MODULI OF STABLE MAPS, GROMOV-WITTEN INVARIANTS AND QUANTUM COHOMOLOGY

ALEX MASSARENTI

Abstract. We introduce moduli spaces of stable maps $\overline{M}_{g,n}(X,\beta)$ for a projective scheme X. Then we define Gromov-Witten invariants as integral on the virtual fundamental class of $\overline{M}_{g,n}(X,\beta)$ and list their fundamental axioms. Finally we introduce big and small quantum cohomology in relations to Gromov-Witten invariants, and study its properties such as the associativity of the quantum product in relation to $WDVV$ equations.

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Introduction

In symplectic topology and algebraic geometry, Gromov-Witten invariants are rational numbers that, in certain situations, count holomorphic curves. The Gromov-Witten invariants may be packaged as a homology or cohomology class, or as the deformed cup product of quantum cohomology. These invariants have been used to distinguish symplectic manifolds that were previously indistinguishable. They also play a crucial role in string theory. They are named for *Mikhail Gromov* and *Edward Witten*.
Gromov-Witten invariants are of interest in string theory. In this theory the elementary particles are made of tiny strings. A string traces out a surface in the spacetime, called the worldsheet of the string. The moduli space of such parametrized surfaces, at least a priori, is infinite-dimensional; no appropriate measure on this space is known, and thus the path integrals of the theory lack a rigorous definition. However in a variation known as *closed A model topological string theory* there are six spacetime dimensions, which constitute a symplectic manifold, and it turns out that the

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worldsheets are necessarily parametrized by pseudoholomorphic curves, whose moduli spaces are only finite-dimensional. Gromov-Witten invariants, as integrals over these moduli spaces, are then path integrals of the theory.

The appropriate moduli spaces were introduced by *M. Kontsevich* in [Ko], these space are denoted by $\overline{M}_{g,n}(X,\beta)$ where X is a projective scheme, and parametrizes holomorphic maps from a n-pointed genus g curves to X whose image has homology class β