbert Scheme) Let $P \in \mathbb{Q}[z]$ be a fixed polynomial. For any S scheme over k consider $\mathbb{P}^N_S = \mathbb{P}^N \times_k S$, and the functor

$$Hilb^N_P : \mathfrak{Sch} \to \mathfrak{Sets},$$

that maps S in the set of subschemes $Y \subseteq \mathbb{P}^N_S$ such that the projection $\pi : Y \to S$ is flat, and for any $s \in S$ the fiber $\pi^{-1}(s)$ is a subscheme of $\mathbb{P}^N$ with Hilbert polynomial P. The functor $Hilb^N_P$ is representable by a scheme $Hilb_P(\mathbb{P}^N)$ projective over k and called the Hilbert Scheme.

To any closed subscheme $Y \subseteq \mathbb{P}^N$ we can associate its structure sheaf $\mathcal{O}_Y$, its ideal sheaf $\mathcal{I}_Y$, and the structure sequence

$$0 \mapsto \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_Y \mapsto 0.$$

Then we can regard the Hilbert scheme as the space parametrizing all the quotients $\mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_Y$, with Hilbert polynomial P.

EXAMPLE 0.13. (Grothendieck's Quot Scheme) As a generalization of the discussion above consider a fixed coherent sheaf $\mathcal{E}$ on $\mathbb{P}^N$. The scheme parametrizing all the quotients $\mathcal{E} \to \mathcal{F} \mapsto 0$ with Hilbert polynomial P is called the Quot Scheme. Grothendieck showed that the local deformation functor of the Quot functor is pro-representable and that the Quot functor is representable by a projective scheme.

EXAMPLE 0.14. (Picard Scheme) Let X be a scheme of finite type over an algebraically closed field k and let $x \in X$ be a fixed point. Consider the functor

$$Pic_{X,x} : \mathfrak{Sch} \to \mathfrak{Sets},$$

that associates to S the group of all invertible shaves $\mathcal{L}$ on $X \times S$, with a fixed isomorphism $\mathcal{L}_{|x} \times S \cong \mathcal{O}_S$.
If X is integral and projective, then this functor is representable by a separated scheme, locally of finite type over k, called the Picard Scheme of X.

EXAMPLE 0.15. (Hilbert-Flag Scheme) Consider a functor that associates to each scheme S a flag $Y_1 \subseteq Y_2 \subseteq ... \subseteq Y_k \subseteq \mathbb{P}^N_S$ of closed subscheme, all flat over S and where the fibers if Y_j have a fixed Hilbert Polynomial P_j for any $j = 1, ..., k$. This functor is representable by a scheme, projective over k, called the Hilbert-Flag Scheme.

1. GIT construction of $\overline{M}_g$

The aim of Geometric invariant theory is to solve the problem of constructing quotient in the framework of algebraic geometry. In this section we collect the main results of this theory, which are fundamental for the construction of moduli spaces. For a detailed discussion see **[MFK]**.
We concentrate on the special case of projective schemes and reductive groups. So let Z be a projective scheme and let G be a reductive group acting on Z. Consider an embedding $Z \to \mathrm{Proj}^r = \mathrm{Proj}(V)$ given by a line bundle $\mathcal{L}$ on Z, so that $Z = \mathrm{Proj}(S)$ for some graded ring S finitely generated over k. When the action of G on Z can be lifted to an action on V we say that there exists

a G-linearization of $\mathcal{L}$, or that G acts linearly with respect to the given embedding. In this case G acts on S and the subring

$$S^G = \{s \in S \mid gs = s \,\forall\, g \in G\} \subseteq S,$$

is called the ring of invariants of S with respect to the action of G. A fundamental theorem in geometric invariant theory ensures that if G is reductive then S^G is a graded algebra, finitely generated over k. In particular for affine schemes we have the following.

THEOREM 1.1. *(Nagata) Let G be a geometrically reductive algebraic group acting rationally on an affine scheme* $\mathrm{Spec}(A)$. *Then A^G is a finitely generated k-algebra.*

The inclusion $S^G \hookrightarrow S$ induces a rational map

$$\pi : \mathrm{Proj}(S) = Z \dashrightarrow Q := \mathrm{Proj}(S^G),\ z \mapsto (f_0(z), ..., f_h(z)),$$

where the f_i's are generators of S^G. The open subset

$$Z^{ss} := \{z \in Z \mid f(z) \neq 0 \text{ for some homogeneous nonconstant} f \in S^G\},$$

that is the locus where π is regular, is called the locus of semi-stable points with respect to the action of G. Now it seems natural to view Q as the quotient of Z^{ss} modulo G. However the fibers of π may fail to be equal to the orbits of G, indeed it may happen that there are non-closed orbits and in this case the closed points of Q will not be in bijective correspondence with the orbits of G. Let M_G be the maximum among the dimensions of all G-orbits in Z^{ss}, this discussion leads us to the following definition

$$Z^s := \{z \in Z^{ss} \mid \overline{O_G(z)} \cap Z^{ss} = O_G(z) \text{ and } \dim(O_G(z)) = M_G\}.$$

The subset Z^s is called the set of stable points with respect to the action of G. We expect that the fibers of $\pi_{|Z^s}$ are equal to orbits of G.

THEOREM 1.2. *(Fundamental Theorem of GIT) Let G be a reductive group acting linearly on a projective scheme $Z = \mathrm{Proj}(S)$. The quotient $Q := \mathrm{Proj}(S^G)$ is a projective scheme and the morphism*

$$\pi : Z^{ss} \to Q$$

satisfies the following properties:

- *For every $x, y \in Z^{ss}$, $\pi(x) = \pi(y)$ if and only if $\overline{O_G(x)} \cap \overline{O_G(y)} \cap Z^{ss} \neq \emptyset$.*
- *(Universal property) If there exists a scheme $Q^{'}$ with a G-invariant morphism $\pi^{'} : Z^{ss} \to Q^{'}$, then there exists a unique morphism $\phi : Q \to Q^{'}$ such that $\pi^{'} = \psi \circ \pi$.*
- *For every $x, y \in Z^s$, $\pi(x) = \pi(y)$ if and only if $O_G(x) = O_G(y)$.*

A quotient satisfying the first and the second properties of Theorem 1.2 is called a *categorical quotient* and denoted by $Z//G$. If in addition the quotient satisfies the third property then it is called a *geometric quotient* and denoted by Z/G.
The most efficient tool to check stability is probably the so called *numerical criterion for stability*. This criterion reduces the study of the action of a reductive group G to the study of the action of its one-parameter subgroups. Let G be a reductive group acting linearly on $\mathrm{Proj}(V)$ and let $Z \subset \mathrm{Proj}(V)$ be a G-invariant subscheme. If G_m denotes k^* with is multiplicative structure and

$$\lambda : G_m \to G$$

is a one-parameter subgroup of G, there exist a basis $\{v_0, ..., v_r\}$ of V and integers $\{w_0, ..., w_r\}$ such that the action of λ on V is given by

$$\lambda(t)v_i = t^{w_i}v_i \;\forall\, t \in G_m,\; 0 \leq i \leq r.$$

If $v = \sum_{i=0}^{r} \alpha_i v_i$ the integers n_j such that the α_j do not vanish are called the λ-weights of v. We denote by $z \in Z$ the point corresponding to the vector $v_z \in V$.

THEOREM 1.3. *(Hilbert-Mumford) The point $z \in Z$ is semi-stable if and only if for any one-parameter subgroup λ of G the λ-weights of v_z are not all positive.*
The point $z \in Z$ is stable if and only if for any one-parameter subgroup λ of G the vector v_z has both positive and negative λ-weights.
The point $z \in Z$ is unstable if and only if there exists a one-parameter subgroup λ of G such that the λ-weights of v_z are all positive.

Construction of $\overline{M}_g$. Fix integers $d \gg 0$, $g \geq 3$ and $N = d - g$. Let $\mathrm{Hilb}_N^{P(x)}$ be the Hilbert scheme finely parametrizing the close subschemes of $\mathbb{P}^N$ with Hilbert polynomial $P(x) = dx - g + 1$. There exists a universal family $\mathcal{H}$ with a tautological polarization $\mathcal{L}$

$$\mathcal{L} \to \mathcal{H} \xrightarrow{\pi} \mathrm{Hilb}_N^{P(x)},$$

such that the fiber $X_h := \pi^{-1}(h)$ is isomorphic to the subscheme of $\mathbb{P}^N$ corresponding to $h \in \mathrm{Hilb}_N^{P(x)}$, and $L_h := \mathcal{L}_{|X_h}$ is isomorphic to the line bundle giving the embedding of X_h in $\mathbb{P}^N$.
Let $X \subset \mathbb{P}^N$ be a curve, we want to construct its Hilbert point in $\mathrm{Hilb}_N^{P(x)}$, and consider the exact sequence

$$0 \mapsto \mathcal{I}_X \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_X \mapsto 0.$$

By a theorem due to *J. P. Serre*, for $m >> 0$, we get the following exact sequence in cohomology

$$0 \mapsto H^0(\mathbb{P}^N, \mathcal{I}_X(m)) \to H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)) \to H^0(X, \mathcal{O}_X(m)) \mapsto 0.$$

Furthermore it can be proven that there exists an integer $\overline{m}$ such that for any $m \geq \overline{m}$ and for any subscheme of $\mathbb{P}^N$ having Hilbert polynomial $P(x)$ the above sequence is exact. This means that the degree m part of the ideal of X, that is $H^0(\mathbb{P}^N, \mathcal{I}_X(m))$, uniquely determines X. We can associate to X a point in the Grassmannian parametrizing $P(m)$-dimensional quotients of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))$ and this correspondence is injective. For any $m \geq \overline{m}$ we get an embedding

$$\phi_m : \mathrm{Hilb}_N^{P(x)} \to \mathbb{P}\Big(\bigwedge^{P(m)} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m))\Big).$$

We have an action of $SL(N+1)$ on $\mathbb{P}(\bigwedge^{P(m)} H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(m)))$ and any embedding ϕ_m determines a linearization of the action of $SL(N+1)$ on $\mathrm{Hilb}_N^{P(x)}$. Our aim is to construct $\overline{M}_g$ as a quotient of a suitable subscheme of $\mathrm{Hilb}_N^{P(x)}$.
Translating the *Hilbert-Mumford criterion* 1.3 in this setting one gets the following theorem:

THEOREM 1.4. *If $d \geq 20(g-1)$ then there are infinitely many linearizations of the action of $SL(N+1)$ on $\mathrm{Hilb}_N^{P(x)}$ such that*

- *(Mumford-Gieseker) if $X \subset \mathbb{P}^N$ is a smooth, connected, non-degenerate curve of genus g and degree d, then its Hilbert point is stable,*

- *(Gieseker) if $h \in \mathrm{Hilb}_N^{P(x)}$ is a $SL(N+1)$-semi-stable point then all connected component of X_h are Deligne-Mumford semi-stable curves.*

Consider now the case $d = r(2g-2)$ for an integer r and fix once and for all an integer m such that Gieseker-Mumford theorem holds. Consider the following subset of $\mathrm{Hilb}_N^{p(m)\ ss}$

$$H = \{h \in \mathrm{Hilb}_N^{p(m)\ ss} \mid \mathcal{L}_{|X_h} \cong \omega_{X_h}^{\otimes r} \text{ and the curve is connected}\}.$$

The $SL(N+1)$-invariant set H parametrizes only DM-stable curves by Gieseker's theorem. In fact, for $r \geq 3$ the dualizing sheaf $\omega_X^{\otimes r}$ is very ample on DM-stable curves and it contracts exactly the destabilizing components of a DM-semi-stable curve.
Finally one can prove that H consists only of $SL(N+1)$-stable points, that it is a closed subscheme of $\mathrm{Hilb}_N^{p(m)\ ss}$ and that the r-th projective canonical model of any stable curve of genus g is an H. At this point it is natural to construct the moduli space of genus g stable curves as the GIT quotient

$$\overline{M}_g := H/SL(N+1).$$

2. The moduli functor of smooth genus g curves is not representable

In this section we will see that the moduli functor of smooth genus g curves is not representable and how the obstructions to its representability came from the automorphisms of the curves.
A family $\pi : \mathcal{C} \to S$ of genus g curves is called *isotrivial* if all its fibers are isomorphic to a fixed curve C. Note that there are isotrivial but non-trivial families of curves, take for instance a ruled surface that is not a product.
Now, assume that the moduli functor of smooth genus g curves is representable by a scheme M_g, and let $\pi : \mathcal{C} \to S$ be an isotrivial family. Then, such a family corresponds to a morphism $f_\pi : S \to M_g$, and the family $\pi : \mathcal{C} \to S$ is the pull-back of the universal curve $\mathcal{U}_g \to M_g$,

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d, "\pi"'] & \mathcal{U}_g \arrow[d, "\pi_g"] \\ S \arrow[r, "f_\pi"] & M_g \end{tikzcd}$$

Now, since $\pi : \mathcal{C} \to S$ is isotrivial $f_\pi(S)$ is a point, and $\mathcal{C} \cong S \times \pi_g^{-1}(f_\pi(S))$ is trivial. We conclude that:
If the moduli functor of smooth genus g curves would be representable then any isotrivial family of curves would be trivial. However, we know that there are isotrivial but non-trivial families of curves. Therefore, moduli functor of smooth genus g curves can not be representable. Let us look at little bit closer to some simple examples.

Curves of genus zero. There is only one smooth curve of genus $g = 0$ over an algebraically closed field k, namely $\mathbb{P}_k^1$. A family of curves of genus zero over a scheme S is a scheme X, smooth and projective over S, whose fibers are curves of genus zero.

PROPOSITION 2.1. *The space $M = \mathrm{Spec}(k)$ is a coarse moduli scheme for curves of genus zero. Furthermore it has a tautological family.*

PROOF. The set $Hom(\mathrm{Spec}(k), \mathrm{Spec}(k))$ consists of a single element and clearly is in a one-to-one correspondence with the set of families over $\mathrm{Spec}(k)$ that consists of the family $\mathbb{P}^1_k \to \mathrm{Spec}(k)$. Clearly $\mathbb{P}^1_k \to \mathrm{Spec}(k)$ is a tautological family. If $X \to S$ is a family there is a unique morphism $S \to M = \mathrm{Spec}(k)$, in this way we get the functorial morphism $\alpha : \mathcal{F} \to Hom(-, M)$.
Now suppose that $\beta : \mathcal{F} \to Hom(-, N)$ is another morphism of functors. In particular the family $\mathbb{P}^1_k \to M$ determines a morphism $e \in Hom(M, N)$. Let $X \to S$ a family over a scheme S of finite type over k. For any closed point $s \in S$ the fiber is $X_s \cong \mathbb{P}^1$, then any closed point s goes to the point $n = e(M) \in N$. Finally, since $H^1(\mathbb{P}_1, T_{\mathbb{P}^1}) = 0$, that is $\mathbb{P}^1$ is rigid, any family of curves of genus zero parametrized by the spectrum of an Artin ring with residue field k is trivial. We conclude that β factors through α. $\square$

Clearly the tautological family is $\mathbb{P}^1 \to \mathrm{Spec}(k)$, that is the unique family over $M = \mathrm{Spec}(k)$. Suppose $M = \mathrm{Spec}(k)$ to be a fine moduli space for the curves of genus zero. Then the universal family is $\mathbb{P}^1 \to \mathrm{Spec}(k)$. Since any other family is obtained by base extension from the universal family it must be trivial i.e. of the form $\mathbb{P}^1 \times_k S \to S$. But the ruled surfaces provide an example of non trivial families of curves of genus zero.
Consider for instance the blow up $Bl_p\mathbb{P}^2$ of $\mathbb{P}^2$ is a point p. The projection $\pi : Bl_p\mathbb{P}^2 \to \mathbb{P}^1$ makes $Bl_p\mathbb{P}^2$ into a ruled surface, but it is not a product. Note that $Pic(Bl_p\mathbb{P}^2) = Pic(\mathbb{P}^1 \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z}$, but on $Bl_p\mathbb{P}^2$ we have a (-1)-curve, the exceptional divisor. Suppose that there is a (-1)-curve $C = (a, b)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. We have $C^2 = (aL + bR)(aL + bR) = 2ab = -1$, a contradiction.

DEFINITION 2.2. *A pointed curve of genus zero over k is a curve of genus zero with a choice of a k-rational point. A family of pointed curves of genus zero is a flat family $X \stackrel{\pi}{\to} S$, whose geometric fibers are curves of genus zero, with a section $\sigma : S \to X$.*

The fact that $\sigma : S \to X$ is a section means that $\pi \circ \sigma = Id_S$. Then for any point $s \in S$ the image $\sigma(s)$ is a point of the fiber $X_s \cong \mathbb{P}^1$ over s. The section σ is sometimes called an S-point of X.
A way to obtain a fine moduli space for the curves of genus zero is to rigidify the curves by taking three distinct points. We know that there is a unique automorphism of $\mathbb{P}^1$ that fixed three distinct points, namely the identity. Consider the families of curves of genus zero with three marked points i.e. the families of $X \to S$, whose fibers are curves of genus zero, with three sections $\sigma_1, \sigma_2, \sigma_3 : S \to X$, such that on each fiber the sections have distinct support. Assume that $X \to S$ is isotrivial. Then we may use the three sections to write an isomorphism between $X \to S$ and $\mathbb{P}^1 \times S \to S$. Therefore, any isotrivial family of smooth genus zero curves endowed with three sections is trivial, and the corresponding moduli functor is represented by $\mathrm{Spec}(k)$. This reflects the fact that a curve X of genus zero with three marked points is rigid i.e. $\mathrm{Aut}(X) = \{Id_X\}$, the corresponding functor is representable by $M = \mathrm{Spec}(k)$ and the universal family is $\mathbb{P}^1 \to \mathrm{Spec}(k)$ with three distinct points, say $[0:1], [1:0], [1:1]$.

Now, we want to understand how to use the automorphisms of the curves in order to construct isotrivial but non-trivial families. Let C be an hyperelliptic curve and let $i : C \to C$ be the hyperelliptic involution. Let X be a $K3$ surface (a smooth projective surface with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$ and irregularity $q = \dim H^1(X, \mathcal{O}_X) = 0$) with a fixed point free involution j such that $Y = X/j$ is an Enriques surface (a smooth surface such that $\omega_Y^{\otimes 2} \cong \mathcal{O}_Y$ and $q = \dim H^1(Y, \mathcal{O}_Y) = 0$). Then $i \times j$ is a fixed point free involution on $C \times X$. Since the action of j on X is free the morphism $C \times X \to X$ induces a family $(C \times X)/(i \times j) \to Y$ with all the fibers

isomorphic to C. We want to show that $(C \times X)/(i \times j) \to Y$ is non-trivial.
Let S be a smooth surface, and consider the Hodge numbers $h^{i,j} = \dim H^j(S, \Omega^i_S)$. They can be arranged in the Hodge diamond:

$$
\begin{array}{ccccc}
 & & h^{0,0} & & \\
 & h^{1,0} & & h^{0,1} & \\
h^{2,0} & & h^{1,1} & & h^{0,2} \\
 & h^{2,1} & & h^{1,2} & \\
 & & h^{2,2} & &
\end{array}
$$

The Hodge diamond of the Enriques surface Y is given by:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
0 & & 10 & & 0 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

In particular $h^{2,0} = \dim H^0(Y, \Omega^2_Y) = 0$, and there are non-zero holomorphic 2-forms on Y. This yields that the product $C \times Y$ does not have non-zero holomorphic 3-forms.
The Hodge diamond of the $K3$ surface X is given by:

$$
\begin{array}{ccccc}
 & & 1 & & \\
 & 0 & & 0 & \\
1 & & 20 & & 1 \\
 & 0 & & 0 & \\
 & & 1 & &
\end{array}
$$

and $h^{2,0} = \dim H^0(X, \Omega^2_X) = 1$. Let $\omega \in H^0(X, \Omega^2_X)$ be a generator. Now, take $\gamma \in H^0(C, \Omega^1_C)$ a non-zero holomorphic 1-form on the curve C. Any invariant 1-form on C would descend to a 1-form on $C/i \cong \mathbb{P}^1$, but $H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1}) = 0$. Therefore, there are not invariant 1-forms on C, and γ is anti-invariant under the involution i. For the same reason, any invariant 2-form on X induces a 2-form on the Enriques surface Y, but $H^0(Y, \Omega^2_Y) = 0$. Then, there are not invariant 2-forms on X. This implies that ω is anti-invariant under j. Consider the product:

$$
\begin{tikzcd}
 & X \times C \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
X & & C
\end{tikzcd}
$$

Since ω and γ and anti-invariant for j and i respectively the 3-form $\pi_1\omega \wedge \pi_2\gamma$ is a non-zero 3-form on $X \times C$ invariant under $i \times j$. Therefore, it induces a non-zero 3-form on the quotient $(C \times X)/(i \times j)$. On the other hand we saw that there are not non-zero 3-forms on the product $C \times Y$. Therefore, the family $(C \times X)/(i \times j) \to Y$ can not be isomorphic to the trivial family $C \times Y \to Y$.

The above construction works in a more general context. Let X be a projective scheme such that $\mathrm{Aut}(X)$ contains a non-trivial finite subgroup G. Let Y be a projective scheme admitting a free G-action, and let $\overline{Y} = Y/G$ be the quotient. Then, G acts freely on $X \times Y$ and the quotient

$\mathcal{X} = (X \times Y)/G$ is a projective scheme. The projection $X \times Y \to Y$ is G-equivariant and induces a morphism $\pi : \mathcal{X} \to \overline{Y}$. We have a commutative diagram

$$\begin{tikzcd} X \times Y \arrow[r, "f"] \arrow[d, "\pi_2"'] & \mathcal{X} \arrow[d, "\pi"] \\ Y \arrow[r, "g"] & \overline{Y} \end{tikzcd}$$

where f and g are étale morphisms and the any fiber of π over a closed point $y \in \overline{Y}$ is isomorphic to X. Since there is a Y-isomorphism $X \times Y \to Y \times_{\overline{Y}} \mathcal{X}$ the above digram is cartesian. Therefore, any section of π induces a section of π_2 that is a morphisms $Y \to X \times Y$ given by $y \mapsto (x, y)$ for some $x \in X$. On the other hand the point $x \in X$ has to be fixed under the action of G. Therefore, since G is finite the morphism $\pi : \mathcal{X} \to \overline{Y}$ admits at most finitely many sections. In particular the family $\pi : \mathcal{X} \to \overline{Y}$ can not be trivial.

3. The Stack $\overline{\mathcal{M}}_{g,n}$

The study of moduli problems introduces a new kind of objects: the so called moduli stacks. We have seen that a moduli problem gives rise to a functor, if the functor is representable we have a fine moduli space, that is a scheme. Sometimes, if it is not representable one can find a coarse moduli space, which parametrizes the isomorphism classes of our objects over a field, but does not describe all the possible families of objects. It happens that the functor related to a moduli problem is not representable by a scheme. We search for a sort of generalized scheme.
A scheme is constructed out of affine schemes by gluing the isomorphism defined on Zariski open subset. In the same spirit consider a collection of schemes $\{X_i\}$, and for each i, j étale morphisms $Y_{i,j} \to X_i$, $Y_{j,i} \to X_j$ and isomorphisms $\phi_{i,j} : Y_{i,j} \to Y_{j,i}$, satisfying a cocycle condition for each i, j, k. We glue together the X_i along the $\phi_{i,j}$. This quotient may not exist in the category of schemes, but it is an *algebraic space*.
Instead of the functor $\mathcal{F}$, which sends any scheme S in the set of isomorphism classes of families $X \to S$, consider a new object $\mathcal{F}$, which to each scheme S assigns the category $\mathcal{F}(S)$ of families and isomorphisms between such families. This object is called a fibered category over the category of schemes. The sheaf axioms for the functor $\mathcal{F}$ are replaced by the *stack axioms* for the fibered category $\mathcal{F}$, which are the following. For any scheme S and any étale covering $\{U_i \to S\}$, consider

$$\mathcal{F}(S) \to \prod \mathcal{F}(U_i) \rightrightarrows \prod \mathcal{F}(U_i \times_S U_j) \rightrightarrows \prod \mathcal{F}(U_i \times_S U_j \times_S U_k).$$

- The fact that the first arrow is injective means that if $a, b \in \mathcal{F}(S)$ and if a_i, b_i are their restriction on $\mathcal{F}(U_i)$, and there is an isomorphism $\phi_i : a_i \to b_i$ such that for each i, j the isomorphisms ϕ_i, ϕ_j restrict to the same isomorphism of $a_{i,j}$ and $b_{i,j}$ on $U_i \times_S U_j$, then there is a unique isomorphism ϕ inducing ϕ_i on each U_i.
- The fact that the sequence is exact at the first middle term means that if we give objects $a_i \in \mathcal{F}(U_i)$ for each i and isomorphisms $\phi_{i,j} : a_i \to a_j$ on $U_i \times_S U_j$ satisfying a cocycle condition on each $U_i \times_S U_j \times_S U_k$, then there exists a unique object $a \in \mathcal{F}(S)$ restricting to each a_i on U_i.

A *Deligne-Mumford stack* is a fibered category $\mathcal{F}$ satisfying the stack axioms, and such that there exists a scheme X and a surjective étale morphism $Hom(-, X) \to \mathcal{F}$. An *Artin stack* is a fibered category $\mathcal{F}$ satisfying the stack axioms, and such that there exists a scheme X and a surjective smooth morphism $Hom(-, X) \to \mathcal{F}$.

The moduli space of curves $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack for any $g \geq 2$. In the paper *The irreducibility of the space of curves of given genus* [**DM**], Deligne and Mumford introduced stacks for the first time, they compactified the stack $\mathcal{M}_g$ adding stable curves, and they proved its irreducibility in any characteristic.

We define a family of pointed curves of genus g parametrized by a scheme S as an object

$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "\sigma_1,\dots,\sigma_n"'] \end{tikzcd}$$

where π is a flat and proper morphism, σ_i is a section of π for any $i = 1, ..., n$, $C_s = \pi^{-1}(s)$ is a nodal connected curve of arithmetic genus g and $\sigma_i(s)$ are distinct smooth points for any $s \in S(k)$. A morphism between two families $C \to S$, $C^{'} \to S$ over S is a morphism of schemes $\phi : C \to C^{'}$ such that the following diagrams

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & C^{'} \arrow[dl, "\pi^{'}"] \\ & S & \end{tikzcd} \qquad \begin{tikzcd} C \arrow[rr, "\phi"] & & C^{'} \\ & S \arrow[ul, "\sigma_i"] \arrow[ur, "\sigma_i^{'}"'] & \end{tikzcd}$$

commute. We consider the pseudofunctor

$$\mathfrak{M}_{g,n} : \mathfrak{Sch} \longrightarrow \mathfrak{Groupoids}$$

mapping a scheme S to the groupoid $\mathfrak{M}_{g,n}(S)$ whose objects are the families parametrized by S and whose morphisms are the isomorphisms between these families. A curve $(C, x_1, ..., x_n) \in \mathrm{Obj}(\mathfrak{M}_{g,n}(\mathrm{Spec}(k)))$ is called a *pre-stable genus g curve*. We denote by $\mathfrak{M}_{g,n}$ the stack associated to this pseudofunctor.

REMARK 3.1. The stack $\mathfrak{M}_{g,n}$ is never a DM-algebraic stack. It contains points representing curves with automorphism groups of positive dimension. Take a smooth curve $(C, x_1, ..., x_n) \in \mathrm{Obj}(\mathfrak{M}_{g,n}(\mathrm{Spec}(k)))$ and consider $(C^{'}, x_1^{'}, ..., x_n^{'})$ where $C^{'} := C \cup \mathbb{P}^1$, $x_i^{'} := x_i$ for $i < n$ and $x_n^{'} := \infty \in \mathbb{P}^1$. Then $C^{'}$ is a nodal connected curve of arithmetic genus $p_a(C^{'}) = g$, but $\dim(\mathrm{Aut}(C^{'})) = 1$.

DEFINITION 3.2. *A pre-stable genus g curve $(C, x_1, ..., x_n)$ with n marked points is called stable if one of the following equivalent conditions are satisfied*
- $\mathrm{Aut}(C, x_1, ..., x_n)$ *is* étale*;*
- $\mathrm{Aut}(C, x_1, ..., x_n)$ *is finite;*
- *Let $\tilde{C} \to C$ be the normalization of C. For any irreducible component $\tilde{C}_i$ of $\tilde{C}$ the inequality $2g(\tilde{C}_i) - 2 + n_i > 0$ holds, where n_i is the number of special points on $\tilde{C}_i$, that are points mapped to a node or to a marked point on C.*

We define $\overline{\mathcal{M}}_{g,n}$ in the same way of the stack $\mathfrak{M}_{g,n}$ but adding the stability condition on the fibers. Clearly we have a natural morphism $\overline{\mathcal{M}}_{g,n} \to \mathfrak{M}_{g,n}$ and if $2g - 2 + n > 0$ there is a morphism $\mathcal{M}_{g,n} \to \overline{\mathcal{M}}_{g,n}$. Both these morphisms are open embeddings.
On the other hand we can construct a category fibered in groupoids in the following way. Let

$g, n \in \mathbb{Z}$ such that $g, n \geq 0$ and $2g - 2 + n > 0$. We define a category $\mathfrak{M}_{g,n}$ over the category of schemes in the following way. $\mathrm{Obj}(\mathfrak{M}_{g,n})$ consists of families

$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "\sigma_1,\dots,\sigma_n"'] \end{tikzcd}$$

where π is a flat and proper morphism, σ_i is a section of π for any $i = 1, ..., n$, $C_s = \pi^{-1}(s)$ is a smooth connected curve of genus g and $\sigma_i(s)$ are distinct smooth points for any $s \in S(k)$.
A morphism between two objects $C \to S$ and $C^{'} \to S^{'}$ is a couple $(\overline{f}, f)$ where $\overline{f} : C \to C^{'}$ and $f : S \to S^{'}$ are morphisms of schemes and the following diagrams

$$\begin{tikzcd} C \arrow[r, "\overline{f}"] \arrow[d, "\pi"'] & C^{'} \arrow[d, "\pi^{'}"] \\ S \arrow[r, "f"'] & S^{'} \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\overline{f}"] & C^{'} \\ S \arrow[r, "f"'] \arrow[u, "\sigma_i"] & S^{'} \arrow[u, "\sigma i^{'}"'] \end{tikzcd}$$

commute. This category is called the *category of n-pointed genus g smooth curves*. The category $\mathfrak{M}_{g,n}$ is a category fibered in groupoids over the category of schemes and this remains true even if the inequality $2g - 2 + n > 0$ does not hold. One can prove that in this category morphisms are a sheaf and that every descend datum is effective.

THEOREM 3.3. *The category fibered in groupoids $\mathfrak{M}_{g,n}$ is a stack.*

PROOF. Consider a scheme S and two families ξ and $\xi^{'}$

$$\begin{tikzcd} C \arrow[d, "\pi"'] \\ S \arrow[u, bend right, "\sigma_1,\dots,\sigma_n"'] \end{tikzcd} \qquad \begin{tikzcd} C^{'} \arrow[d, "\pi^{'}"'] \\ S \arrow[u, bend right, "\sigma^{'}_1,\dots,\sigma^{'}_n"'] \end{tikzcd}$$

parametrized by S. We define a functor

$$F : \mathfrak{Sch}/S \longrightarrow \mathfrak{Sets}$$

sending $f : X \to S$ to $Mor(f^*\xi, f^*\xi^{'})$. By applying the universal property of the fiber product we get the following diagrams

$$\begin{tikzcd} X \arrow[dr, "\sigma_{i,X}"] \arrow[drr, bend left, "\sigma_i \circ f"] \arrow[ddr, bend right, "Id_X"'] & & \\ & C_X := C \times_S X \arrow[r] \arrow[d, "\pi_X"'] & C \arrow[d, "\pi"] \\ & X \arrow[r, "f"] & S \end{tikzcd} \qquad \begin{tikzcd} X \arrow[dr, "\sigma^{'}_{i,X}"] \arrow[drr, bend left, "\sigma^{'}_i \circ f"] \arrow[ddr, bend right, "Id_X"'] & & \\ & C^{'}_X := C^{'} \times_S X \arrow[r] \arrow[d, "\pi^{'}_X"'] & C^{'} \arrow[d, "\pi^{'}"] \\ & X \arrow[r, "f"] & S \end{tikzcd}$$

To give a morphism $f^*\xi \to f^*\xi^{'}$ is equivalent to giving a morphism $\tilde{f} : C_X \to C^{'}_X$ such that $\sigma_{i,X} = \sigma^{'}_{i,X} \circ \tilde{f}$, $\pi_X = \pi^{'}_X \circ \tilde{f}$, and $\tilde{f}$ makes the diagram over the identity cartesian. That is $\tilde{f}$ is an isomorphism. Now, let $\{X_i \to X\}$ be an étale cover, and consider isomorphisms $\tilde{f}_i : C_{X_i} \to C^{'}_{X_i}$

such that $\tilde{f}_{i|C_{X_{i,j}}}$ and $\tilde{f}_{j|C_{X_{i,j}}}$ are naturally isomorphic. Since $\{C_{x_i} \to C_X\}$ is an étale cover and morphisms form a sheaf in the étale topology, the $\tilde{f}_i$ glue to a morphism $\tilde{f} : C_X \to C'_X$. The morphism $\tilde{f}$ commutes with $\pi_X, \sigma_{X,i}, \pi'_X, \sigma'_{X,i}$, since this is true for the $\tilde{f}_i$ and morphisms are a sheaf in the étale topology. Furthermore we can define $\tilde{g}^{-1}$ étale locally and then glue. This proves that morphisms are a sheaf.
Now, let S be a scheme, $\{S_i \to S\}$ an étale cover, ξ_i objects $C_i \to S_i$, and $\phi_{i,j} : C_{i|S_{i,j}} \to C_{j|S_{i,j}}$ isomorphisms. Using the $\phi_{i,j}$ we can glue the ξ_i to a global ξ over S, by descent theory we obtain a morphism $\pi : C \to S$. To construct the sections consider the composition

$$S_i \xrightarrow{\sigma_{S_i,j}} C_i \longrightarrow C$$

which agree locally and glue to define global sections $\sigma_{i,S} : S \to C$. Since $\{S_i \to S\}$ is an étale cover, and the ground field is algebraically closed, any morphism $\mathrm{Spec}(K) \to S$ factors through at least one of the $S_i \to S$. Then the fibers of π are genus g connected curves. Finally, since smoothness and properness are local in the target even in the Zariski topology the morphism π is smooth and proper. This proves that every descent datum is effective. □

LEMMA 3.4. *Let $(C, \{x_1, ..., x_n\})$ be a n-pointed genus g pre-stable curve. The sheaf $\omega_C(x_1 + ... + x_n)$ is ample if and only if $(C, \{x_1, ..., x_n\})$ is stable.*

PROOF. An invertible sheaf $\mathcal{L}$ on a proper curve C is ample if and only if it has positive degree on every irreducible component of C. Let C_i be an irreducible component of C. We have $\deg(\omega_C(x_1 + ... + x_n)_{|C_i|}) = \deg(\omega_{C|C_i}) + m_{C_i} = \deg(\omega_{C_i}) + \sharp(C_i \cap C_i^c) + m_{C_i} = 2p_a(C_i) - 2 + \sharp(C_i \cap C_i^c) + m_{C_i} = 2p_a(C_i) - 2 + n_{C_i}$, where m_{C_i}, n_{C_i} are respectively the number of marked and special points on C_i. Now, $\deg(\omega_C(x_1 + ... + x_n)_{|C_i|}) > 0$ for any i if and only if $2p_a(C_i) - 2 + n_{C_i} > 0$ for any i if only if $(C, \{x_1, ..., x_n\})$ is stable. □

DEFINITION 3.5. *Let X be a scheme, and G be a group scheme acting on X. The quotient stack $[X/G]$ is defined as the category whose objects are of the type*

$$\begin{array}{ccc} P & \longrightarrow & X \\ \downarrow & & \\ S & & \end{array}$$

where $P \to S$ is a principal G-bundle, $P \to X$ is a G-equivariant morphism, and whose morphisms are isomorphisms of principal G-bundle commuting with maps to X.

Let $\pi : C \to S$ be a family of stable curves of genus g. By Lemma 3.4 the relative dualizing sheaf $\omega_{C/S}$ is relatively ample. The r-th power $\omega_{C/S}^{\otimes r}$ is relatively ample, and $\pi_*\omega_{C/S}^{\otimes r}$ is locally free of rank $N + 1 = h^0(\omega_{C/S}^{\otimes r}) = (2r - 1)(g - 1)$ on S. Therefore any genus g stable curve can be embedded in $\mathbb{P}^N$ using the sections of $\omega_{C/S}^{\otimes r}$. The Hilbert polynomial of such a curve is determined by $\deg(P) = 1, P(0) = 1 - g, P(1) = \chi(\omega_{C/S}^{\otimes r})$. We can write $P(z) = Az + B$, then $P(0) = B = 1 - g$, and $P(1) = A = \chi(\omega_{C/S}^{\otimes r})$. Then

$$P(z) = (2rz - 1)(g - 1).$$

Let $\mathrm{Hilb}^P(\mathbb{P}^N)$ be the Hilbert scheme parametrizing subschemes of $\mathbb{P}^N$ with Hilbert polynomial P. There is a closed subscheme H of $\mathrm{Hilb}^P(\mathbb{P}^N)$ parametrizing m-canonically embedded stable curves. To give a morphism $S \to H$ is equivalent to give a closed subscheme $i : C \hookrightarrow \mathbb{P}^N \times S$ such that the projection $\pi : C \to S$ is a family of genus g stable curves, and there exists an isomorphism $\phi : \mathbb{P}(\pi_* \omega_{C/S}^{\otimes r}) \to \mathbb{P}^N \times S$ making the diagram

$$\begin{tikzcd} C \arrow[rr, "\phi"] \arrow[dr, "i"'] & & \mathbb{P}(\pi_* \omega_{C/S}^{\otimes r}) \arrow[dl] \\ & \mathbb{P}^N \times S & \end{tikzcd}$$

commutative. Finally there is a natural action of $\mathrm{Aut}(\mathbb{P}^N) = PGL(N+1)$ on H given by

$$PGL(N+1) \times H \to H,\ (\sigma, \alpha : C \hookrightarrow \mathbb{P}^N \times S) \mapsto (\sigma^{-1} \circ \alpha : C \hookrightarrow \mathbb{P}^N \times S).$$

THEOREM 3.6. *For $g \geq 2$ there is an equivalence of stacks*

$$\overline{\mathcal{M}}_g \cong [H/PGL(N+1)].$$

PROOF. Let $\pi : C \to S$ be a family of genus g stable curves. We have a canonical projective bundle $P_\pi := \mathbb{P}(\pi_* \omega_{C/S}^{\otimes r}) \to S$. Let $E := Isom_S(P_\pi, \mathbb{P}_S^N)$ be the S-scheme parametrizing isomorphisms from P_π to $\mathbb{P}_S^N$. The group $PGL(N+1)$ acts on E by

$$PGL(N+1) \times E \to E,\ (\sigma, \phi) \mapsto \sigma^{-1} \circ \phi.$$

and E is a $PGL(N+1)$-principal bundle. Now, consider the pull-back

$$\begin{tikzcd} C_E = C \times_S E \arrow[r, "\pi_E"] \arrow[d] & E \arrow[d] \\ C \arrow[r, "\pi"] & S \end{tikzcd}$$

since the projection $E \times_S E \to E$ has a section $\Delta : E \to E \times E$, the $\mathbb{P}^N$-bundle $P_{\pi_E} := \mathbb{P}(\pi_{E*} \omega_{C_E/E}^{\otimes m})$ is trivial, and we have an isomorphism $\xi_E : \mathbb{P}_{\pi_E} \to \mathbb{P}_S^N \times_S E$. Let $i_E : C_E \to \mathbb{P}_{\pi_E}$ be the canonical embedding, the composition $\xi_E \circ_E : C_E \to \mathbb{P}_S^N \times_S E$ gives a family of stable curves in $\mathbb{P}^N$, corresponding to a morphism $f_\pi : E \to H$, which clearly is $PGL(N+1)$-equivariant.
Now, consider a morphism

$$\begin{tikzcd} C' \arrow[r, "\phi"] \arrow[d, "\pi'"'] & C \arrow[d, "\pi"] \\ S' \arrow[r, "\psi"] & S \end{tikzcd}$$

in $\overline{\mathcal{M}}_g$. We have a canonical isomorphism $\pi'_* \omega_{C'/S'} \cong \phi^* \pi_* \omega_{C/S}$ and two cartesian squares

$$\begin{tikzcd} \mathbb{P}(\omega_{C'/S'}^{\otimes m}) \arrow[r] \arrow[d] & \mathbb{P}(\omega_{C/S}^{\otimes m}) \arrow[d] \\ S' \arrow[r, "\psi"] & S \end{tikzcd} \quad \begin{tikzcd} E' \arrow[r, "f_{\phi'}"] \arrow[d] & E \arrow[d] \\ S' \arrow[r, "\phi"] & S \end{tikzcd}$$

where $f_{\phi'}$ is compatible with f_π and $f_{\pi'}$. Then we get the following:

- an objects $\pi : C \to S$ to

$$\begin{tikzcd} E \arrow[r, "f_\pi"] \arrow[d] & H \\ S & \end{tikzcd}$$

- a morphism

$$\begin{tikzcd} C^{'} \arrow[r, "\phi"] \arrow[d, "\pi^{'}"'] & C \arrow[d, "\pi"] \\ S^{'} \arrow[r, "\psi"] & S \end{tikzcd}$$

to a morphism

$$\begin{tikzcd} E^{'} \arrow[r, "f_{\phi^{'}}"] \arrow[d] & E \arrow[d] \\ S^{'} \arrow[r, "\phi"] & S \end{tikzcd}$$

This defines a morphism of stacks

$$F : \overline{\mathcal{M}}_g \to [H/PGL(N+1)].$$

On the other hand given a morphism $S \to H$ we have a corresponding family $\pi_S : C \to S$ of genus g stable curves embedded in $\mathbb{P}^N_S$. By forgetting the embedding $C \hookrightarrow \mathbb{P}^N_S$ we obtain an object in $\overline{\mathcal{M}}_g$, furthermore morphisms in the same $PGL(N+1)$-orbit are sent to the same object of $\overline{\mathcal{M}}_g$. So we get a morphism

$$G : [H/PGL(N+1)] \to \overline{\mathcal{M}}_g.$$

Take an object $\xi := (E^{'}/S \to H)$ in $[H/PGL(N+1)]$, and let $\tilde{\pi}_{E^{'}} : C^{'} \to E^{'}$ be the family induced by the $PGL(N+1)$-equivariant morphism $E^{'} \to H$. If $\mathcal{H} \to H$ is the universal family then $\tilde{\pi}_{E^{'}} : C^{'} \to E^{'}$ is the pull-back of $\mathcal{H} \to H$ by the morphism $E^{'} \to H$. Furthermore if $E \to E^{'}$ we can consider the pull-back $\tilde{C}_E \to E$ and the following diagram

$$\begin{tikzcd} \tilde{C}_E \arrow[r] \arrow[d] & \tilde{C} \arrow[r] \arrow[d, "\tilde{\pi}_{E^{'}}"] & \mathcal{H} \arrow[d] \\ E \arrow[r] & E^{'} \arrow[r] \arrow[d] & H \\ & S & \end{tikzcd}$$

The scheme $\tilde{C}_E$ carries a natural $PGL(N+1)$-action. By descent theory $C = \tilde{C}_E/PGL(N+1)$ exists as a scheme, and there is a morphism $\pi : C \to S$ such that the base extension $\pi_{E^{'}} : C \times_S E^{'} \to E^{'}$ is exactly $\tilde{\pi}_{E^{'}} : \tilde{C} \to E^{'}$:

$$\begin{tikzcd} \tilde{C} \arrow[r, "\sim"] \arrow[dr, "\tilde{\pi}_{E^{'}}"'] & C \times_S E^{'} \arrow[r] \arrow[d, "\pi_{E^{'}}"] & C \arrow[d, "\pi"] \\ & E^{'} \arrow[r] & S \end{tikzcd}$$

The family $\pi : C \to S$ is exactly $G(\xi) \in \overline{\mathcal{M}}_g$. If $E = Isom_S(P_\pi, \mathbb{P}^N_S)$ where $P_\pi = \mathbb{P}(\pi_*\omega^{\otimes m}_{C/S})$ we get that $F \circ G(\xi)$ is isomorphic to ξ, that is $F \circ G \cong Id$. Finally, from the construction it is clear that $G \circ F \cong Id$. □

PROPOSITION 3.7. *For any $g \geq 2$ the stack $\overline{\mathcal{M}}_g$ is a Deligne-Mumford stack.*

PROOF. Since a genus $g \geq 2$ stable curve over an algebraically closed field has a finite and reduced automorphism group the stabilizers of the geometric points of $\overline{\mathcal{M}}_g$ are finite and reduced. So $\overline{\mathcal{M}}_g$ is a DM stack. □

4. Details on algebraic Curves

In this section we recall some well known results on algebraic curves and their automorphisms. Finally, using deformation theory we prove that $\overline{\mathcal{M}}_g$ is as smooth stack.

Grothendieck Spectral Sequence. We begin recalling the notion of *five terms exact sequence* or *exact sequence of low degree terms* associated to a spectral sequence. Let

$$E_2^{h,k} \Longrightarrow H^n(A)$$

be a spectral sequence whose terms are non trivial only for $h, k \geq 0$. Then this is an exact sequence

$$0 \mapsto E_2^{1,0} \to H^1(A) \to E_2^{0,1} \to E_2^{2,0} \to H^2(A).$$

The *Grothendieck spectral sequence* is an algebraic tool to express the derived functors of a composition of functors $\mathcal{G} \circ \mathcal{F}$ in terms of the derived functors of $\mathcal{F}$ and $\mathcal{G}$.
Let $\mathcal{F} : \mathcal{C}_1 \to \mathcal{C}_2$ and $\mathcal{G} : \mathcal{C}_2 \to \mathcal{C}_3$ be two additive covariant functors between abelian categories. Suppose that $\mathcal{G}$ is left exact and that $\mathcal{F}$ takes injective objects of $\mathcal{C}_1$ in $\mathcal{G}$-acyclic objects of $\mathcal{C}_2$. Then there exists a spectral sequence for any object A of $\mathcal{C}_1$

$$E_2^{h,k} = (R^h\mathcal{G} \circ R^k\mathcal{F})(A) \Longrightarrow R^{h+k}(\mathcal{G} \circ \mathcal{F})(A).$$

The corresponding exact sequence of low degrees is the following

$$0 \mapsto R^1\mathcal{G}(\mathcal{F}(A)) \to R^1(\mathcal{G}\mathcal{F}(A)) \to \mathcal{G}(R^1\mathcal{F}(A)) \to R^2\mathcal{G}(\mathcal{F}(A)) \to R^2(\mathcal{G}\mathcal{F})(A).$$

As a special case of the Grothendieck spectral sequence we get the *Leray spectral sequence*. Let $f : X \to Y$ be a continuous map between topological spaces. We take $\mathcal{C}_1 = \mathfrak{Ab}(X)$ and $\mathcal{C}_2 = \mathfrak{Ab}(Y)$ to be the categories of sheaves of abelian groups over X and Y respectively. Then we take $\mathcal{F}$ to be the direct image functor $f_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ and $\mathcal{G} = \Gamma_Y : \mathfrak{Ab}(Y) \to \mathfrak{Ab}$ to be the global section functor, where $\mathfrak{Ab}$ is the category of abelian groups. Note that

$$\Gamma_Y \circ f_* = \Gamma_X : \mathfrak{Ab}(X) \to \mathfrak{Ab}$$

is the global section functor on X. By Grothendieck's spectral sequence we know that $(R^h\Gamma_Y \circ R^k f_*)(\mathcal{E}) \Longrightarrow R^{h+k}(\Gamma_Y \circ f_*)(\mathcal{E}) = R^{h+k}\Gamma_X(\mathcal{E})$ for any $\mathcal{E} \in \mathfrak{Ab}(X)$, that is

$$H^h(Y, R^k f_*\mathcal{E}) \Longrightarrow H^{h+k}(X, \mathcal{E}).$$

The exact sequence of low degrees looks like

$$0 \mapsto H^1(Y, f_*\mathcal{E}) \to H^1(X, \mathcal{E}) \to H^0(Y, R^1 f_*\mathcal{E}) \to H^2(Y, f_*\mathcal{E}) \to H^2(X, \mathcal{E}).$$

Finally we work out the *spectral sequence of Ext functors*. Let $\mathcal{E} \in \mathfrak{Coh}(X)$ be a coherent sheaf on a scheme X. Consider the functor

$$\mathcal{H}om(\mathcal{E}, -) : \mathfrak{Coh}(X) \to \mathfrak{Coh}(X),\ \mathcal{Q} \mapsto \mathcal{H}om(\mathcal{E}, \mathcal{Q}),$$

and the global section functor

$$\Gamma_X : \mathfrak{Coh}(X) \to \mathfrak{Ab},\ \mathcal{Q} \mapsto \Gamma_X(\mathcal{Q}).$$

Note that $\Gamma_X \circ \mathcal{H}om(\mathcal{E}, -) = Hom(\mathcal{E}, -)$. By Grothendieck spectral sequence we have $(R^h\Gamma_X \circ R^k\mathcal{H}om(\mathcal{E}, -))(\mathcal{Q}) \Longrightarrow R^{h+k}(Hom(\mathcal{E}, -)(\mathcal{Q})$ for any $\mathcal{Q} \in \mathfrak{Coh}(X)$, that is

$$H^h(X, \mathcal{E}xt^k(\mathcal{E}, \mathcal{Q})) \Longrightarrow Ext^{h+k}(\mathcal{E}, \mathcal{Q}).$$

The corresponding sequence of low degrees is

$$0 \mapsto H^1(X, \mathcal{H}om(\mathcal{E}, \mathcal{Q})) \to Ext^1(\mathcal{E}, \mathcal{Q}) \to H^0(X, \mathcal{E}xt^1(\mathcal{E}, \mathcal{Q})) \to H^2(X, \mathcal{H}om(\mathcal{E}, \mathcal{Q})) \to Ext^2(\mathcal{E}, \mathcal{Q}).$$

Deformations of Schemes. Let X be a smooth scheme of finite type over k. We define the deformation functor $Def_X : \mathfrak{Art} \to \mathfrak{Sets}$ of X sending an Artin ring A to the set of couples $(X_A \overset{\pi_A}{\to} \mathrm{Spec}(A), \phi)$ modulo isomorphism, where π_A is a smooth morphism, $\phi : X \to X_0$ is an isomorphism, X_0 is defined by the cartesian diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_A \arrow[d] \\ \mathrm{Spec}(k) \arrow[r] & \mathrm{Spec}(A) \end{tikzcd}$$

and (X_A, ϕ), (X'_A, ϕ') are isomorphic if there is an isomorphism $\alpha : X_A \to X'_A$ such that the diagram

$$\begin{tikzcd} X_A \arrow[rr, "\alpha"] \arrow[dr, "\pi_A"'] & & X'_A \arrow[dl, "\pi'_A"] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

commutes and $\phi' = \alpha \circ \phi$.

THEOREM 4.1. *For any semi-small exact sequence* $0 \mapsto I \to A \to B \mapsto 0$ *in* $\mathfrak{Art}$, *let* $T^iDef_X = H^i(X, T_X)$, *then*

(1) *there exists a functorial exact sequence*

$$T^1Def_X \otimes I \to Def_X(A) \to Def_X(B) \to T^2Def_X \otimes I;$$

(2) *for any* $(X_A, \pi_A, \phi) \in Def_X(A)$, *let* $G = Stab(X_A) \subseteq T^1Def_X \otimes I$, *we have a functorial exact sequence*

$$0 \mapsto T^0Def_X \otimes I \to \mathrm{Aut}(X_A) \to \mathrm{Aut}(X_B) \to G \mapsto 0.$$

Now let X be any scheme over k. Consider the exact sequence of low degree for Ext functors with sheaves Ω_X and $\mathcal{O}_X$. We have

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to Ext^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to H^2(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)).$$

The set of deformations of X over the dual numbers $D = \frac{k[\epsilon]}{\epsilon^2}$ is in one-to-one correspondence with the group $Ext^1(\Omega_X, \mathcal{O}_X)$. Then we get the sequence

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to Def_X(D) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \to H^2(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)).$$

Differentials and Ext groups. Let X be a smooth scheme and let Y be a closed subscheme with ideal sheaf $\mathcal{I}$. We have an exact sequence of sheaves

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0,$$

where the first map is the differential. Furthermore Y is smooth if and only if

- Ω_Y is locally free,
- the sequence is also exact on the left

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

In this case the sheaf $\mathcal{I}$ is locally generated by $Codim(Y, X)$ elements, and its is locally free of rank $Codim(Y, X)$ on Y.

REMARK 4.2. Let $Y \subseteq X$ be an hypersurface not necessarily smooth. We can associate to Y a Cartier divisor $\{(\mathcal{U}_i, f_i)\}$, and the ideal sheaf $\mathcal{I}$ is locally generated by f_i on $\mathcal{U}_i$. Furthermore $\mathcal{O}_X(Y)$ is the sheaf locally generated by f_i^{-1} on $\mathcal{U}_i$. We conclude that $\mathcal{O}_X(-Y) \cong \mathcal{I}$ is locally free. If $Y \subseteq X$ is a reduced hypersurface, then $\mathcal{I}$ is locally free of rank one. We have the differential $d : \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y$, if f is a local generator of $\mathcal{I}$ then df is a local generator of $Im(d)$, since Y is reduced then $df \neq 0$, $Im(d)$ is locally free of rank one, and the map d is injective. So we have again an exact sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Let $f = f(x_1, ..., x_n)$, with $n = dim(X)$, be a local equation for Y in X. Then $df = \frac{\partial f}{\partial x_1}dx_1 + ... + \frac{\partial f}{\partial x_n}$. Since Y is reduced the differential is injective, furthermore $\mathcal{I}/\mathcal{I}^2$ is locally free of rank one and $\Omega_X \otimes \mathcal{O}_Y$ is locally free of rank n. Applying $Hom(-, \mathcal{O}_Y)$ to the sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0,$$

we obtain

$$0 \mapsto Hom(\Omega_Y, \mathcal{O}_Y) \to Hom(\Omega_{X|Y}, \mathcal{O}_Y) \to Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \to Ext^1(\Omega_Y, \mathcal{O}_Y) \to Ext^1(\Omega_{X|Y}, \mathcal{O}_Y).$$

REMARK 4.3. Let X be a noetherian scheme such that any coherent sheaf on X is quotient of a locally free sheaf i.e. $Coh(X)$ has enough locally free objects. We define the homological dimension of $\mathcal{F} \in Coh(X)$, denoted by $hd(\mathcal{F})$, to be the least length of a locally free resolution of $\mathcal{F}$ or ∞ if there is no finite one. Clearly $\mathcal{F}$ is locally free if and only if $hd(\mathcal{F}) = 1$ if and only if $Ext^1(\mathcal{F}, \mathcal{G}) = 0$ far any $\mathcal{G} \in Mod(X)$. Furthermore $hd(\mathcal{F}) \leq n$ if and only if $Ext^i(\mathcal{F}, \mathcal{G}) = 0$ for any $i > n$ and $\mathcal{G} \in Mod(X)$. Finally $hd(\mathcal{F}) = Sup_{x \in X}(pd_{\mathcal{O}_x}\mathcal{F}_x)$, where pd is the projective dimension.

In our case $\Omega_{X|Y}$ is locally free, and by the preceding remark $Ext^1(\Omega_{X|Y}, \mathcal{O}_Y) = 0$. Then we get the exact sequence

$$0 \mapsto Hom(\Omega_Y, \mathcal{O}_Y) \to Hom(\Omega_{X|Y}, \mathcal{O}_Y) \to Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y) \to Ext^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

Consider now the special case $X = \mathbb{A}^n$ and $Y = \mathrm{Spec}(A)$, where $A = k[x_1, ..., x_n]/(f)$. The map $Hom(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \to Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is the transpose of the differential $d : \mathcal{I}/\mathcal{I}^2 \to \Omega_{\mathbb{A}^n|Y}$. Furthermore $Hom(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \cong A^n$ and $Hom(\mathcal{I}/\mathcal{I}^2) \cong A$. We can write the map $Hom(\Omega_{\mathbb{A}^n|Y}, \mathcal{O}_Y) \to Hom(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ as

$$\phi : A^n \to A,\ (\alpha_1, ..., \alpha_n) \mapsto \alpha_1 \frac{\partial f}{\partial x_1} + ... + \alpha_n \frac{\partial f}{\partial x_n}.$$

We rewrite our exact sequence as

$$0 \mapsto Hom(\Omega_Y, \mathcal{O}_Y) \to A^n \to A \to Ext^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

Then $Im(\phi) = (\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n}) \subseteq A$, and $Ext^1(\Omega_Y, \mathcal{O}_Y) \cong A/(\frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n})$.
Now let $Y = C \subseteq \mathbb{A}^2$ be a nodal curve. In an *é*tale neighborhood of the node we can assume $C = \mathrm{Spec}(A)$, where $A = k[x,y]/(xy)$. From the preceding discussion we get $Ext^1(\Omega_C, \mathcal{O}_C) \cong A/(x,y) \cong k$. So $Ext^1(\Omega_C, \mathcal{O}_C)_p = 0$ if p is a smooth point of C and $Ext^1(\Omega_C, \mathcal{O}_C)_p = k$ if $p \in Sing(C)$. Furthermore

$$\mathcal{E}xt^1(\Omega_C, \mathcal{O}_X) \cong \sum_{p \in Sing(C)} \mathcal{O}_p.$$

Curves of Genus One. An *elliptic curve* over an algebraically closed field is a smooth projective curve of genus one.
Let X be an elliptic curve and let $P \in X$ be a point, consider the linear system $|2P|$ on X. Since the curve is not rational $|2P|$ has no base points, and since $deg(K - 2P) = 2g - 2 - 2 = -2 < 0$ the divisor $|2P|$ is non-special i.e. $h^0(K - 2P) = 0$. By Riemann-Roch theorem $h^0(2P) = deg(2P) - g + 1 = 2$. Then the linear system $|2P|$ defines a morphism $f : X \to \mathbb{P}^1$ of degree 2 on $\mathbb{P}^1$. Now by Riemann-Hurwitz theorem we have

$$2g - 2 = deg(f)(2g_{\mathbb{P}^1} - 2) + deg(R_f),$$

then $deg(R_f) = 2 \cdot deg(f) = 4$, and f is ramified in four points and clearly P is one of them. If x_1, x_2, x_3, ∞ are the four branch points in $\mathbb{P}^1$, then there is a unique automorphism of $\mathbb{P}^1$ sending x_1 to 0, x_1 to 1, and leaving ∞ fixed, namely $y = \frac{x - x_1}{x_2 - x_1}$. After this change of coordinates we can assume that f is branched over $0, 1, \lambda, \infty \in \mathbb{P}^1$, whit $\lambda \in k$, $\lambda \neq 0, 1$.
We define the j-invariant of the elliptic curve X by

$$j = j(\lambda) = 2^8 \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

It is well known that over an algebraically closed field k with $char(k) \neq 2$ the scalar $j(X)$ depends only on X. Furthermore two elliptic curves $X, X^{'}$ are isomorphic if and only if $j(X) = j(X^{'})$, and every element of k is the j-invariant of some elliptic curve. Then there is a one-to-one correspondence with the set of elliptic curves up to isomorphism and $\mathbb{A}^1_k$ given by $X \mapsto j(X)$.

DEFINITION 4.4. *A family of elliptic curves over a scheme S is a flat morphism of schemes $X \to S$ whose fibers are smooth curves of genus one, with a section $\sigma : S \to X$. In particular, an elliptic curve is a smooth curve C of genus one with a rational point $P \in C$.*

Consider the functor $\mathcal{F} : \mathfrak{Sch} \to \mathfrak{Sets}$ where $\mathcal{F}(S)$ is the set of families of elliptic curves over S modulo isomorphism. One can prove that $\mathcal{F}$ does not have a fine moduli space, but the affine line $\mathbb{A}^1_k$ is a coarse moduli space for $\mathcal{F}$.
Now a natural question is how to compactify this coarse moduli space to obtain a complete moduli space. In addition to elliptic curves we admit also irreducible nodal curve of arithmetic genus $p_a = 1$ with a fixed nonsingular point. We consider families $X \to S$ whose fibers are elliptic curves or pointed nodal curve, then taking $j(C) = \infty$ for the nodal curve the projective line $\mathbb{P}^1$ becomes a coarse moduli space.
Let C be a reduced, irreducible curve with $p_a = 1$ and such that $Sing(C)$ is a node. Such a curve

can be embedded in $\mathbb{P}^2$ as the nodal cubic $C = Z(y^2z - x^3 + x^2z)$. Consider the low degrees exact sequence for Ext functors,

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_C, \mathcal{O}_C)) \to Ext^1(\Omega_C, \mathcal{O}_C) \to H^0(X, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) \to H^2(X, \mathcal{H}om(\Omega_C, \mathcal{O}_C)).$$

Since $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is concentrated at the singular point of C we know that $H^0(X, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$ is a 1-dimensional k-vector space. Now we consider the sheaf $\mathcal{H}om(\Omega_C, \mathcal{C}) = T_C$.
Recall that if X is a smooth variety and $Y \subseteq X$ is a closed irreducible subscheme defined by the sheaf of ideals $\mathcal{I}$, then there is an exact sequence

$$\mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Furthermore Y is smooth if and only if

- the sheaf Ω_Y is locally free, and
- the sequence above is also exact on the left

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_X \otimes \mathcal{O}_Y \to \Omega_Y \mapsto 0.$$

Consider the sequence for a general subscheme Y and apply the functor $\mathcal{H}om(-, \mathcal{O}_Y)$. We obtain

$$0 \mapsto T_Y \to T_{X|Y} \to N_{Y/X} \to \mathcal{E}xt^1(\Omega_Y, \mathcal{O}_Y) \mapsto 0.$$

For our nodal curve C in $\mathbb{P}^2$ we have

$$0 \mapsto T_C \to T_{\mathbb{P}^2|C} \to N_{C/\mathbb{P}^2} \to \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C) \mapsto 0.$$

We know that $N_{C/\mathbb{P}^2} = \mathcal{O}_C(C) = \mathcal{O}_C(3)$, let D be the divisor associated to $\mathcal{O}_C(3)$. Since C is a local complete intersection the dualizing sheaf ω° is an invertible sheaf. We define the canonical divisor as the divisor corresponding to ω° with support in C_{reg}. Since there are no regular differentials on C we have $deg(K - D) < 0$. By Riemann-Roch theorem for singular curves we get

$$h^0(N_{C/\mathbb{P}^2}) = deg(D) + 1 - p_a = 9 + 1 - 1 = 9.$$

Consider now the Euler sequence

$$0 \mapsto \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3} \to T_{\mathbb{P}^2} \mapsto 0.$$

Tensorizing by $\mathcal{O}_C$ we get

$$0 \mapsto \mathcal{O}_C \to \mathcal{O}_C(1)^{\oplus 3} \to T_{\mathbb{P}^2|C} \mapsto 0.$$

Using the dualizing sheaf $\omega_C^\circ \cong \mathcal{O}_C$, and Serre duality we get $h^1(\mathcal{O}_C(1)) = h^0(\mathcal{O}_C(-1)) = 0$. The cohomology sequence looks like

$$0 \mapsto H^0(C, \mathcal{O}_C) \to H^0(C, \mathcal{O}_C(1)^{\oplus 3}) \to H^0(C, T_{\mathbb{P}^2|C}) \to H^1(C, \mathcal{O}_C) \mapsto 0,$$

so $h^0(T_{\mathbb{P}^2|C}) = 9$. Furthermore the map $H^0(C, N_{C/\mathbb{P}^2}) \to H^0(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C))$ is surjective since the former parametrizes the embedded deformations of C as a subscheme of $\mathbb{P}^2$ and the latter parametrizes the abstract deformations of the node. We conclude that $h^0(T_C) > 0$. Let $\sigma \in H^0(C, T_C)$ be a nonzero section, we have an exact sequence $0 \mapsto \mathcal{O}_C \xrightarrow{\sigma} T_C \to R \mapsto 0$. The cokernel R is not zero, because T_C is not locally free. Then $\check{T_C}$ is a proper subsheaf of $\mathcal{O}_C$, using the dualizing sheaf $\omega_C^\circ \cong \mathcal{O}_C$ and Serre duality we get $h^1(T_C) = h^0(\check{T_C}) = 0$. We conclude that $Def(C)$ is one-dimensional.

Automorphisms of Curves. The only curve of genus one is $\mathbb{P}^1$, and its automorphism group is $PGL(2)$ which is an open subset of $\mathbb{P}^3$. If we choose one or two marked points in $\mathbb{P}^1$ the automorphism group remains infinite of dimension two and one respectively. However a well known theorem in projective geometry asserts that if we fix three marked points the automorphism group is trivial.
We will see that an elliptic curve has infinitely many automorphisms, but if we choose a marked point then its automorphism group is finite. Finally we will prove that any curve X of genus $g \geq 2$ has finitely many automorphisms, and we will give a bound on the cardinality on $\mathrm{Aut}(X)$.
Recall that an elliptic curve X has a group structure, more precisely if we fix a point on X then we get a bijective correspondence between the points of X and the divisors of degree zero in $Cl^0(X)$, so any translation $X \times X \to X$ gives an automorphism of X. Clearly if we choose a marked point $p \in X$, then the only possible translation is the identity, in this way the automorphism group becomes finite.

PROPOSITION 4.5. *Let E be an elliptic curve over k with a marked point. The automorphism group* $\mathrm{Aut}(E)$ *is a finite group of order dividing* 24. *More precisely*
- *if* $j(E) \neq 0, 1728$, *then* $|\mathrm{Aut}(E)| = 2$,
- *if* $j(E) = 1728$ *and* $char(k) \neq 2, 3$, *then* $|\mathrm{Aut}(E)| = 4$,
- *if* $j(E) = 0$ *and* $char(k) \neq 2, 3$, *then* $|\mathrm{Aut}(E)| = 6$,
- *if* $j(E) = 0, 1728$ *and* $char(k) = 3$, *then* $|\mathrm{Aut}(E)| = 12$,
- *if* $j(E) = 0, 1728$ *and* $char(k) = 2$, *then* $|\mathrm{Aut}(E)| = 24$.

PROOF. We consider the case $char(k) \neq 2, 3$. Then E can be realized as a plane smooth cubic and can be written in Weierstrass form
$$y^2 = x^3 + \alpha x + \beta,$$
furthermore every automorphism of E is of the form
$$x = u^2 x', \; y = u^3 y',$$
for some $u \in \underline{k}^*$. Such a substitution will give an automorphism if and only if
$$u^{-4}\alpha = \alpha, \; u^{-6}\beta = \beta.$$
If $\alpha \cdot \beta = 0$ then $j(E) \neq 0, 1728$, the only possibilities are $u = \pm 1$. If $\beta = 0$ then $j(E) = 1728$, and u satisfies $u^4 = 1$, so $\mathrm{Aut}(E)$ is cyclic of order 4. If $\alpha = 0$ then $j(E) = 0$, and u satisfies $u^6 = 1$, so $\mathrm{Aut}(E)$ is cyclic of order 6. $\square$

PROPOSITION 4.6. *Any smooth curve X of genus $g \geq 2$ has finitely many automorphisms.*

Before proving the proposition we recall some general facts about canonically embedded varieties.

REMARK 4.7. (*Canonically Embedded Varieties*) Let $f : X \to Y$ be a dominant morphism between smooth varieties. The pullback $f^* : f^*\Omega_Y \to \Omega_X$ defines a canonical morphisms between the cotangent sheaves, and since pullback commutes with maximal exterior powers we get a canonical morphism $f^* : f^*\omega_Y \to \omega_X$ of the canonical sheaves. In particular if $X = Y$ and $f \in \mathrm{Aut}(X)$, since $f^*\omega_X \cong \omega_X$, we get an automorphism f^* of ω_X. Then an automorphism of X induces an automorphism of ω_X, and an automorphism on the vector space of the its global section $H^0(X, \omega_X)$. Suppose now that ω_X is ample, then $\omega_X^{\otimes n}$ is very ample for some $n \geq 0$. Any automorphism of X induces also an automorphism of $\omega_X^{\otimes n}$. Let $\phi : X \to \mathbb{P}(H^0(X, \omega_X^{\otimes n})^*)$ be the corresponding embedding. Then we have an action of $\mathrm{Aut}(X)$ on $\mathbb{P}(H^0(X,\omega_X^{\otimes n})^*)$, and any $f \in \mathrm{Aut}(X)$ induces an automorphism of $\mathbb{P}(H^0(X,\omega_X^{\otimes n})^*) = \mathbb{P}^N$. We have seen that if X has ample canonical sheaf then $\mathrm{Aut}(X)$ is a closed algebraic subgroup of $PGL(N+1)$. Clearly the same argument works if X has ample anticanonical sheaf.

PROOF. Recall that if $f : X \to Y$ is a morphism of schemes, with X separated and Y smooth, and Def_f is the deformation functor of f, then $T^1 Def_f = H^0(X, f^*T_Y)$. In particular for $f = Id_X : X \to X$ we get $T^1_{Id_X} Def_{Id_X} = T_{Id_X}\mathrm{Aut}(X) = H^0(X, T_X)$, and $h^0(X,T_X) = 0$ since X is a curve of genus $g \geq 2$. The curve X has canonical ample sheaf, and by the preceding remark we can embed $\mathrm{Aut}(X)$ in $PGL(N+1) \subseteq \mathbb{P}^{(N+1)^2-1}$ as closed subscheme. Since the tangent space of $\mathrm{Aut}(X)$ has dimension zero we conclude that $\mathrm{Aut}(X)$ is a finite set of points. □

In the following proposition we give a bound on the number of automorphisms of a curve of genus $g \geq 2$.

PROPOSITION 4.8. *Let X be a projective curve of genus $g \geq 2$, then the group* $\mathrm{Aut}(X)$ *is finite and* $|\mathrm{Aut}(X)| \leq 84(g-1)$.

PROOF. Let $W(X)$ be the set of Weierstrass points of X, we know that $W(X)$ is finite. If $\phi \in \mathrm{Aut}(X)$ is a non trivial automorphism then ϕ has at most $2g+2$ fixed points. Since the set of Weierstrass points is fixed by the group $\mathrm{Aut}(X)$ we have a morphism

$$F : \mathrm{Aut}(X) \to Perm(W(X)),$$

where $Perm(W(X))$ is the group of permutations of $W(X)$. If X is non hyperelliptic there are more than $2g+2$ Weierstrass points on X and there is a unique automorphism that leaves more that $2g+2$ points fixed, the identity. So $\ker(F) = \{Id_X\}$.
If X is hyperelliptic then any automorphism in the subgroup (J) generated by the involution $J : X \to X$ fixes the Weierstrass points, but since $J^2 = Id_X$ this subgroup is finite. We conclude that F is a morphism of $\mathrm{Aut}(X)$ into a finite group and with finite kernel, then the group $\mathrm{Aut}(X)$ is finite.
Let $G = \mathrm{Aut}(X)$ and $|G| = n$, consider the projection $\pi : X \to X/G$. For any $\overline{x} \in X/G$ we have $\pi^{-1}(\overline{x}) = \{x \in X \mid \pi(x) = \overline{x}\} = \{x \in X \mid \exists\, g \in G,\, g(x) = \overline{x}\} = \{g^{-1}(\overline{x}),\, g \in G\}$, then π is a morphism of degree n. The map π is branched only at fixed point of G. Let $P_1, ..., P_s$ be a maximal sets of ramification points of X lying over distinct points of X/G, and let r_i be the index of ramification of P_i. Recall that if $P \in X$ is a ramification point, and r is its ramification index, then the fiber $\pi^{-1}(\pi(P))$ consists of exactly $\frac{n}{r}$ points, each having ramification index r, essentially because X is a covering space for X/G. So in the fiber of any P_j there are $\frac{n}{r_j}$ points each with ramification index r_j. Then the degree of the ramification divisor is

$$deg(R_\pi) = \sum_{j=1}^{s}(r_j - 1)\frac{n}{r_j} = n\sum_{j=1}^{s}(1 - \frac{1}{r_j}).$$

By Riemann-Hurwitz formula we get $2g-2 = n(2\alpha-2) + n\sum_{j=1}^{s}(1-\frac{1}{r_j})$, where α is the genus of X/G. Then

$$\frac{2g-2}{n} = 2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}).$$

Note that since $r_j \geq 2$ we have $\frac{1}{2} \leq 1 - \frac{1}{r_j} < 1$. Since we may assume $n > 1$ it is clear that $g > \alpha$. Now we have to analyze the expression $2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j})$.

- If $\alpha \geq 2$ we obtain $2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}) \geq 2 - \sum_{j=1}^{s}(1 - \frac{1}{r_j}) \geq 2$, so $\frac{2g-2}{n} \geq 2$ and
$$n \leq g - 1.$$
- If $\alpha = 1$ then $2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}) = \sum_{j=1}^{s}(1 - \frac{1}{r_j}) \geq \frac{1}{2}$, so $\frac{2g-2}{n} \geq \frac{1}{2}$ and
$$n \leq 4(g - 1).$$
- If $\alpha = 0$ then $2\alpha - 2 + \sum_{j=1}^{s}(1 - \frac{1}{r_j}) = \sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2$. Since $\sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2 > 0$ and $1 - \frac{1}{r_j} < 1$, we conclude that $s \geq 3$.
  - If $s \geq 5$, then $\sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2 \geq \frac{1}{2}$, so $\frac{2g-2}{n} \geq \frac{1}{2}$ and
  $$n \leq 4(g - 1).$$
  - If $r = 4$ then the r_j cannot be all equal to 2, otherwise we would have $\frac{2g-2}{n} = 0$, so $g = 1$. Then at least one is ≥ 3 and gives$\sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2 \geq 3(1 - \frac{1}{2}) + (1 - \frac{1}{3}) - 2 = \frac{1}{6}$, so $\frac{2g-2}{n} \geq \frac{1}{6}$ and
  $$n \leq 12(g - 1).$$
  - In the case $s = 3$ we can assume without loss of generality $2 \leq r_1 \leq r_2 \leq r_3$. We have $r_3 > 3$ otherwise $\sum_{j=1}^{s}(1 - \frac{1}{r_j}) - 2 < 0$. Then $r_2 \geq 3$.
  If $r_3 \geq 7$ then $n \leq 84(g - 1)$.
  If $r_3 = 6$ and $r_1 = 2$ then $r_2 \geq 4$ and $n \leq 24(g - 1)$.
  If $r_3 = 6$ and $r_1 \geq 3$ then $n \leq 12(g - 1)$.
  If $r_3 = 5$ and $r_1 = 2$ then $r_2 \geq 4$ and $n \leq 40(g - 1)$.
  If $r_3 = 5$ and $r_1 \geq 3$ then $n \leq 15(g - 1)$.
  If $r_3 = 4$ then $r_1 \geq 3$ and $n \leq 24(g - 1)$.

$\square$

To compactify the coarse moduli space M_g Deligne and Mumford introduces *stable* curves. We have seen that $T_{Id_X} \operatorname{Aut}(X) = H^0(X, T_X)$, an element of this space is called an *infinitesimal automorphism*.

DEFINITION 4.9. *A reduced, connected, projective curve X, having at most nodes as singularities is said to be stable if $H^0(X, T_X) = 0$, i.e. X has no infinitesimal automorphisms.*

Clearly for a curve X of genus $g \geq 2$ the following are equivalent,
- X has no infinitesimal automorphisms,
- $H^0(X, T_X) = 0$,
- $\operatorname{Aut}(X)$ is finite.

By the preceding discussion any smooth curve of genus $g \geq 2$ is stable.

Consider the local infinitesimal deformation functor of $\mathcal{F}$ for a stable curve X of genus $g \geq 2$,
$$Def_X : \mathfrak{Art} \to \mathfrak{Sets},$$
which associates to any Artin local algebra A the set of isomorphism classes $Y \to \operatorname{Spec}(A)$ of families of curves of genus g over $\operatorname{Spec}(A)$, with a fixed isomorphism $Y_0 \to X$, where $Y_0 \to \operatorname{Spec}(k)$

is the central fiber of Y. Note that the isomorphism $Y_0 \to X$ is not unique, indeed we can recover any other isomorphism composing with an automorphism of X, and the set of such isomorphisms is a principal homogeneous space under the action of $\mathrm{Aut}(X)$. The following remark will be important in order to prove that $\overline{\mathcal{M}}_g$ is smooth.

REMARK 4.10. Let X be a proper scheme and let Def_X be its deformation functor. Then $T^i_{Def_X} = Ext^i(L^\bullet_X, \mathcal{O}_X)$, where $L^\bullet_X$ is the cotangent complex of X. If X has only local complete intersection singularities the $L^\bullet_X$ coincides with Ω_X in degree zero. Recall that from the spectral sequence of Ext groups we have

$$H^q(X, \mathcal{E}xt^p(\Omega_X, \mathcal{O}_X)) \Rightarrow Ext^{p+q}(\Omega_X, \mathcal{O}_X).$$

Consider the special case where $X = C$ is a nodal curve and $p+q = 2$. Then

- $H^0(C, \mathcal{E}xt^2(\Omega_C, \mathcal{O}_C)) = 0$ because Ω_C admits a locally free resolution of length one. Indeed take an embedding $C \to Y$ of Y in a smooth surface, then we have an exact sequence

$$0 \mapsto \mathcal{I}/\mathcal{I}^2 \to \Omega_Y \otimes \mathcal{O}_C \to \Omega_C \mapsto 0.$$

- $H^1(C, \mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)) = 0$ because $\mathcal{E}xt^1(\Omega_C, \mathcal{O}_C)$ is supported on $Sing(C)$ which is zero dimensional.
- $H^2(C, \mathcal{H}om(\Omega_C, \mathcal{O}_C)) = 0$ because $dim(C) = 1$.

We conclude that $Ext^2(\Omega_C, \mathcal{O}_C) = T^2 Def_C = 0$.

Heuristically, Riemann computed that $\dim(M_g) = 3g-3$. By Riemann-Hurwitz formula to any collection of $2d+2g-2$ points on $\mathbb{P}^1$ corresponds a curve X with a finite morphism $\phi : X \to \mathbb{P}^1$ of degree d. To give such a morphism is equivalent to choose a divisor D of degree d on X (i.e. d distinct points on X) and a element in $H^0(X, \mathcal{O}_X(D))$. If we consider divisors of degree $d > 2g-2$, by Riemann-Roch we get $h^0(D) = d - g + 1$. Then we have to subtract $\dim(\mathrm{Aut}(X))$ but a curve of genus $g \geq 2$ as only a finite number of automorphism. We conclude that

$$\dim(M_g) = 2d + 2g - 2 - (d + d - g + 1) = 3g - 3.$$

In what follows we rigorously prove this fact by arguments of deformation theory.

THEOREM 4.11. *(Smoothness of $\overline{\mathcal{M}}_g$) Let X be a stable curve of arithmetic genus $g \geq 2$. Then the functor of local infinitesimal deformations Def_X of X is pro-representable by a regular local ring of dimension $3g-3$. In other words $\overline{\mathcal{M}}_g$ is a smooth Deligne-Mumford stack of dimension*

$$dim(\overline{\mathcal{M}}_g) = 3g - 3.$$

PROOF. The functor Def_X is pro-representable since X is projective and does not have infinitesimal automorphism. Furthermore $T^2 Def_X = H^2(X, T_X) = 0$ since $dim(X) = 1$, then there are no obstructions to deforming X and the local ring representing Def_X is regular. Furthermore from remark 4.10 we get $Ext^2(\Omega_X, \mathcal{O}_X) = T^2 Def_X = 0$ for a nodal curve. Then in any case the deformation functor of X is unobstructed. So far we have proved that $\overline{\mathcal{M}}_g$ is a smooth DM stack. To compute its dimension we distinguish two cases.

- If X is a smooth curve, and $0 \mapsto I \to A \to B \mapsto 0$ is a semi-small exact sequence in $\mathfrak{Art}$, then there is a functorial exact sequence

$$H^1(X, T_X) \otimes I \to Def_X(A) \to Def_X(B) \to H^2(X, T_X) \otimes I.$$

On a curve $T_X = \omega_X^{\vee}$, where ω_X is the canonical sheaf of X. Then $deg(T_X) = 2-2g$, and since $h^0(T_X) = 0$, by Riemann-Roch theorem we get $h^0(T_X) - h^1(T_X) = 2-2g-g+1 = 3-3g$, and $h^1(T_X) = 3g-3$. We conclude that in a point $x \in \overline{\mathcal{M}}_g$ corresponding to the isomorphism class of a smooth curve X, the tangent space $T_x\overline{\mathcal{M}}_g$ has dimension $3g-3$.

- Now consider the case where X is a stable nodal curve. We have a sequence

$$0 \mapsto H^1(X, \mathcal{H}om(\Omega_X, \mathcal{O}_X)) \to Ext^1(\Omega_X, \mathcal{O}_X) \to H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) \mapsto 0,$$

there being no H^2 on a curve. We denote by δ the number of nodes in X. Since the sheaf Ω_X is locally free on the smooth locus of X, the sheaf $\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))$ is just k at each node, then $\dim(H^0(X, \mathcal{E}xt^1(\Omega_X, \mathcal{O}_X))) = \delta$. The curve X is l.c.i, then the dualizing sheaf ω_X is an invertible sheaf, and since $\omega_X \cong \Omega_X$ on the open set of regular points, we have an injective morphism $\omega_X^{\vee} \to \mathcal{H}om(\Omega_X, \mathcal{O}_X)$, and an exact sequence

$$0 \mapsto \omega_X^{\vee} \to \mathcal{H}om(\Omega_X, \mathcal{O}_X) \to \mathcal{O}_Z \mapsto 0,$$

where $Z = Sing(X)$. Since X is stable $h^0(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) = 0$, by the cohomology exact sequence we get $h^0(\omega_X^{\vee}) = 0$, and

$$0 \mapsto H^0(X, \mathcal{O}_Z) \to H^1(X, \omega_X^{\vee}) \to H^1(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) \mapsto 0.$$

By Riemann-Roch for singular curves we get $h^1(\omega_X^{\vee}) = 3g-3$, and since $h^0(\mathcal{O}_Z) = \delta$ we get $h^1(\mathcal{H}om(\Omega_X, \mathcal{O}_X)) = 3g-3-\delta$. Finally

$$dim(Ext^1(\Omega_X, \mathcal{O}_X)) = h^1(T_X) + h^0(\mathcal{E}xt^1(\Omega_X, \mathcal{O}_X)) = 3g-3-\delta+\delta = 3g-3.$$

We conclude that any point of $\overline{\mathcal{M}}_g$ is smooth and $\overline{\mathcal{M}}_g$ is a smooth stack of dimension $3g-3$. $\square$

REMARK 4.12. Theorems 3.6 and 4.11 hold also for $n > 0$. That is $\overline{\mathcal{M}}_{g,n}$ is a smooth DM-stack of dimension $3g-3+n$ for any g, n such that $2g-2+n > 0$. The notation is more convoluted but the proofs work exactly in the same way.

Nodal curves. The arithmetic genus g of a connected curve C is defined as $g = h^1(C, \mathcal{O}_C)$. Suppose that C has at most nodal singularities. Let $C = \bigcup_{i=1}^{\gamma} C_i$ be the irreducible components decomposition of C, and set $\delta := \sharp\,\mathrm{Sing}(C)$. Let

$$\nu : \overline{C} = \bigsqcup_{i=1}^{\gamma} \overline{C_i} \to C$$

be the normalization of C. The associated morphism $\mathcal{O}_C \hookrightarrow \mathcal{O}_{\overline{C}}$ on the structure sheaves yield the following sequence in cohomology

$$0 \mapsto H^0(C, \mathcal{O}_C) \to H^0(\overline{C}, \mathcal{O}_{\overline{C}}) \to \mathbb{C}^{\delta} \to H^1(C, \mathcal{O}_C) \to H^1(\overline{C}, \mathcal{O}_{\overline{C}}) \mapsto 0.$$

We get a formula for the arithmetic genus g of C

$$g = h^1(\overline{C}, \mathcal{O}_{\overline{C}}) + \delta - \gamma + 1 = \sum_{i=1}^{\gamma} g_i + \delta - \gamma + 1$$

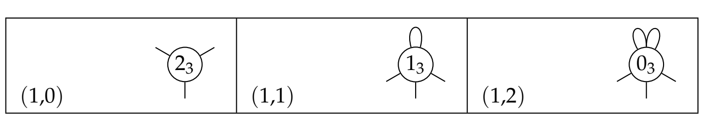
where $g_i = h^1(\overline{C_i}, \mathcal{O}_{\overline{C_i}})$ is the geometric genus of C_i.

DEFINITION 4.13. *A stable n-pointed curve is a complete connected curve C that has at most nodal singularities, with an ordered collection $x_1, ..., x_n \in C$ of distinct smooth points of C, such that the $(n+1)$-tuple $(C, x_1, ..., x_n)$ has finitely many automorphisms.*

This finiteness condition is equivalent to say that every rational component of the normalization of C has at least 3 points lying over singular or marked points of C.
Moduli spaces of smooth algebraic curves have been defined and then compactified adding stable curves by *Deligne* and *Mumford* in [**DM**]. Furthermore *Deligne* and *Mumford* proved that, if $2g - 2 + n > 0$, there exists a coarse moduli space $\overline{M}_{g,n}$ parametrizing isomorphism classes of n-pointed stable curves of arithmetic genus g, and this space is an irreducible projective variety of dimension $3g - 3 + n$.

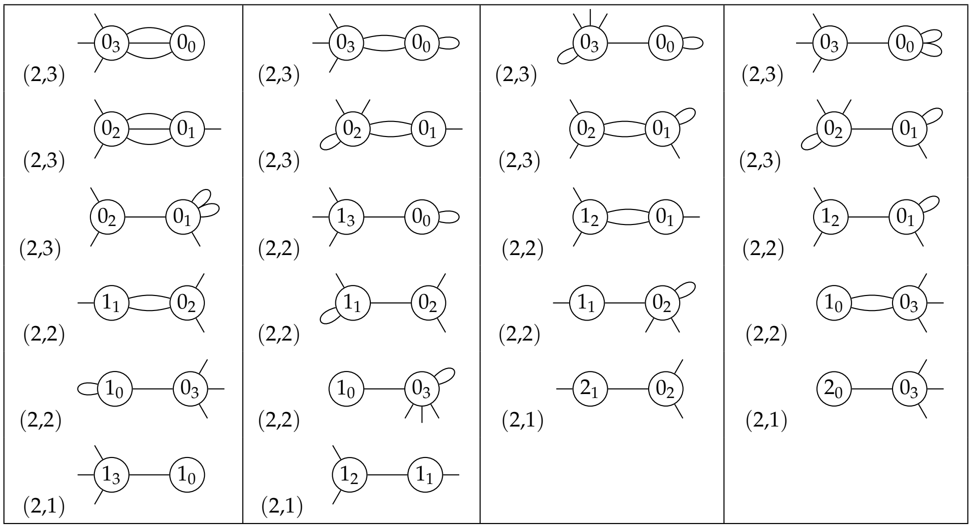
Boundary of $\overline{M}_{g,n}$ and dual graphs. The points in the boundary $\partial\overline{M}_{g,n}$ of the moduli space $\overline{M}_{g,n}$ represent isomorphisms classes of singular pointed stable curves. The geometry of such curves is encoded in a graph, called the dual graph. The boundary has a stratification whose loci, called strata, parametrize curves of a certain topological type and with a fixed configuration of the marked points.
Each nodal curve has an associated graph. This allows to represent nodal curves in a very simple way and translate some issues related to nodal curves in the language of graph theory.
Let C be a connected nodal curve with γ irreducible components and δ nodes. The dual graph Γ_C of C is the graph whose vertices represent the irreducible components of C and whose edges represent nodes lying on two components.
More precisely, each irreducible component is represented by a vertex labeled by two numbers: the genus and the number of marked points of the component. An edge connecting two vertices means that the two corresponding components intersect in the node corresponding to the edge. A loop on a vertex means that the corresponding component has a self-intersection.
Recently, *S. Maggiolo* and *N. Pagani* developed a software package, called `boundary`, that generates all stable dual graphs for prescribed values of g, n whose detailed description can be found in [**MP**]. We will use this package to generate graphs needed in this paper.
We denote by Δ_{irr} the locus in $\overline{M}_{g,n}$ parametrizing irreducible nodal curves with n marked points, and by $\Delta_{i,P}$ the locus of curves with a node which divides the curve into a component of genus i containing the points indexed by P and a component of genus $g - i$ containing the remaining points.
The closures of the loci Δ_{irr} and $\Delta_{i,P}$ are the irreducible components of the boundary $\partial\overline{M}_{g,n}$, see [**HM**, Chapter 2].

REMARK 4.14. The number of different classes in the boundary grows very fast with g and n. For example, in $\overline{M}_{2,3}$ we have three different class of stable irreducible curves, whose graph are the following: [1]



[1] The couple of numbers before each graph are respectively the number of components of the curve and the codimension of the corresponding stratum in $\overline{M}_{2,3}$.

while the graphs of stable curves with two irreducible components are the following:



Furthermore there are 163 other graphs representing curves with 3, 4 or 5 irreducible components.

Forgetful morphisms and the universal curve. For any $i = 1, ..., n$ there is a canonical forgetful morphism

$$\pi_i : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n-1}$$

forgetting the i-th marked point. If $g > 2$ and $[C, x_1, ..., \hat{x}_i, ..., x_n] \in \overline{M}_{g,n-1}$ is a general point the fiber

$$\pi_i^{-1}([C, x_1, ..., \hat{x}_i, ..., x_n]) \cong C$$

is isomorphic to C. However π_i is not the universal curve. Indeed if $(C, x_1, ..., \hat{x}_i, ..., x_n)$ has non trivial automorphism group then $\pi_i^{-1}([C, x_1, ..., \hat{x}_i, ..., x_n])$ is not isomorphic to C but to the quotient of C by the automorphism group of the pointed curve $(C, x_1, ..., \hat{x}_i, ..., x_n)$. For example the moduli space $\overline{M}_{g,1}$ with the forgetful morphism $\pi : \overline{M}_{g,1} \rightarrow \overline{M}_g$ at first glance seems to play the role of the universal curve over $\overline{M}_g$. However, on closer examination one realizes that $\pi^{-1}([C]) \cong C$ if and only if $[C] \in \overline{M}_g^0$, the locus of automorphism free curves. It is well known that the set-theoretic fiber of $\pi : \overline{M}_{g,1} \rightarrow \overline{M}_g$ over $[C] \in \overline{M}_g$ is the quotient $C/\operatorname{Aut}(C)$. For example over an open subset of $\overline{M}_2$ the fibration $\pi : \overline{M}_{2,1} \rightarrow \overline{M}_2$ is a $\mathbb{P}^1$-bundle and this is true even scheme-theoretically.
The situation is different if instead of considering the moduli space $\overline{M}_{g,1}$ we consider the Deligne-Mumford moduli stack $\overline{\mathcal{M}}_{g,1}$. In fact, in this case the fiber $\pi^{-1}([C])$ is isomorphic to C and via the morphism $\pi : \overline{\mathcal{M}}_{g,1} \rightarrow \overline{\mathcal{M}}_g$ the stack $\overline{\mathcal{M}}_{g,1}$ is the universal curve over $\overline{\mathcal{M}}_g$.
Note that if $n \geq 2$ the fiber $\pi_i^{-1}([C, x_1, ..., \hat{x}_i, ..., x_n])$ always intersects the boundary of $\overline{M}_{g,n}$. In fact the points of the fiber corresponding to marked points represent singular curves with two irreducible components: C itself and a $\mathbb{P}^1$ with two marked points and intersecting C in a point. In the same way for any $I \subseteq \{1, ..., n\}$ we have a forgetful map $\pi_I : \overline{M}_{g,n} \rightarrow \overline{M}_{g,n-|I|}$. The map π_i has sections $s_{i,j} : \overline{M}_{g,n-1} \rightarrow \overline{M}_{g,n}$ defined by sending the point $[C, x_1, ..., \hat{x}_i, ..., x_n]$ to the isomorphism class of the n-pointed genus g curve obtained by attaching at $x_j \in C$ a $\mathbb{P}^1$ with two marked points labeled by x_i and x_j.

Divisor classes on $\overline{\mathcal{M}}_{g,n}$. Let us briefly recall the definitions of classes λ and ψ_i on $\overline{\mathcal{M}}_{g,n}$. Consider the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ forgetting one of the marked points and its sections $\sigma_1, ..., \sigma_n : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n+1}$. Let ω_π be the relative dualizing sheaf of the morphism π. The Hodge class is defined as

$$\lambda := c_1(\pi_*(\omega_\pi)).$$

The classes ψ_i are defined as

$$\psi_i := \sigma_i^*(c_1(\omega_\pi))$$

for any $i = 1, ..., n$. Finally we denote by δ_{irr} and $\delta_{i,P}$ the boundary classes on $\overline{\mathcal{M}}_{g,n}$.

5. Cohomology classes on the moduli space of curves

Let $\pi : \overline{\mathcal{C}}_g \to \mathcal{M}_g$ be the universal curve over the stack $\mathcal{M}_g$, and let

$$\gamma = c_1(\omega)$$

be the first Chern class of the relative dualizing sheaf $\omega := \omega_\pi$ of π. We define the classes $k_i \in H^{2i}(\mathcal{M}_g)$ as

$$k_i = \pi_*(\gamma^{i+1}).$$

The Hodge bundle over $\mathcal{M}_g$ is defined as $\mathbb{E} = \pi_*\omega$ and its Chern classes are usually denoted by $\lambda_i = c_i(\mathbb{E})$. The fiber of $\mathbb{E}$ over a point $[C] \in \mathcal{M}_g$ is the space $H^0(C, \omega_C)$ of regular differential forms on C. Therefore $\mathbb{E}$ has rank g. The difference $T_{\mathcal{C}_g} - T_{\mathcal{M}_g}$ in $K(\mathcal{C}_g)$ is the relative tangent bundle. Therefore

$$\frac{td(\mathcal{C}_g)}{\pi^* td(\mathcal{M}_g)} = td(\omega^\vee) = 1 - \frac{\gamma}{2} + \frac{\gamma^2}{12} - \frac{\gamma^4}{720} + ...$$

Furthermore we have

$$ch(\omega) = 1 + \gamma + \frac{\gamma^2}{2} + \frac{\gamma^3}{6} + ...$$

and by Grothendieck-Riemann-Roch we get

$$ch(\pi_!\omega) = \pi_*\left(\frac{ch(\omega) \cdot td(\mathcal{C}_g)}{\pi_* td(\mathcal{M}_g)}\right) = \pi_*\left(1 + \frac{\gamma}{2} + \frac{\gamma^2}{12} + ...\right)$$

Now, $\pi_!(\omega) = \pi_*\omega - R^1\pi_*\omega$ and $R^1\pi_*\omega = \mathcal{O}_{\mathcal{M}_g}$. Therefore we have

$$ch(\mathbb{E}) - 1 = \mathrm{rank}(\mathbb{E}) - 1 + c_1(\mathbb{E}) + \frac{c_1(\mathbb{E})^2 - 2c_2(\mathbb{E})}{2} + ... = \pi_*\left(\frac{\gamma}{2} + \frac{\gamma^2}{12} + ...\right)$$

Since γ has degree $2g - 2$ on a fiber of π we find

$$\mathrm{rank}(\mathbb{E}) = \pi_*\left(\frac{\gamma}{2}\right) + 1 = g - 1 + 1 = g.$$

Furthermore

$$c_1(\mathbb{E}) = \pi_*\left(\frac{\gamma^2}{12}\right) = \frac{k_1}{12}.$$

Note that the degree three component of $\frac{ch(\omega)\cdot td(\mathcal{C}_g)}{\pi_* td(\mathcal{M}_g)}$ is $\frac{\gamma^3}{6} - \frac{\gamma^3}{4} + \frac{\gamma^3}{12} = 0$. Therefore $\frac{c_1(\mathbb{E})^2 - 2c_2(\mathbb{E})}{2} = 0$ and

$$c_2(\mathbb{E}) = \frac{c_1(\mathbb{E})^2}{2} = \frac{k_1^2}{288}.$$

In general the Chern classes of the Hodge bundle can be expressed as polynomials in the tautological classes k_i.

Now, we want to derive some relations on the compactification $\overline{\mathcal{M}}_g$. By [**HM**, Proposition 3.93] a relation among divisor classes on $\overline{\mathcal{M}}_g$ corresponds to the same relation among the associated divisor classes on the base B of any family $X \to B$ of stable curves, where B and the general fiber are smooth curves. Let $f : X \to B$ be such a family and let t be a local parameter on B. We consider a minimal resolution $r : Y \to X$ of the singularities of X. The family $g = f \circ r : Y \to B$ is a family of semi-stable curves. We summarize the situation in the following diagram.

$$\begin{tikzcd} Y \arrow[rr, "r"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & B & \end{tikzcd}$$

Each note p of a fiber of $X \to B$ satisfying $xy = t^m$ has been replaced by a chain of $m-1$ smooth, rational curves. We conclude that $g : Y \to B$ is a family of semi-stable curves, with smooth total space, and over a node of a fiber of $X \to B$ with equation $xy = t^m$ we have now $m-1$ nodes.

Note that each exceptional component over a node of a fiber of $X \to B$ is a smooth rational component intersecting the rest of the fiber on two points. Therefore the canonical bundle of such a component is trivial and the relative dualizing sheaf of the family $g : Y \to B$ is trivial on the exceptional divisor of the resolution r. Therefore

$$\omega_{Y/B} = r^*\omega_{X/B}, g_*\omega_{Y/B} = f_*\omega_{X/B} \text{ and } g_*(c_1(\omega_{Y/B})^2) = f_*(c_1(\omega_{X/B})^2).$$

Let $p \in Y$ be a node of a fiber over o point $[C] \in B$. We have an injective morphism

$$\begin{array}{ccc} \mathcal{O}_Y \langle dt \rangle & \longrightarrow & \mathcal{O}_Y \langle dx, dy \rangle \\ dt & \longmapsto & xdy + ydx \end{array}$$

which is the injection $\pi^* T_B^\vee \to T_Y^\vee$. The cokernel is the relative cotangent sheaf

$$\Omega_{Y/B} = \frac{\mathcal{O}_Y \langle dx, dy \rangle}{\langle xdy + ydx \rangle}.$$

This sheaf is an invertible sheaf on $Y \setminus Z$, where Z is the locus of nodes of fibers of $Y \to B$. The relative dualizing sheaf is the unique invertible sheaf ω such that $\omega_{|Y\setminus Z} \cong \Omega_{Y/B|Y\setminus Z}$. We can write $\omega = \mathcal{O}_Y \langle \alpha \rangle$ with $\alpha = \frac{dx}{x} - \frac{dy}{y}$. Furthermore we have $x\alpha = 2dx$ and $y\alpha = -2dy$. Therefore $\Omega = \Omega_{Y/B} = \mathcal{I}_Z \otimes \omega$. Let ξ be the class of the singular locus Z. The exact sequence

$$0 \mapsto \mathcal{I}_Z \to \mathcal{O}_Y \to \mathcal{O}_Z \mapsto 0$$

yields $ch(\mathcal{I}_Z) = 1 - \xi$. We have

$$ch(\Omega) = ch(\omega) \cdot ch(\mathcal{I}_Z) = \left(1 + \gamma + \frac{\gamma^2}{2} + ...\right) \cdot (1 - \xi)$$

that is

$$ch(\Omega) = 1 + c_1(\Omega) + \frac{c_1(\Omega)^2 - 2c_2(\Omega)}{2} + ... = 1 + \gamma + \left(\frac{\gamma^2}{2} - \xi\right) + ...$$

Therefore $c_1(\Omega) = \gamma = c_1(\omega)$ and indeed Ω and ω differ on a codimension two locus. Furthermore $\frac{c_1(\Omega)^2 - 2c_2(\Omega)}{2} = \frac{\gamma^2}{2} - \xi$ yields

$$c_2(\Omega) = \xi.$$

We conclude that

$$td(Y/B) = 1 - \frac{\gamma}{2} + \frac{\gamma^2 + \xi}{12} + ...$$

and by Grothendieck-Riemann-Roch

$$c_1(g_* \omega_{Y/B}) = g_* \left(\frac{\gamma^2 + \xi}{12} \right) = g_* \left(\frac{c_1(\omega_{Y/B})^2 + \xi}{12} \right).$$

Now, coming back to the family $f : X \to B$ we get

$$\lambda = \frac{k_1 + \delta}{12}$$

where δ is the locus corresponding to Z. Finally on the moduli space $\overline{M}_g$ we have

$$12\lambda - k_1 = \Delta_0 + \frac{1}{2}\Delta_1 + \Delta_2 + ... + \Delta_{\lfloor \frac{g}{2} \rfloor}$$

where $\frac{1}{2}\Delta_1$ is the divisor parametrizing elliptic tails and the rational coefficient $\frac{1}{2}$ keeps trace of the elliptic involution of the elliptic tail fixing the attachment point.

The canonical class. On the smooth locus of the moduli space $\overline{M}_g$ we can consider the bundle Ω generated by regular differential forms of top degree $3g-3$. We define the canonical bundle of $\overline{M}_g$ as the unique $\mathbb{Q}$-line bundle restricting to Ω on the smooth locus.
The cotangent space to the stack $\overline{\mathcal{M}}_g$ at a point $[C]$ is $H^0(C, \Omega_C \otimes \omega_C)$. The canonical class of $\overline{\mathcal{M}}_g$ is given by associating to a family $f : X \to B$ of stable curves the class $K_{\overline{\mathcal{M}}_g}(f) = f_*(\Omega_{X/B} \otimes \omega_{X/B})$. Note that the higher direct images of $f_*(\Omega_{X/B} \otimes \omega_{X/B})$ vanish. We have

$$ch(\Omega_{X/B} \otimes \omega_{X/B}) = 1 + 2\gamma + \frac{4\gamma^2 - 2\xi}{2} = 1 + \gamma + 2\gamma^2 - \xi$$

and by Grothendieck-Riemann-Roch we have

$$\begin{aligned} ch(f_*(\Omega_{X/B} \otimes \omega_{X/B})) &= f_* \left(\left(1 + 2\gamma + 2\gamma^2 - \xi\right) \cdot \left(1 - \tfrac{\gamma}{2} + \tfrac{\gamma^2+\xi}{12}\right)\right) \\ &= f_* \left(1 - \tfrac{\gamma}{2} + \tfrac{\gamma^2+\xi}{12} + 2\gamma - \gamma^2 + 2\gamma^2 - \xi\right) \\ &= f_* \left(1 + \tfrac{3}{2}\gamma + \tfrac{13}{12}\gamma^2 - \tfrac{11}{12}\xi\right) \\ &= \tfrac{3}{2}(2g-2) + \left(\tfrac{13}{12}k_1 - \tfrac{11}{12}\delta\right). \end{aligned}$$

Recalling that $k_1 = 12\lambda - \delta$ we conclude

$$\begin{aligned} ch(f_*(\Omega_{X/B} \otimes \omega_{X/B})) &= \tfrac{3}{2}(2g-2) + \left(\tfrac{13}{12}k_1 - \tfrac{11}{12}\delta\right) \\ &= 3g - 3 + \tfrac{13}{12}(12\lambda - \delta) - \tfrac{11}{12}\delta \\ &= 3g - 3 + 13\lambda - 2\delta. \end{aligned}$$

In particular the canonical class of the stack $\overline{\mathcal{M}}_g$ is

$$K_{\overline{\mathcal{M}}_g} = 13\lambda - 2\delta.$$

Let $\pi : \overline{\mathcal{M}}_g \to \overline{M}_g$ be the canonical morphism between the stack an the coarse moduli space. The morphism π is ramified along the divisor $\Delta_1 \subset \overline{M}_g$ parametrizing elliptic tails. We have

$$\pi^* K_{\overline{M}_g} = K_{\overline{\mathcal{M}}_g} + \delta_1$$

and recalling that any point of Δ_1 has automorphism group of order two we conclude that

$$K_{\overline{M}_g} = 13\lambda - 2\Delta + \frac{1}{2}\Delta_1.$$

6. Moduli spaces of weighted pointed curves and Kapranov's construction of $\overline{M}_{0,n}$

In **[Ha]** *B. Hassett* introduced new compactifications $\overline{\mathcal{M}}_{g,A[n]}$ of the moduli stack $\mathcal{M}_{g,n}$ and $\overline{M}_{g,A[n]}$ for the coarse moduli space $M_{g,n}$, by assigning rational weights $A = (a_1, ..., a_n)$, $0 < a_i \leq 1$ to the markings. In genus zero some of these spaces appear as intermediate steps of the blow-up construction of $\overline{M}_{0,n}$ developed by *M. Kapranov* in **[Ka]**, while in higher genus they may be related to the LMMP on $\overline{M}_{g,n}$.
We work over an algebraically closed field of characteristic zero. Let S be a Noetherian scheme and g, n two non-negative integers. A family of nodal curves of genus g with n marked points over S consists of a flat proper morphism $\pi : C \to S$ whose geometric fibers are nodal connected curves of arithmetic genus g, and sections $s_1, ..., s_n$ of π. A collection of input data $(g, A) := (g, a_1, ..., a_n)$ consists of an integer $g \geq 0$ and the weight data: an element $(a_1, ..., a_n) \in \mathbb{Q}^n$ such that $0 < a_i \leq 1$ for $i = 1, ..., n$, and

$$2g - 2 + \sum_{i=1}^{n} a_i > 0.$$

DEFINITION 6.1. *A family of nodal curves with marked points $\pi : (C, s_1, ..., s_n) \to S$ is stable of type (g, A) if*

- *the sections $s_1, ..., s_n$ lie in the smooth locus of π, and for any subset $\{s_{i_1}, ..., s_{i_r}\}$ with non-empty intersection we have $a_{i_1} + ... + a_{i_r} \leq 1$,*
- *$\omega_\pi(\sum_{i=1}^n a_i s_i)$ is π-relatively ample, where ω_π is the relative dualizing sheaf.*

B. Hassett in [**Ha**, Theorem 2.1] proved that given a collection (g, A) of input data, there exists a connected Deligne-Mumford stack $\overline{\mathcal{M}}_{g,A[n]}$, smooth and proper over $\mathbb{Z}$, representing the moduli problem of pointed stable curves of type (g, A). The corresponding coarse moduli scheme $\overline{M}_{g,A[n]}$ is projective over $\mathbb{Z}$.
Furthermore, by [**Ha**, Theorem 3.8] a weighted pointed stable curve admits no infinitesimal automorphisms, and its infinitesimal deformation space is unobstructed of dimension $3g - 3 + n$. Then $\overline{\mathcal{M}}_{g,A[n]}$ is a smooth Deligne-Mumford stack of dimension $3g - 3 + n$.

REMARK 6.2. Since $\overline{\mathcal{M}}_{g,A[n]}$ is smooth as a Deligne-Mumford stack the coarse moduli space $\overline{M}_{g,A[n]}$ has finite quotient singularities, that is étale locally it is isomorphic to a quotient of a smooth scheme by a finite group. In particular, $\overline{M}_{g,A[n]}$ is normal.

For fixed g, n, consider two collections of weight data $A[n], B[n]$ such that $a_i \geq b_i$ for any $i = 1, ..., n$. Then there exists a birational *reduction morphism*

$$\rho_{B[n],A[n]} : \overline{M}_{g,A[n]} \to \overline{M}_{g,B[n]}$$

associating to a curve $[C, s_1, ..., s_n] \in \overline{M}_{g,A[n]}$ the curve $\rho_{B[n],A[n]}([C, s_1, ..., s_n])$ obtained by collapsing components of C along which $\omega_C(b_1 s_1 + ... + b_n s_n)$ fails to be ample, where ω_C denote the dualizing sheaf of C.
Furthermore, for any g consider a collection of weight data $A[n] = (a_1, ..., a_n)$ and a subset

$A[r] := (a_{i_1}, ..., a_{i_r}) \subset A[n]$ such that $2g - 2 + a_{i_1} + ... + a_{i_r} > 0$. Then there exists a *forgetful morphism*

$$\pi_{A[n],A[r]} : \overline{M}_{g,A[n]} \to \overline{M}_{g,A[r]}$$

associating to a curve $[C, s_1, ..., s_n] \in \overline{M}_{g,A[n]}$ the curve $\pi_{A[n],A[r]}([C, s_1, ..., s_n])$ obtained by collapsing components of C along which $\omega_C(a_{i_1}s_{i_1} + ... + a_{i_r}s_{i_r})$ fails to be ample. For the details see [**Ha**, Section 4].

In the following we will be especially interested in the boundary of $\overline{M}_{g,A[n]}$. The boundary of $\overline{M}_{g,A[n]}$, as for $\overline{M}_{g,n}$, has a stratification whose loci, called strata, parametrize curves of a fixed topological type and with a fixed configuration of the marked points.
We denote by Δ_{irr} the locus in $\overline{M}_{g,A[n]}$ parametrizing irreducible nodal curves with n marked points, and by $\Delta_{i,P}$ the locus of curves with a node which divides the curve into a component of genus i containing the points indexed by P and a component of genus $g - i$ containing the remaining points. Note that in $\overline{M}_{g,A[n]}$ may appear boundary divisors parametrizing smooth curves. For instance, as soon as there exist two indices i, j such that $a_i + a_j \leq 1$ we get a boundary divisor whose general point represents a smooth curve where the marked points labelled by i and j collide.

Kapranov's blow-up constructions. We follow [**Ka**]. Let $(C, x_1, ..., x_n)$ be a genus zero n-pointed stable curve. The dualizing sheaf ω_C of C is invertible, see [**Kn**]. By [**Kn**, Corollaries 1.10 and 1.11] the sheaf $\omega_C(x_1 + ... + x_n)$ is very ample and has $n - 1$ independent sections. Then it defines an embedding $\phi : C \to \mathbb{P}^{n-2}$. In particular, if $C \cong \mathbb{P}^1$ then $\deg(\omega_C(x_1 + ... + x_n)) = n - 2$, $\omega_C(x_1 + ... + x_n) \cong \phi^*\mathcal{O}_{\mathbb{P}^{n-2}}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n-2)$, and $\phi(C)$ is a degree $n - 2$ rational normal curve in $\mathbb{P}^{n-2}$. By [**Ka**, Lemma 1.4] if $(C, x_1, ..., x_n)$ is stable the points $p_i = \phi(x_i)$ are in linear general position in $\mathbb{P}^{n-2}$.
This fact combined with a careful analysis of limits in $\overline{M}_{0,n}$ of 1-parameter families in $M_{0,n}$ led *M. Kapranov* to prove the following theorem [**Ka**, Theorem 0.1]:

THEOREM 6.3. *Let $p_1, ..., p_n \in \mathbb{P}^{n-2}$ be points in linear general position, and let $V_0(p_1, ..., p_n)$ be the scheme parametrizing rational normal curves through $p_1, ..., p_n$. Consider $V_0(p_1, ..., p_n)$ as a subscheme of the Hilbert scheme $\mathcal{H}$ parametrizing subschemes of $\mathbb{P}^{n-2}$. Then*

- *$V_0(p_1, ..., p_n) \cong M_{0,n}$.*
- *Let $V(p_1, ..., p_n)$ be the closure of $V_0(p_1, ..., p_n)$ in $\mathcal{H}$. Then $V(p_1, ..., p_n) \cong \overline{M}_{0,n}$.*

Kapranov's construction allows to translate many issues of $\overline{M}_{0,n}$ into statements on linear systems on $\mathbb{P}^{n-3}$. Consider a general line $L_i \subset \mathbb{P}^{n-2}$ through p_i. There is a unique rational normal curve C_{L_i} through $p_1, ..., p_n$, and with tangent direction L_i in p_i. Let $[C, x_1, ..., x_n] \in \overline{M}_{0,n}$ be a stable curve, and let $\Gamma \in V_0(p_1, ..., p_n)$ be the corresponding curve. Since $p_i \in \Gamma$ is a smooth point considering the tangent line $T_{p_i}\Gamma$, with some work [**Ka**], we get a morphism

$$f_i : \overline{M}_{0,n} \to \mathbb{P}^{n-3},\ [C, x_1, ..., x_n] \mapsto T_{p_i}\Gamma.$$

Furthermore, f_i is birational and it defines an isomorphism on $M_{0,n}$. The birational maps $f_j \circ f_i^{-1}$

$$\begin{tikzcd} & \overline{M}_{0,n} \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ \mathbb{P}^{n-3} \arrow[rr, dashed, "f_j \circ f_i^{-1}"] & & \mathbb{P}^{n-3} \end{tikzcd}$$

are standard Cremona transformations of $\mathbb{P}^{n-3}$ [**Ka**, Proposition 2.12]. For any $i = 1, ..., n$ the class Ψ_i is the line bundle on $\overline{M}_{0,n}$ whose fiber on $[C, x_1, ..., x_n]$ is the tangent line $T_{p_i}C$. From the previous description we see that the line bundle Ψ_i induces the birational morphism $f_i : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$, that is $\Psi_i = f_i^*\mathcal{O}_{\mathbb{P}^{n-3}}(1)$. In [**Ka**] Kapranov proved that Ψ_i is big and globally generated, and that the birational morphism f_i is an iterated blow-up of the projections from p_i of the points $p_1, ..., \hat{p}_i, ...p_n$ and of all strict transforms of the linear spaces they generate, in order of increasing dimension.

CONSTRUCTION 6.4. [**Ka**] More precisely, fix $(n-1)$-points $p_1, ..., p_{n-1} \in \mathbb{P}^{n-3}$ in linear general position.

(1) Blow-up the points $p_1, ..., p_{n-2}$, the strict transforms of the lines $\langle p_i, p_j \rangle$ for $i, j = 1, ..., n-2$, the strict transforms of the linear spaces spanned by the subsets of cardinality $n-4$ of $\{p_1, ..., p_{n-2}\}$.

(2) Blow-up p_{n-1}, the strict transforms of the lines spanned by pairs of points including p_{n-1} but not p_{n-2},..., the strict transforms of the linear spaces spanned by the subsets of cardinality $(n-4)$ of $\{p_1, ..., p_{n-1}\}$ containing p_{n-1} but not p_{n-2}.

$\vdots$

(r) Blow-up the strict transforms of all the linear spaces spanned by subsets of the form $\{p_{n-1}, p_{n-2}, ..., p_{n-r+1}\}$, so that the order of the blow-ups in compatible by the partial order on the subsets given by inclusion.

$\vdots$

($n-3$) Blow-up the strict transforms of the codimension two linear space spanned by the subset $\{p_{n-1}, p_{n-2}, ..., p_4\}$.

The composition of these blow-ups is the morphism $f_n : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ induced by the psi-class Ψ_n. Identifying $\overline{M}_{0,n}$ with $V(p_1, ..., p_n)$, and fixing a general $(n-3)$-plane $H \subset \mathbb{P}^{n-2}$, the morphism f_n associates to a curve $C \in V(p_1, ..., p_n)$ the point $T_{p_n}C \cap H$.

We denote by $W_{r,s}[n]$, where $s = 1, ..., n-r-2$, the variety obtained at the r-th step once we finish blowing-up the subspaces spanned by subsets S with $|S| \leq s+r-2$, and by $W_r[n]$ the variety produced at the r-th step. In particular, $W_{1,1}[n] = \mathbb{P}^{n-3}$ and $W_{n-3}[n] = \overline{M}_{0,n}$.

In [**Ha**, Section 6.1], Hassett interprets the intermediate steps of Construction 6.4 as moduli spaces of weighted rational curves. Consider the weight data

$$A_{r,s}[n] := (\underbrace{1/(n-r-1), ..., 1/(n-r-1)}_{(n-r-1)-\text{times}}, s/(n-r-1), \underbrace{1, ..., 1}_{r-\text{times}})$$

for $r = 1, ..., n-3$ and $s = 1, ..., n-r-2$. Then $W_{r,s}[n] \cong \overline{M}_{0,A_{r,s}[n]}$, and the Kapranov's map $f_n : \overline{M}_{0,n} \rightarrow \mathbb{P}^{n-3}$ factorizes as a composition of reduction morphisms

$$\rho_{A_{r,s-1}[n],A_{r,s}[n]} : \overline{M}_{0,A_{r,s}[n]} \rightarrow \overline{M}_{0,A_{r,s-1}[n]},\ s = 2, ..., n-r-2,$$
$$\rho_{A_{r,n-r-2}[n],A_{r+1,1}[n]} : \overline{M}_{0,A_{r+1,1}[n]} \rightarrow \overline{M}_{0,A_{r,n-r-2}[n]}.$$

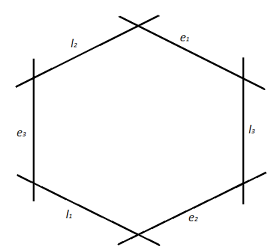
REMARK 6.5. The Hassett's space $\overline{M}_{0,A_{1,n-3}[n]}$, that is $\mathbb{P}^{n-3}$ blown-up at all the linear spaces of codimension at least two spanned by subsets of $n-2$ points in linear general position, is the Losev-Manin's moduli space $\overline{L}_{n-2}$ introduced by *A. Losev* and *Y. Manin* in [**LM**], see [**Ha**, Section 6.4]. The space $\overline{L}_{n-2}$ parametrizes $(n-2)$-pointed chains of projective lines $(C, x_0, x_\infty, x_1, ..., x_{n-2})$ where:

- C is a chain of smooth rational curves with two fixed points x_0, x_∞ on the extremal components,
- $x_1, ..., x_{n-2}$ are smooth marked points different from x_0, x_∞ but non necessarily distinct,
- there is at least one marked point on each component.

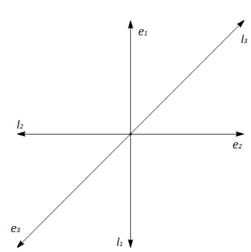
By [**LM**, Theorem 2.2] there exists a smooth, separated, irreducible, proper scheme representing this moduli problem. Note that after the choice of two marked points in $\overline{M}_{0,n}$ playing the role of x_0, x_∞ we get a birational morphism $\overline{M}_{0,n} \rightarrow \overline{L}_{n-2}$ which is nothing but a reduction morphism.
For example, $\overline{L}_1$ is a point parametrizing a $\mathbb{P}^1$ with two fixed points and a free point, $\overline{L}_2 \cong \mathbb{P}^1$, and $\overline{L}_3$ is $\mathbb{P}^2$ blown-up at three points in general position, that is a Del Pezzo surface of degree six, see [**Ha**, Section 6.4] for further generalizations.
For example consider Del Pezzo surface of degree six $\overline{M}_{0,A_{1,2}[5]} \cong \overline{L}_3 \cong \mathcal{S}_6$. Let us say that $\mathcal{S}_6$ is the blow-up of $\mathbb{P}^2$ at the coordinate points p_1, p_2, p_3 with exceptional divisors e_1, e_2, e_3 and let us denote by $l_i = \langle p_j, p_k \rangle$, $i \neq j, k$, $i = 1, 2, 3$ the three lines generated by p_1, p_2, p_3.
Such surface can be realized as the complete intersection in $\mathbb{P}^2 \times \mathbb{P}^2$ cut out by the equations $x_0 y_0 = x_1 y_1 = x_2 y_2$. The six lines are given by $e_i = \{x_j = x_k = 0\}$, $l_i = \{y_j = y_k = 0\}$ for $i \neq j, k$, $i = 1, 2, 3$. The torus $T = (\mathbb{C}^*)^3/\mathbb{C}^*$ acts on $\mathbb{P}^2 \times \mathbb{P}^2$ by

$$(\lambda_0, \lambda_1, \lambda_2) \cdot ([x_0 : x_1 : x_2], [y_0 : y_1 : y_2]) = ([\lambda_0 x_0 : \lambda_1 x_1 : \lambda_2 x_2], [\lambda_0^{-1} y_0 : \lambda_1^{-1} y_1 : \lambda_2^{-1} y_2]).$$

This torus action stabilizes $\mathcal{S}_6$. Furthermore S_2 acts on $\mathcal{S}_6$ by the transpositions $x_i \leftrightarrow y_i$, and S_3 acts on $\mathcal{S}_6$ by permuting the two sets of homogeneous coordinates separately. The action of S_3 corresponds to the permutations of the three points of $\mathbb{P}^2$ we are blowing-up, while the S_2-action is the switch of roles of exceptional divisors between the sets of lines $\{e_1, e_2, e_3\}$ and $\{l_1, l_2, l_3\}$. These six lines are arranged in a hexagon inside $\mathcal{S}_6$



which is stabilized by the action of $S_3 \times S_2$. The fan of $\mathcal{S}_6$ is the following



where the six 1-dimensional cones correspond to the toric divisors e_1, l_3, e_2, l_1, e_3 and l_2. It is clear from the picture that the fan has many symmetries given by permuting $\{e_1, e_2, e_3\}$, $\{l_1, l_2, l_3\}$ and switching e_i with l_i for $i = 1, 2, 3$.

EXAMPLE 6.6. Let $n=5$, and fix $p_1,...,p_4 \in \mathbb{P}^2$ points in general position. Kapranov's map f_5 is as follows: blow-up p_1,p_2,p_3 and then blow-up p_4.
At the step $r=1,s=1$ we get $\overline{M}_{0,A_{1,1}[n]} = \mathbb{P}^2$ and the weights are

$$A_{1,1}[5] := (1/3,1/3,1/3,1/3,1).$$

While for $r=2,s=1$ we get $\overline{M}_{0,A_{2,1}[n]} \cong \overline{M}_{0,5}$, indeed in this case the weight data are

$$A_{2,1}[5] := (1/2,1/2,1/2,1,1).$$

Note that as long as all the weights are strictly greater than $1/3$, Hassett's space is isomorphic to $\overline{M}_{0,n}$ because at most two points can collide, so the only components that get contracted are rational tail components with exactly two marked points. Since these have exactly three special points they have no moduli and contracting them does not affect the coarse moduli space even though it does change the universal curve, see also [**Ha**, Corollary 4.7]. In our case $\overline{M}_{0,A_{2,1}[5]} \cong \overline{M}_{0,5}$.
We have only one intermediate step, namely $r=1,s=2$. The moduli space $\overline{M}_{0,A_{1,2}[5]}$ parametrizes weighted pointed curves with weight data

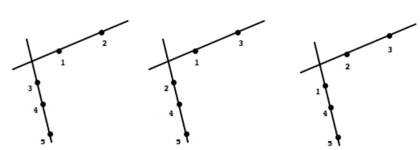
$$A_{1,2}[5] := (1/3,1/3,1/3,2/3,1).$$

Since $a_4+a_i=1$ for $i=1,2,3$ and $a_4+a_5>1$ the point p_4 is allowed to collide with p_1,p_2,p_3 but not with p_5 which has not yet been blown-up. Kapranov's map $f_5:\overline{M}_{0,5} \to \mathbb{P}^2$ factorizes as

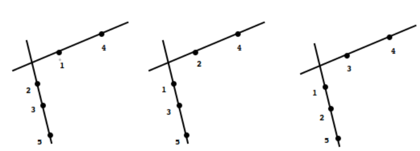
$$\begin{tikzcd}
\overline{M}_{0,5} \cong \overline{M}_{0,A_{2,1}[5]} \arrow[dd, "f_5"'] \arrow[dr, "\rho_1"] & \\
 & \overline{M}_{0,A_{1,2}[5]} \arrow[dl, "\rho_2"] \\
\mathbb{P}^2 \cong \overline{M}_{0,A_{1,1}[5]} &
\end{tikzcd}$$

where ρ_1,ρ_2 are the corresponding reduction morphisms. Let us analyze these two morphisms.

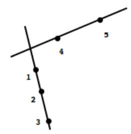
- Given $(C,s_1,...,s_5) \in \overline{M}_{0,A_{2,1}[5]}$ the curve $\rho_1(C,s_1,...,s_5)$ is obtained by collapsing components of C along which $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{2}{3}s_4 + s_5$ fails to be ample. So it contracts the 2-pointed components of the following curves:



along which $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{2}{3}s_4 + s_5$ is anti-ample, and the 2-pointed components of the following curves:

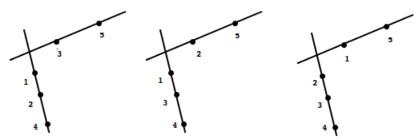


along which $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{2}{3}s_4 + s_5$ is nef but not ample. However all the contracted components have exactly three special points, and therefore they do not have moduli. This affects only the universal curve but not the coarse moduli space.
Finally $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{2}{3}s_4 + s_5$ is nef but not ample on the 3-pointed component of the curve



In fact this corresponds to the contraction of the divisor $E_{5,4} = f_5^{-1}(p_4)$.

- The morphism ρ_2 contracts the 3-pointed components of the curves



along which $K_C + \frac{1}{3}s_1 + \frac{1}{3}s_2 + \frac{1}{3}s_3 + \frac{1}{3}s_4 + s_5$ has degree zero. This corresponds to the contractions of the divisors $E_{5,3} = f_5^{-1}(p_3)$, $E_{5,2} = f_5^{-1}(p_2)$ and $E_{5,1} = f_5^{-1}(p_1)$.

EXAMPLE 6.7. Now, let us consider the case $n = 6$. Construction 6.4 is as follows:

- $r = 1, s = 1$, gives $\mathbb{P}^3$,
- $r = 1$, $s = 2$, we blow-up the points $p_1, ..., p_4 \in \mathbb{P}^3$ and get the Hassett's space with weights $A_{1,2}[6] := (1/4, 1/4, 1/4, 1/4, 1/2, 1)$,
- $r = 1$, $s = 3$, we blow-up the strict transforms of the lines $\langle p_i, p_j \rangle$, $i, j = 1, ..., 4$, and get the Hassett's space with weights $A_{1,3}[6] := (1/4, 1/4, 1/4, 1/4, 3/4, 1)$,
- $r = 2, s = 1$, we blow-up the point p_5, and get the Hassett's space with weights $A_{2,1}[6] := (1/3, 1/3, 1/3, 1/3, 1, 1)$,
- $r = 2$, $s = 2$, we blow-up the strict transforms of the lines $\langle p_i, p_5 \rangle$, $i, j = 1, ..., 3$, and get the Hassett's space with weights $A_{2,2}[6] := (1/3, 1/3, 1/3, 2/3, 1, 1)$,
- $r = 3$, $s = 1$, we blow-up the strict transform of the line $\langle p_4, p_5 \rangle$ and get the Hassett's space with weights $A_{3,1}[6] := (1/2, 1/2, 1/2, 1, 1, 1)$, that is $\overline{M}_{0,6}$.

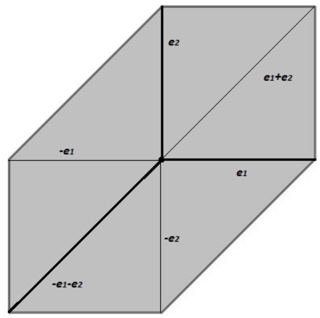
7. $\overline{M}_{0,n}$ is not a Mori Dream Space for $n > 133$ (following Castravet and Tevelev)

In [**HK**, Question 3.2] Y. Hu and S. Keel asked if $\overline{M}_{0,n}$ is a Mori Dream Space. If $n = 4, 5$ this is well known because $\overline{M}_{0,4} \cong \mathbb{P}^1$ and $\overline{M}_{0,5}$ is a Del Pezzo surface of degree five. By [**HK**] $\overline{M}_{0,n}$ is log Fano if and only if $n \leq 6$. In particular $\overline{M}_{0,6}$ is a Mori Dream Space. For $g \geq 1$ it is know that:

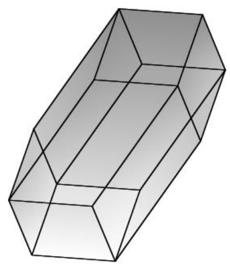
- in characteristic zero $\overline{M}_{g,n}$ is not a Mori Dream Space for $g \geq 3, n \geq 1$. This was proven in [**Ke**] by providing a nef but not semiample divisor on $\overline{M}_{g,n}$;
- in [**CC**] D. Chen and I. Coskun proved that $\overline{M}_{1,n}$ is not a Mori Dream Space for $n \geq 3$ because it has infinitely many extremal effective divisors.

REMARK 7.1. The step $r = 1, s = n - 3$ of Construction 6.4 is the Losev-Manin's space $\overline{L}_{n-2}$ [**Ha**, Section 6.4]. This space is a toric variety of dimension $n - 3$. It is the last toric variety in

Construction 6.4. For instance L_3 is a Del Pezzo surface of degree six. The following picture represents the corresponding polyhedron.



The space $\overline{L}_4$ is the blow-up of $\mathbb{P}^3$ at four general points and along the strict transform of the six lines joining them. The corresponding polyhedron is the following.



Note that both the polyhedra are very symmetric.

In a way $\overline{M}_{0,n}$ is very close to a toric variety. This is one of the reasons that led to conjecture that $\overline{M}_{0,n}$ is a Mori Dream Space.

THEOREM 7.2. [**CT1**, Theorem 1.3] *Let $n = a+b+c+8$ where a, b, c are positive coprime integers. If $Bl_e\overline{L}_{n-3}$ is a Mori Dream Space then $Bl_e\mathbb{P}(a,b,c)$ is a Mori Dream Space.*

PROOF. Let $e_1, ..., e_{n-2}$ be vectors in $\mathbb{R}^{n-3}$ such that $e_1 + ... + e_{n-2} = 0$. Let N be the lattice generated by $e_1, ..., e_{n-2}$, and consider the fan Σ_{n-2} spanned by the primitive lattice vectors $\sum_{i\in I} e_i$ for each subset $I \subset S = \{1, ..., n-2\}$ with $1 \leq |I| \leq n-3$. The toric variety associated to this fan is the Losev-Manin space $\overline{L}_{n-2} = X(\Sigma_{n-2})$.
Let us consider a partition $S = S_1 \cup S_2 \cup S_3$ into subsets of order $a+2, b+2, c+2$. Then $n = a+b+c+8$. We fix $n_i \in S_i$ for $i = 1,2,3$, and consider the sublattice spanned by the vectors

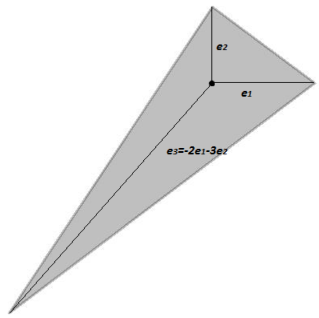
$$e_{n_i} + e_r, \quad \text{for} \quad r \in S_i \setminus \{n_i\},\ i = 1,2,3. \tag{7.1}$$

Let $N' = N/N''$ be the quotient and let $\pi : N \to N'$ be the projection. Then N' is a lattice, it is spanned by the vectors $\pi(e_{n_i})$ for $i = 1,2,3$, and $a\pi(e_{n_1}) + b\pi(e_{n_2}) + c\pi(e_{n_3}) = 0$.

EXAMPLE 7.3. Take $a = 1, b = 2, c = 3$, and $S_1 = \{e_1, e_2, e_3\}, S_2 = \{e_4, e_5, e_6, e_7\}, S_3 = \{e_8, e_9, e_{10}, e_{11}, e_{12}\}$. The we take $e_{n_1} = e_1, e_{n_2} = e_4, e_{n_3} = e_8$. Clearly $N' = N/N''$ is generated by $\pi(e_1), \pi(e_4), \pi(e_8)$. Since $\pi(e_1) = -\pi(e_i)$ for $i = 2,3$, $\pi(e_4) = -\pi(e_i)$ for $i = 5,6,7$, and $\pi(e_8) = -\pi(e_i)$ for $i = 9,10,11,12$, the relation $\sum_{i=1}^{12} e_i = 0$ gives $\pi(e_1) - \pi(e_1) - \pi(e_1) + \pi(e_4) - \pi(e_4) - 2\pi(e_4) + \pi(e_8) - \pi(e_8) - 3\pi(e_8) = -(\pi(e_1) + 2\pi(e_4) + 3\pi(e_8)) = 0$. Therefore

$$\pi(e_1) + 2\pi(e_4) + 3\pi(e_8) = 0.$$

It follows that the toric surface with lattice N' and rays spanned by $\pi(e_{n_i})$ for $i = 1,2,3$ is the weighted projective plane $\mathbb{P}(a,b,c)$. For instance the following is the fan of $\mathbb{P}(1,2,3)$.



Let N_j, for $j = 1, ..., n-4$, be the lattice obtained by taking the quotient of N by a sublattice spanned by the first $j-1$ vectors of the sequence 7.1. Let Γ_j be a sets of rays obtained by projecting the rays of the fan of $\overline{L}_{n-2}$, and $X_j = X(\Gamma_j)$. Mote that $N_{n-4} = N'$ and we have a regular map $X_{n-4} \to \mathbb{P}(a,b,c)$ obtained forgetting all vector of Γ_{n-4} except the $\pi(e_{n_i})$ for $i = 1,2,3$. Since this map is an isomorphism on the torus it induces a birational morphism $Bl_eX_{n-4} \to Bl_e\mathbb{P}(a,b,c)$, where e is the identity of the torus. In this way we get a sequence of toric morphism

$$X_1 \to X_2 \to ... \to X_{n-4} \to \mathbb{P}(a,b,c).$$

Note that X_1 has the same rays of $\overline{L}_{n-2}$ and therefore is a small modification of $\overline{L}_{n-2}$ which is an isomorphism on the torus. Then Bl_eX_1 is a small modification of $Bl_e\overline{L}_{n-2}$. □

Next we consider the following theorem.

THEOREM 7.4. [**CT1**, Theorem 1.1] *There exists a small $\mathbb{Q}$- factorial projective modification $\widetilde{L}_{n-2}$ of $Bl_e\overline{L}_{n-2}$, and surjective morphisms*

$$\widetilde{L}_{n-2} \to \overline{M}_{0,n} \to Bl_e\overline{L}_{n-3}.$$

In particular, by Proposition 0.5, if $\overline{M}_{0,n}$ is a Mori Dream Space then $Bl_e\overline{L}_{n-3}$ is a Mori Dream Space, if $Bl_e\overline{L}_{n-2}$ is a Mori Dream Space then $\overline{M}_{0,n}$ is a Mori Dream Space.

In particular, if $\overline{M}_{0,n}$ is a Mori Dream Space then $Bl_e\overline{L}_{n-2}$ is a Mori Dream Space, and by Theorem 7.2 $Bl_e\mathbb{P}(a,b,c)$ is a Mori Dream Space. Now, the key ingredient is the following result due to S. Goto, K. Nishida, and K. Watanabe.

THEOREM 7.5. [**GNW**] *Assume* $\mathrm{char}(k) = 0$. *If $(a,b,c) = (7h-3, 5h^2-2h, 8h-3)$, with $h \geq 4$ and $3 \nmid h$, then $Bl_e\mathbb{P}(a,b,c)$ is not a Mori Dream Space.*

An immediate consequence of Theorems 7.2, 7.4 and 7.5 is the following.

THEOREM 7.6. [**CT1**, Corollary 1.4] *Assume* $\mathrm{char}(k) = 0$. *Then $\overline{M}_{0,n}$ is not a Mori Dream Space for $n > 133$.*

PROOF. We have $n(h) = a+b+c+8 = 7h-3+5h^2-2h+8h-3+8 = 5h^2+13h+2$. So $n(4) = 134$. Therefore $\overline{M}_{0,134}$ is not a Mori Dream Space. If $n > 135$ we have a surjective forgetful morphism $\pi_i : \overline{M}_{0,n} \to \overline{M}_{0,134}$. Therefore, by Proposition 0.5, $\overline{M}_{0,n}$ is not a Mori Dream Space for $n \geq 134$. □

7.1. A problem by Hassett. Let S be a Noetherian scheme and g, n two non-negative integers. A family of nodal curves of genus g with n marked points over S consists of a flat proper morphism $\pi : C \to S$ whose geometric fibers are nodal connected curves of arithmetic genus g, and sections $s_1, ..., s_n$ of π. A collection of input data $(g, A) := (g, a_1, ..., a_n)$ consists of an integer $g \geq 0$ and the weight data: an element $(a_1, ..., a_n) \in \mathbb{Q}^n$ such that $0 < a_i \leq 1$ for $i = 1, ..., n$, and

$$2g - 2 + \sum_{i=1}^{n} a_i > 0.$$

DEFINITION 7.7. *A family of nodal curves with marked points $\pi : (C, s_1, ..., s_n) \to S$ is stable of type (g, A) if*

- *the sections $s_1, ..., s_n$ lie in the smooth locus of π, and for any subset $\{s_{i_1}, ..., s_{i_r}\}$ with non-empty intersection we have $a_{i_1} + ... + a_{i_r} \leq 1$,*
- *$\omega_\pi(\sum_{i=1}^{n} a_i s_i)$ is π-relatively ample, where ω_π is the relative dualizing sheaf.*

B. Hassett in [**Ha**, Theorem 2.1] proved that given a collection (g, A) of input data, there exists a connected Deligne-Mumford stack $\overline{\mathcal{M}}_{g,A[n]}$, smooth and proper over $\mathbb{Z}$, representing the moduli problem of pointed stable curves of type (g, A). The corresponding coarse moduli scheme $\overline{M}_{g,A[n]}$ is projective over $\mathbb{Z}$.
For fixed g, n, consider two collections of weight data $A[n], B[n]$ such that $a_i \geq b_i$ for any $i = 1, ..., n$. Then there exists a birational *reduction morphism*

$$\rho_{B[n],A[n]} : \overline{M}_{g,A[n]} \to \overline{M}_{g,B[n]}$$

associating to a curve $[C, s_1, ..., s_n] \in \overline{M}_{g,A[n]}$ the curve $\rho_{B[n],A[n]}([C, s_1, ..., s_n])$ obtained by collapsing components of C along which $\omega_C(b_1 s_1 + ... + b_n s_n)$ fails to be ample, where ω_C denote the dualizing sheaf of C. For the details see [**Ha**, Section 4].
In the following we will be especially interested in the boundary of $\overline{M}_{0,A[n]}$. We consider a partition $I \cup J = \{1, ..., n\}$, such that $|I|, |J| \geq 2$ and $a_{i_1} + ... + a_{i_r} > 1$, $a_{j_1} + ... + a_{j_{n-r}} > 1$ where $I = \{i_1, ..., i_r\}$, $J = \{j_1, ..., j_{n-r}\}$. We denote by $D_{I,J}(A)$ the divisor in $\overline{M}_{0,A[n]}$ whose general point corresponds to a nodal curve with two irreducible components with marked points $x_{i_1}, ..., x_{i_r}$ on one component and $x_{j_1}, ..., x_{j_{n-r}}$ on the other.
Furthermore, for any partition with $I = \{i_1, i_2\}$ and $i_1 + i_2 \leq 1$ corresponds to a divisor $D_{I,J}(A)$ as well. Such a divisor parametrizes curves where the marked points x_{i_1}, x_{i_2} coincides. Note that these curves are note necessarily nodal. In [**Ha**] Hassett proposed the following problem:

PROBLEM 7.8. [**Ha**, Problem 7.1] *Let $A[n]$ be a vector of weights and consider the moduli space $\overline{M}_{0,A[n]}$. Do there exist rational numbers $\alpha_{I,J}$ such that*

$$K_{\overline{M}_{0,A[n]}} + \sum_{I,J} \alpha_{I,J} D_{I,J}(A)$$

is ample and the pair $(\overline{M}_{0,A[n]}, \sum_{I,J} \alpha_{I,J} D_{I,J}(A))$ is log canonical?

In [**Ha**, Sections 7.1, 7.2, 7.3, Remark 8.5] Hassett provides examples in which Problem 7.8 admits a positive answer. By taking advantage of Proposition 4.8 we are able to provide two new classes of examples. Let us recall the following construction due to *M. Kapranov* [**Ka**].

CONSTRUCTION 7.9. Fixed $(n-1)$-points $p_1, ..., p_{n-1} \in \mathbb{P}^{n-3}$ in linear general position:

(1) Blow-up the points $p_1, ..., p_{n-1}$,

(2) Blow-up the strict transforms of the lines $\langle p_{i_1}, p_{i_2}\rangle$, $i_1, i_2 = 1, ..., n-1$,

$\vdots$

(k) Blow-up the strict transforms of the $(k-1)$-planes $\langle p_{i_1}, ..., p_{i_k}\rangle$, $i_1, ..., i_k = 1, ..., n-1$,

$\vdots$

$(n-4)$ Blow-up the strict transforms of the $(n-5)$-planes $\langle p_{i_1}, ..., p_{i_{n-4}}\rangle$, $i_1, ..., i_{n-4} = 1, ..., n-1$.

Now, consider Hassett's spaces $X_k[n] := \overline{M}_{0,A_k[n]}$ for $k = 1, ..., n-4$, such that

- $a_i + a_n > 1$ for $i = 1, ..., n-1$,
- $a_{i_1} + ... + a_{i_r} \leq 1$ for each $\{i_1, ..., i_r\} \subset \{1, ..., n-1\}$ with $r \leq n-k-2$,
- $a_{i_1} + ... + a_{i_r} > 1$ for each $\{i_1, ..., i_r\} \subset \{1, ..., n-1\}$ with $r > n-k-2$.

Then $X_k[n]$ is isomorphic to the variety obtained at the step k of the blow-up construction. Therefore the variety Y appearing in Proposition 6.4 is isomorphic to $\overline{M}_{0,n+3}$ and the boundary divisor of the log resolution $\pi : Y \to X^n_{n+2}$ is nothing but the total boundary divisor of $\overline{M}_{0,n+3}$. Furthermore X^n_{n+2} is isomorphic to the Hassett's space $X_1[n+3]$. Therefore, by Proposition 6.5 the Hassett's space $X_1[n+3]$ is log Fano.

Now, let us consider the space $\overline{M}_{0,A_1[n]} = X_1[n]$ obtained at the first step of Construction 7.9. In our notations this is $X^{n-3}_{n-1} = Bl_{p_1,...,p_{n-1}}\mathbb{P}^{n-3}$. This fix ideas $\overline{M}_{0,A_1[n]}$ can be realized taking

$$A_1[n] = (1/(n-3), ..., 1/(n-3), 1).$$

PROPOSITION 7.10. *For the moduli spaces $\overline{M}_{0,A_1[n]}$ Problem 7.8 admits a positive answer.*

PROOF. The blow-up morphism $\overline{M}_{0,A_1[n]} \to \mathbb{P}^{n-3}$ is nothing but the reduction morphism $\rho : \overline{M}_{0,A_1[n]} \to \mathbb{P}^{n-3}$ given by $(1/(n-3), ..., 1/(n-3), 1) \mapsto (1/(n-2), ..., 1/(n-2), 1)$. We have $n-1$ partitions of the type $I = (\hat{\imath}, n)$, $J = (1, ..., \hat{\imath}, ..., n-1)$. The $(n-1)$ divisors $D_{I,J}$ corresponding to these partitions are contracted to a point by ρ and are nothing but the $n-1$ exceptional divisors of the blow-up. Furthermore, we have $\binom{n-1}{2}$ divisors $D_{I,J}(A)$ with $I = \{\hat{\imath}_1, \hat{\imath}_2\}$, $J = \{j_1, ..., \hat{\imath}_1, ..., \hat{\imath}_2, ..., j_{n-3}, n\}$, and therefore $x_{\imath_1} = x_{\imath_2}$. These divisors are mapped by ρ to the $\binom{n-1}{2}$ hyperplanes spanned subsets of cardinality $n-3$ of $\{p_1, ..., p_{n-1}\}$. As usual we denote by H the pullback of the hyperplane class of $\mathbb{P}^{n-3}$ and by $E_1, ..., E_{n-1}$ the exceptional divisors. Then we have

$$K_{\overline{M}_{0,A_1[n]}} = -(n-2)H + (n-4)(E_1 + ... + E_{n-1}).$$

Let $H_{i_1,...,i_{n-3}}$ be the strict transform of the hyperplane $\langle p_{i_1}, ..., p_{i_{n-3}}\rangle$. Using the same notations of Problem 7.8 we take $\alpha_{I,J} = \alpha$ for any $D_{I,J}(A)$ of type

$$I = \{\hat{\imath}_1, \hat{\imath}_2\}, \quad J = \{j_1, ..., \hat{\imath}_1, ..., \hat{\imath}_2, ..., j_{n-3}, n\}$$

and $\alpha_{I,J} = \beta$ for any $D_{I,J}(A)$ of type

$$I = (\hat{\imath}, n), \quad J = (1, ..., \hat{\imath}, ..., n-1).$$

Then

$$\sum_{I,J} \alpha_{I,J} D_{I,J}(A) = \alpha(H_{1,...,n-3} + ... + H_{3,...,n-1}) + \beta(E_1 + ... + E_{n-1})$$

and since $H_{i_1,...,i_{n-3}} = H - E_{i_1} - ... - E_{i_{n-3}}$ we get

$$K_{\overline{M}_{0,A_1[n]}} + \sum_{I,J} \alpha_{I,J} D_{I,J}(A) = \left(\alpha\binom{n-1}{2} - n + 2\right) H - \left(\alpha\binom{n-2}{2} - n - \beta + 4\right) \sum_{i=1}^{n-1} E_i.$$

Now, in the notations of Section 4.3 we have

$$(K_{\overline{M}_{0,A_1[n]}} + \sum_{I,J}\alpha_{I,J}D_{I,J}(A))\cdot R_i = \frac{\alpha}{2}(n-2)(n-3) - n - \beta + 4, \tag{7.2}$$

and

$$(K_{\overline{M}_{0,A_1[n]}} + \sum_{I,J}\alpha_{I,J}D_{I,J}(A))\cdot L_{i,j} = \frac{\alpha}{2}(n-2)(5-n) + 2\beta + n - 6. \tag{7.3}$$

In particular for $\alpha = \frac{2}{n-2}$ and $\beta = \frac{2}{3}$ we have that both (7.2) and (7.3) are strictly positive. Therefore, by Proposition 4.8 for $\alpha = \frac{2}{n-2}$ and $\beta = \frac{2}{3}$ the divisor $(K_{\overline{M}_{0,A_1[n]}} + \sum_{I,J}\alpha_{I,J}D_{I,J}(A)$ is ample.
Let $\overline{\rho} : \overline{M}_{0,n} \to \overline{M}_{0,A_1[n]}$ be the reduction morphism obtained by composition of the blow-ups in Construction 7.9. By Proposition 6.4 the morphism $\overline{\rho}$ is a log resolution of the pair $(\overline{M}_{0,A_1[n]}, D)$, where $D = \alpha\sum_{i_1,...,i_{n-3}} H_{i_1,...,i_{n-3}} + \beta\sum_i E_i$.
We have $\rho_h = \binom{n-1}{h+1}$ h-planes spanned by subsets of cardinality $h+1$ of $\{p_1,...,p_{n-1}\}$. Let E_j^h for $j = 1,...,\rho_h$ be the exceptional divisors over them. Then we have

$$K_{\overline{M}_{0,n}} = \overline{\rho}^*K_{\overline{M}_{0,A_1[n]}} + \sum_{h=1}^{n-5}(n-h-4)(E_1^h + ... + E_{\rho_h}^h).$$

Furthermore, through any such an h-plane there are $\binom{n-h-2}{n-h-4}$ of the $H_{i_1,...,i_{n-3}}$'s. Proceeding as in the proof of Proposition 6.4 we may write:

$$\overline{\rho}^*(D) = \sum_{h=1}^{n-5}\alpha\binom{n-h-2}{2}(E_1^h + ... + E_{\rho_h}^h) + \alpha\sum_{i_1,...,i_{n-3}}\widetilde{H}_{i_1,...,i_{n-3}} + \beta\sum_i\widetilde{E}_i$$

where $\widetilde{H}_{i_1,...,i_{n-3}}$ and $\widetilde{E}_i$ are respectively the strict transforms in $\overline{M}_{0,n}$ of $H_{i_1,...,i_{n-3}}$ and E_i. Finally

$$\begin{aligned} K_{\overline{M}_{0,n}} = \ & \overline{\rho}^*(K_{\overline{M}_{0,A_1[n]}} + D) + \textstyle\sum_{h=1}^{n-5}\left(n-h-4-\alpha\binom{n-h-2}{2}\right)(E_1^h + ... + E_{\rho_h}^h) \\ & -\alpha\textstyle\sum_{i_1,...,i_{n-3}}\widetilde{H}_{i_1,...,i_{n-3}} - \beta\sum_i\widetilde{E}_i. \end{aligned}$$

To conclude it is enough to observe that for $\alpha = \frac{2}{n-2}$ and $\beta = \frac{2}{3}$ all the discrepancies are greater than -1. Therefore the pair $(\overline{M}_{0,A_1[n]}, D)$ is klt and in particular log canonical. □

Now, let us consider Construction 6.4. The moduli space $\overline{M}_{0,A_{1,2}[n]}$ with weights

$$A_{1,2}[n] = (1/(n-2),...,1/(n-2),2/(n-2),1)$$

is the blow-up $X_{n-2}^{n-3} = Bl_{p_1,...,p_{n-2}}\mathbb{P}^{n-3}$.

PROPOSITION 7.11. *Problem 7.8 admits a positive answer for the moduli spaces $\overline{M}_{0,A_{1,2}[n]}$ as well.*

PROOF. In this case the divisors $D_{I,J}$ are the following:

- the $n-2$ exceptional divisors $E_1,...,E_{n-2}$,
- the strict transforms $H_{i_1,...,i_{n-3}}$ of the $n-2$ hyperplanes spanned by subsets of cardinality $n-3$ of $\{p_1,...,p_{n-2}\}$,
- the strict transforms $\Lambda_{j_1,...,j_{n-4}}$ of the $\binom{n-2}{2}$ hyperplanes spanned by subsets of cardinality $n-4$ of $\{p_1,...,p_{n-2}\}$ and p_{n-1}.

We consider the divisor

$$D = \sum_{I,J} \alpha_{I,J} D_{I,J}(A) = \frac{2}{n-2} \sum_{i_1,...,i_{n-3}} H_{i_1,...,i_{n-3}} + \frac{2}{n-2} \sum_{j_1,...,j_{n-4}} \Lambda_{j_1,...,j_{n-4}} + \frac{2}{3} \sum_{i=1}^{n-2} E_i.$$

We proceed as in the proof of Proposition 7.10. Now, $H_{i_1,...,i_{n-3}} = H - E_{i_1} - ... - E_{i_{n-3}}$ and $\Lambda_{j_1,...,j_{n-4}} = H - E_{j_1} - ... - E_{j_{n-4}}$, through each of the p_i's for $i = 1, ..., n-2$ there are $\binom{n-3}{n-4} = n-3$ of the $H_{i_1,...,i_{n-3}}$'s and $\binom{n-3}{2}$ of the $\Lambda_{i_1,...,i_{n-3}}$'s. Therefore, we may write:

$$D = (n-1)H + \left(\frac{2}{3} - \frac{2(n-3)}{n-2} - \frac{2}{n-2}\binom{n-3}{2}\right) \sum_{i=1}^{n-2} E_i = (n-1)H - \frac{3n-11}{3} \sum_{i=1}^{n-2} E_i$$

and

$$K_{\overline{M}_{0,A_{1,2}[n]}} + D = (-n+2+n-1)H + \left(n-4+\frac{11-3n}{3}\right) \sum_{i=1}^{n-2} E_i = H - \frac{1}{3} \sum_{i=1}^{n-2} E_i.$$

Now, $(K_{\overline{M}_{0,A_{1,2}[n]}} + D) \cdot R_i = (K_{\overline{M}_{0,A_{1,2}[n]}} + D) \cdot L_{i,j} = \frac{1}{3}$ and by Proposition 4.8 the divisor $(K_{\overline{M}_{0,A_{1,2}[n]}} + D)$ is ample.

Now, let $\pi_{n-1} : X_{n-1}^{n-3} \to X_{n-2}^{n-3}$ be the blow-up of p_{n-1} and consider the composition

$$\begin{tikzcd} \overline{M}_{0,n} \arrow[r, "\overline{\rho}"] \arrow[rr, bend right, "\widetilde{\rho}"'] & X_{n-1}^{n-3} = \overline{M}_{0,A_1[n]} \arrow[r, "\pi_{n-1}"] & X_{n-2}^{n-3} = \overline{M}_{0,A_{1,2}[n]} \end{tikzcd}$$

where $\overline{\rho}$ is the log resolution used in the proof of Proposition 7.10. Then $\widetilde{\rho}$ is a log resolution of the pair $(\overline{M}_{0,A_{1,2}[n]}, D)$. Let E_{n-1} be the exceptional divisor over p_{n-1}, E_j^h be the $\gamma_h = \binom{n-2}{h+1}$ exceptional divisors over the h-planes spanned by subsets of cardinality $h+1$ of $\{p_1, ..., p_{n-2}\}$, and $\overline{E}_j^h$ be the $\overline{\gamma}_h = \binom{n-2}{h}$ exceptional divisors over the h-planes spanned by subsets of cardinality h of $\{p_1, ..., p_{n-2}\}$ and p_{n-1}. Now, note that

- the point p_{n-1} is contained in any $\Lambda_{j_1,...,j_{n-4}}$ and we have $\binom{n-2}{2}$ of them,
- any h-plane spanned by subsets of cardinality $h+1$ of $\{p_1, ..., p_{n-2}\}$ is contained in $n-h-3$ of the $H_{i_1,...,i_{n-3}}$'s and in $\binom{n-h-3}{2}$ of the $\Lambda_{j_1,...,j_{n-4}}$'s,
- any h-plane spanned by subsets of cardinality h of $\{p_1, ..., p_{n-2}\}$ and p_{n-1} is contained in $\binom{n-h-2}{2}$ of the $\Lambda_{j_1,...,j_{n-4}}$'s.

Therefore, we have

$$\begin{aligned} \widetilde{\rho}^* D = \; & \tfrac{2}{n-2}\tbinom{n-2}{2} E_{n-1} + \tfrac{2}{n-2} \textstyle\sum_{h=1}^{n-5} \left(n-h-3+\tbinom{n-h-3}{2}\right)(E_1^h + ... + E_{\gamma_h}^h) + \\ & \tfrac{2}{n-2} \textstyle\sum_{h=1}^{n-5} \tbinom{n-h-2}{2}(\overline{E}_1^h + ... + \overline{E}_{\overline{\gamma}_h}^h) + \widetilde{D}. \end{aligned}$$

Now, since

$$K_{\overline{M}_{0,n}} = \widetilde{\rho}^* K_{\overline{M}_{0,A_{1,2}[n]}} + (n-4)E_{n-1} + \sum_{h=1}^{n-5} (n-h-4)(E_1^h + ... + E_{\gamma_h}^h + \overline{E}_1^h + ... + \overline{E}_{\gamma_h}^h)$$

we get

$$\begin{aligned} K_{\overline{M}_{0,n}} = \ & \widetilde{\rho}^*(K_{\overline{M}_{0,A_{1,2}[n]}}+D)+\left(n-4-\tfrac{2}{n-2}\tbinom{n-2}{2}\right)E_{n-1}+\\ & \textstyle\sum_{h=1}^{n-5}\left(n-h-4-\tfrac{2}{n-2}\left(n-h-3+\tbinom{n-h-3}{2}\right)\right)(E_1^h+...+E_{\gamma_h}^h)+\\ & \textstyle\sum_{h=1}^{n-5}\left(n-h-4-\tfrac{2}{n-2}\tbinom{n-h-2}{2}\right)(\overline{E}_1^h+...+\overline{E}_{\gamma_h}^h)-\widetilde{D} \end{aligned}$$

where $\widetilde{D}$ is the strict transform of D in $\overline{M}_{0,n}$. The discrepancies are all greater or equal than -1 hence the pair $(\overline{M}_{0,A_{1,2}[n]}, D)$ is log canonical. □

Finally, we observe that for 3-fold Hassett's spaces a little improvement is at hand. The moduli space $\overline{M}_{0,6}$ is a log Fano 3-fold, see [**HK**]. By Proposition 1.2 it is a Mori Dream Space. See [**?**] for a direct proof of this last fact and the detailed description of $\mathrm{Cox}(\overline{M}_{0,6})$.

PROPOSITION 7.12. *Any 3-fold Hassett's space $\overline{M}_{0,A[6]}$ is log Fano.*

PROOF. By [**Ha**, Theorem 4.1] there exists a birational reduction morphism $\rho : \overline{M}_{0,6} \to \overline{M}_{0,A[6]}$. Now, it is enough to recall that by [**HK**] $\overline{M}_{0,6}$ is log Fano, and to apply [**GOST**, Corollary 1.3] to the morphism ρ. □

An immediate consequence of Proposition 7.12 is that the following varieties are log Fano.

- The blow-up of $\mathbb{P}^3$ in four general points, along the strict transforms of the lines spanned by them, and in a fifth general point. Indeed, by Construction 6.4 this variety con be realized by taking $A[6] = (1/3, 1/3, 1/3, 1/3, 1, 1)$.
- The blow-up of $\mathbb{P}^3$ in five general points, and along the strict transforms of the lines spanned by them. By Construction 7.9 this is $\overline{M}_{0,6}$ itself.
- The blow-up X_1 of $\mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^1_3$ in $p_1 = ([0:1],[0:1],[0:1])$, $p_2 = ([1:0],[1:0],[1:0])$, and $p_3 = ([1:1],[1:1],[1:1])$. By [**Ha**, Section 6.3], X_1 is isomorphic to $\overline{M}_{0,A_2[6]}$ with $A_1[6] = (2/3, 2/3, 2/3, 1/6, 1/6, 1/6)$.
- Consider the projections $\pi_i : \mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^1_3 \to \mathbb{P}^1_i$, and define $F_0 = \bigcup_{i=1}^3 \pi_i^{-1}([0:1])$, $F_1 = \bigcup_{i=1}^3 \pi_i^{-1}([1:0])$, $F_\infty = \bigcup_{i=1}^3 \pi_i^{-1}([1:1])$. Let Δ_2 be the union of the 2-dimensional diagonals of $\mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^1_3$. Then we have X_2 the blow-up of X_1 along the strict transform of $\Delta_2 \cap (F_0 \cup F_1 \cup F_\infty)$. By [**Ha**, Section 6.3], X_2 can be realized taking $A_2[6] = (2/3, 2/3, 2/3, 1/3, 1/3, 1/3)$.
- Finally, the blow-up X_3 of X_2 along the strict transform of the 1-dimension diagonal Δ_1 of $\mathbb{P}^1_1 \times \mathbb{P}^1_2 \times \mathbb{P}^1_3$. Again by [**Ha**, Section 6.3] this is $\overline{M}_{0,6}$.

PROPOSITION 7.13. *Let us consider the points $q_1 = ([1:0],...,[1:0])$, $q_2 = ([0:1],...,[0:1])$, $q_3 = ([1:1],...,[1:1]) \in (\mathbb{P}^1)^{n-3}$. There exits a small transformation*

$$f : X^{n-3}_{n-1} \dashrightarrow Y^{n-3}_3 = Bl_{q_1,q_2,q_3}(\mathbb{P}^1)^{n-3}.$$

In particular, Y^{n-3}_3 is log Fano.

PROOF. First of all, note that $\rho(X^{n-3}_{n-1}) = \rho(Y^{n-3}_3) = n$. We may assume $p_1 = [1:0:...:0]$, $p_2 = [0:1:...:0]$,..., $p_{n-2} = [0:...:0:1]$, $p_{n-1} = [1:1:...:1]$. Let us consider $X^{n-3}_{n-2} = Bl_{p_1,...,p_{n-2}}\mathbb{P}^{n-3}$ and $Y^{n-3}_2 = Bl_{q_1,q_2}(\mathbb{P}^1)^{n-3}$. These are both toric varieties. Let $e_1, ..., e_{n-3}$ be the standard basis vectors of the co-character lattice of $(k^*)^{n-3}$. The rays of the fan of $\mathbb{P}^{n-3}$

are $e_1, ..., e_{n-3}$ and $-e_1 - ... - e_{n-3}$. By blowing-up $p_1, ..., p_{n-2}$ we add the rays $-e_1, ..., -e_{n-3}$ and $e_1 + ... + e_{n-3}$. On the other hand the rays of $(\mathbb{P}^1)^{n-3}$ are $e_1, ..., e_{n-3}, -e_1, ..., -e_{n-3}$, and the blow-up of q_1, q_2 corresponds to introduce the two rays $e_1 + ... + e_{n-3}$ and $-e_1 - ... - e_{n-3}$. We see that the toric fans of X^{n-3}_{n-2} and Y^{n-3}_2 are the same. Therefore, X^{n-3}_{n-2} and Y^{n-3}_2 are isomorphic in codimension one.
Now, consider the points $p_1, ..., p_{n-3}$. We have $n-3$ codimension two linear subspaces $H^{n-5}_{i_1,...,i_{n-4}} = \langle p_{i_1}, ..., p_{i_{n-4}} \rangle$. For any choice of $i_1, ..., i_{n-4}$ we define $\{j_1, j_2\} = \{0, ..., n-3\} \setminus \{i_1 - 1, ..., i_{n-4} - 1\}$. Then, the projection from $H^{n-5}_{i_1,...,i_{n-4}}$ is the rational map

$$\begin{array}{cccc} \pi_{i_1,...,i_{n-4}}: & \mathbb{P}^{n-3} & \dashrightarrow & \mathbb{P}^1 \\ & [x_0 : ... : x_{n-3}] & \mapsto & [x_{j_1} : x_{j_2}] \end{array}$$

We get a rational map

$$\begin{array}{cccc} g: & \mathbb{P}^{n-3} & \dashrightarrow & (\mathbb{P}^1)^{n-3} \\ & x = [x_0 : ... : x_{n-3}] & \mapsto & (\pi_{1,...,n-4}(x), ..., \pi_{2,...,n-3}(x)) \end{array}$$

Note that the hyperplane $W = \langle p_1, ..., p_{n-3} \rangle = \{x_{n-3} = 0\}$ is mapped by g to the point $q_1 = ([1:0], ..., [1:0]) \in (\mathbb{P}^1)^{n-3}$. Furthermore, this is the only divisor contracted by g. Therefore, blowing-up $q_1 \in (\mathbb{P}^1)^{n-3}$ we get a small transformation $g_1 : X^{n-3}_{n-3} = Bl_{p_1,...,p_{n-3}}\mathbb{P}^{n-3} \dashrightarrow Y^{n-3}_1 = Bl_{q_1}(\mathbb{P}^1)^{n-3}$ fitting in the following diagram:

$$\begin{tikzcd} X^{n-3}_{n-3} \arrow[r, dashed, "g_1"] \arrow[d, "\phi_{n-3}"'] & Y^{n-3}_1 \arrow[d, "\psi_1"] \\ \mathbb{P}^{n-3} \arrow[r, dashed, "g"] & (\mathbb{P}^1)^{n-3} \end{tikzcd}$$

Note that g_1 maps the strict transform $\widetilde{W}$ of W to the exceptional divisor E_{q_1}, while the exceptional divisors $E_{p_1}, ..., E_{p_{n-3}}$ are mapped to the strict transforms of the $n-3$ divisors in $(\mathbb{P}^1)^{n-3}$ obtained by fixing one the factors.
Furthermore, $g([0 : ... : 0 : 1]) = ([0:1], ..., [0:1])$ and $g([1 : ... : 1]) = ([1:1], ..., [1:1])$. Let $\mathcal{U} \subset X^{n-3}_{n-3}$ and $\mathcal{V} \subset Y^{n-3}_1$ be the two open subsets on which g_1 is an isomorphism. Now, by applying the universal property of the blow-up [**Har**, Corollary 7.15] we get that $g_{1|\mathcal{U}}$ lifts to an isomorphism $f : Bl_{p_{n-2},p_{n-1}}\mathcal{U} \to Bl_{q_2,q_3}\mathcal{V}$. Since g_1 is an isomorphism in codimension one we conclude that f induces a small transformation $f : X^{n-3}_{n-1} \dashrightarrow Y^{n-3}_3$ mapping $E_{p_{n-2}}$ to E_{q_2}, and $E_{p_{n-3}}$ to E_{q_3}.
To conclude that Y^{n-3}_3 is log Fano it is enough to recall that by Theorem 6.5 X^{n-3}_{n-1} is log Fano, and to apply Lemma 1.15 to the small map $f : X^{n-3}_{n-1} \dashrightarrow Y^{n-3}_3$. $\square$

Finally, we observe that by [**Ha**, Section 6.3] the variety $Y^{n-3}_3 = Bl_{q_1,q_2,q_3}(\mathbb{P}^1)^{n-3}$ can be interpreted as an Hassett's space $\overline{M}_{0,A[n]}$ with $A[n] = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \frac{1}{3(n-4)}, ..., \frac{1}{3(n-4)})$.

8. $\overline{M}_{0,6}$ is weak Fano

In order to understand the intersection numbers appearing in this section it is useful to keep in mind Section 2. Let us recall the Krapranov's blow-up construction of $\overline{M}_{0,6}$.

CONSTRUCTION 8.1. Let $p_1, ..., p_5 \in \mathbb{P}^3$ be points in linear general position. We consider:

- $\pi_1 : X \to \mathbb{P}^3$ the blow-up of $p_1, ..., p_5$,
- $\pi_2 : Y \to X$ the blow-up of the strict transforms of the lines $\langle p_i, p_j\rangle$, $i, j = 1, ..., 5$,

Then $Y \cong \overline{M}_{0,6}$, and the morphism $f_6 = \pi_1 \circ \pi_2 : \overline{M}_{0,6} \to \mathbb{P}^3$ is induced by the psi-call Ψ_6 on $\overline{M}_{0,6}$.

By [**KMc**, Theorem 1.2] the Mori Cone $NE(\overline{M}_{0,6})$ of $\overline{M}_{0,6}$ is generated by classes of vital curves. Let us denote by E_i and $E_{i,j}$ the exceptional divisors over p_i and the strict transform of $\langle p_i, p_j\rangle$ respectively.
In the first blow-up X the strict transforms of the lines $\langle p_i, p_j\rangle$ intersects the exceptional divisor E_i over p_i in four points q_j for $j \neq i$. Therefore, after blowing-up all the strict transforms of the lines the divisor E_i in $\overline{M}_{0,6}$ is isomorphic to the blow-up of $\mathbb{P}^2$ in four points. We denote by $L^i_{h,k}$ the strict transform in E_i of the line spanned by q_h and q_k, and by R^i_h the exceptional divisor over q_h. So, in any exceptional divisor, we get 10 vital curves: 6 of type $L^i_{h,k}$ and 4 of type R^i_h.
Now, for any line $\langle p_i, p_j\rangle \subset \mathbb{P}^3$ we have three planes $\langle p_i, p_j, p_k\rangle$ for $k \neq i, j$ containing this line. The strict transforms of the three planes intersects the exceptional divisor $E_{i,j}$ in three vital curves $\sigma^k_{i,j}$. Therefore, we have $\binom{5}{2} \cdot 3 = 30$ of them.
Note that R^i_j is numerically equivalent to R^j_i for any i, j because the are fibers of the same ruling of $E_{i,j}$. Furthermore, the $\sigma^k_{i,j}$'s for $k \neq i, j$ are all numerical equivalent because they are fibers of the other ruling of $E_{i,j}$. We conclude that $NE(\overline{M}_{0,6})$ is a polyhedral cone generated by 50 extremal rays.

LEMMA 8.2. *For any i we have:*

$$H^2 \cdot E_i = H \cdot E_i^2 = 0,\ E_i^3 = 1.$$

Furthermore $H \cdot E_{i,j}^2 = -1$, $H^2 \cdot E_{i,j} = 0$ for any i, j, and

$$E_i \cdot E_{h,k}^2 = \begin{cases} -1 & if\ i \in \{h,k\}, \\ 0 & if\ i \notin \{h,k\}. \end{cases}$$

Finally $E_i^2 \cdot E_{h,k} = 0$ for any i, h, k.

PROOF. We will denote by E_i both the exceptional divisor over p_i in X and its strict transform in Y. Let H_i be the strict transform of a general plane through p_i. Then $H_i = H - E_i$ and $H_i^3 = H^3 - 3H^2 \cdot E_i + 3H \cdot E_i^2 - E_i^3$, $H_i^3 = H^2 \cdot E_i = H \cdot E_i^2 = 0$ yield $E_i^3 = H^3 = 1$.
Now, let us consider the following diagram:

$$\begin{tikzcd} E_{i,j} \arrow[r, "j"] \arrow[d, "\pi_E"'] & Y \arrow[d, "\pi"] \\ L_{i,j} \arrow[r, "i"] & X \end{tikzcd}$$

where $\pi_E = \pi_{|E_{i,j}}$. We have $(H - E_{i,j})^2 = H^3 - H^2 \cdot (H - E_{i,j}) = 0$. Therefore,

$$\begin{aligned} H \cdot E_{i,j}^2 &= \pi^* H \cdot j_* E_{i,j}^2 = j_*(E_{i,j}^2 \cdot \pi_E^* i_* H) = -1, \\ H^2 \cdot E_{i,j} &= \pi^* H^2 \cdot j_* E_{i,j} = j_*(E_{i,j} \cdot \pi_E^* i^* H^2) = 0, \\ E_i \cdot E_{i,j}^2 &= \pi^* E_i \cdot E_{i,j}^2 = j_*(E_{i,j}^2 \cdot \pi_E^* i^* E_i) = -1, \\ E_i^2 \cdot E_{i,j} &= \pi^* E_i^2 \cdot E_{i,j} = j_*(E_{i,j} \cdot \pi_E^* i^* E_i^2) = 0. \end{aligned}$$

Finally $(H-E_{i,j})^3 = H^3 - 3H^2\cdot E_{i,j} + 3H\cdot E_{i,j} - E_{i,j}^3 = H_{i,j}^3 = 0$ yields $E_{i,j}^3 = -2$. □

PROPOSITION 8.3. *The moduli space $\overline{M}_{0,6}$ is weak Fano.*

PROOF. The anti-canonical bundle is given by

$$-K_{\overline{M}_{0,6}} = 4H - 2\sum_{i=1}^{5} E_i - \sum_{i,j=1}^{5} E_{i,j}.$$

First we consider the curves of type $L^i_{h,k}$. We have

$$L^i_{h,k}\cdot E_t = \begin{cases} -1 & if\ i=t, \\ 0 & if\ i\neq t. \end{cases}$$

Furthermore,

$$L^i_{h,k}\cdot E_{s,t} = \begin{cases} 1 & \text{if } s{=}i \text{ and } t\in\{h,k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, $L^i_{h,k}\cdot H = 0$, and

$$-K_{\overline{M}_{0,6}}\cdot L^i_{h,k} = -2(-1) - (1+1) = 0.$$

Now, let us consider a curve of type R^i_j. Then $R^i_j\cdot H = R^i_j\cdot E_k = 0$ for any i,j,k, and

$$R^i_j\cdot E_{h,k} = \begin{cases} -1 & \text{if } \{i,j\}=\{h,k\}, \\ 0 & \text{otherwise.} \end{cases}$$

This yields

$$-K_{\overline{M}_{0,6}}\cdot R^i_j = 1.$$

Finally, we consider a curve of type $\sigma_{i,j}$. Note that the normal bundle of the strict transform of a line $L_{i,j} = \langle p_i, p_j\rangle$ is $\mathcal{N}_{L_{i,j}} = \mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore, $\mathcal{O}_{E_{i,j}}(E_{i,j}) = \mathcal{O}_{E_{i,j}}(-1,-1)$. This yields

$$\sigma_{i,j}\cdot E_{h,k} = \begin{cases} -1 & \text{if } \{i,j\}=\{h,k\}, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore $\sigma_{i,j}\cdot H = 1$ and

$$\sigma_{i,j}\cdot E_h = \begin{cases} 1 & \text{if } h\in\{i,j\}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore

$$-K_{\overline{M}_{0,6}}\cdot\sigma_{i,j} = 4 - 2(1+1) - (-1) = 1.$$

This means that $-K_{\overline{M}_{0,6}}$ is nef. Now, by the formulas in Lemma 8.2 we get that $(-K_{\overline{M}_{0,6}})^3 > 0$ which implies that $-K_{\overline{M}_{0,6}}$ is big. □

Bibliography

[AH] J. Alexander, A. Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom. 4, 1995, no. 2, 201-222.

[BPV] W. Barth, C. Peters, A. van de Ven, *Compact complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete Springer-Verlag, Berlin, 1984.

[Be] A. Bertram, *Moduli of Rank-2 Vector Bundles, Theta divisors, and the geometry of curves in projective space*, J. Differential Geom. 35, 1992, 429-469.

[BCHM] C. Birkar, P. Cascini, C. Hacon, J. McKernan, *Existence of minimal models for varieties of log general type*, Journal of the American Mathematical Society, vol. 23, no. 2, 2010, 405-468.

[BL] J. Blanch, S. Lamy, *Weak Fano threefolds obtained by blowing-up a space curve and construction of Sarkisov links*, Proc. Lond. Math. Soc, 2012, no. 5, 1047-1075.

[BCS] P. Bürgisser, M. Clausen, M.A. Shokrollahi, *Algebraic Complexity Theory*, Vol. 315 of Grund. der Math. Wiss, Springer, Berlin, 1997.

[CG] P. Cascini, Y. Gongyo, *On the anti-canonical ring and varieties of Fano type*, arXiv:1306.4689v3.

[CT1] A. M. Castravet, J. Tevelev, *$\overline{M}_{0,n}$ is not a Mori Dream Space*, arXiv:1311.7673v2.

[CT2] A. M. Castravet, J. Tevelev, *Hilbert's 14th problem and Cox rings*, Compositio Math. 142 (2006), 1479-1498.

[CC] D. Chen, I. Coskun, *Extremal effective divisors on the moduli space of n-pointed genus one curves*, arXiv:1304.0350.

[CGLM] P. Comon, G. Golub, L. Lim, B. Mourrain, *Symmetric tensors and symmetric tensor rank*, SIAM J. Matrix Anal. Appl, 2008, no. 3, 1254-1279.

[DM] P. Deligne, D. Mumford, *The irreducibility of the space of curves of given genus*, Inst. Hautes Études Sci. Publ. Math. 36, 1969, 75-109.

[EH] D. Eisenbud, J. Harris, *3264 & All That Intersection Theory in Algebraic Geometry*, http://isites.harvard.edu/fs/docs/icb.topic720403.files/book.pdf.

[FH] W. Fulton, J. Harris, *Representation theory. A first course*, Graduate Texts in Mathematics, Readings in Mathematics 129, 1991, New York: Springer-Verlag.

[Ful] W. Fulton, *Introduction to toric varieties*, Princeton Univ. Press, 1993.

[GOST] Y. Gongyo, S. Okawa, A. Sannai, S. Takagi, *Characterization of varieties of Fano type via singularities of Cox rings*, J. Algebraic Geom, 24, 2015, 159-182.

[GNW] S. Goto, K. Nishida, and K. Watanabe, *Non-Cohen-Macaulay symbolic blow-ups for space monomial curves and counterexamples to Cowsik's question*, Proc. Amer. Math. Soc. 120 (1994), no. 2, 383-392.

[Harr] J. Harris, *Algebraic Geometry: A first course*, Springer-Verlag Graduate Texts in Mathematics, 133, 1993.

[HM] J. Harris, I. Morrison, *Moduli of Curves*, volume 187 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1998.

[Har] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, no. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Ha] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Advances in Mathematics 173 (2003), Issue 2, 316-352.

[Hi] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero*, I, II. Ann. of Math. (2) 79 (1964), 109-203; ibid. (2), 79, 1964, 205-326.

[HK] Y. Hu, S. Keel, *Mori dream spaces and GIT*, Michigan Math. J. 48 (2000), 331-348.

[Kan] V. Kanev, *Chordal varieties of Veronese varieties and catalecticant matrices*, J. Math. Sci, New York, 1999, no. 1, 1114-1125.

[Ka] M. Kapranov, *Veronese curves and Grothendieck-Knudsen moduli spaces $\overline{M}_{0,n}$*, Jour. Alg. Geom. 2 (1993), 239-262.

[Ke] S. KEEL, *Basepoint freeness for nef and big line bundles in positive characteristic*, Ann. of Math. 2, 149, (1999), no. 1, 253-286.

[KMc] S. KEEL, J. MCKERNAN, *Contractible Extremal Rays on $\overline{M}_{0,n}$*, arXiv:alg-geom/9607009v1.

[Kn] F. KNUDSEN, *The projectivity of the moduli space of stable curves II: the stack $M_{g,n}$*, Math. Scand. 52 (1983), 161-199.

[La] R. K. LAZARSFELD, *Positivity in Algebraic Geometry I*, Berlin, Springer, ISBN 3-540-22533-1.

[LO] J. M. LANDSBERG, G. OTTAVIANI, *Equations for secant varieties of Veronese and other varieties*, Annali di Matematica Pura ed Applicata, 2011.

[LM] A. LOSEV, Y. MANIN, *New moduli spaces of pointed curves and pencils of flat connections*, Michigan Math. J. Volume 48, Issue 1, 2000, 443-472.

[MP] S. MAGGIOLO, N. PAGANI, *Generating stable modular graphs*, Journal of Symbolic Computation, Volume 46, Issue 10, October 2011, 1087-1097.

[MM] A. MASSARENTI, M. MELLA, *On the automorphisms of Hassett's moduli spaces*, arXiv:1307.6828v1.

[Mu1] S. MUKAI, *Geometric realization of T-shaped root systems and counterexamples to Hilbert's fourteenth problem*, Algebraic transformation groups and algebraic varieties, Encyclopaedia of Mathematical Sciences, vol. 132 (Springer, Berlin, 2004), 123-130.

[MFK] D. MUMFORD, J. FOGARTY AND F. KIRWAN, *Geometric invariant theory*, Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2), 34. Springer-Verlag, Berlin, 1994, xiv+292 pp.

[Ok] S. OKAWA, *On images of Mori dream spaces*, arXiv:1104.1326.

[Re] M. REID, *Young person's guide to canonical singularities*, Proc. Sympos. Pure Math., 46, Providence, R.I.: American Mathematical Society, 345-414.

[Te] A. TERRACINI, *Sulle V_k per cui la varieta degli S_h $(h+1)$-seganti ha dimensione minore dell'ordinario*, Rend. Circ. Mat. Palermo, 31, 1911, 392-396.

[Ve] P. VERMEIRE, *On the Regularity of Powers of Ideal Sheaves*, Compositio Mathematica, 131, 2002, 161-172.

[Ve1] P. VERMEIRE, *Singularities of the secant variety*, Journal of Pure and Applied Algebra, 213, 2009, 1129-1132.

[Wa] E. Waring, *Meditationes Algebricae*, J. Archdlated from the Latin by D.Weeks, American Mathematical Sociey, Providence, RI, 1991.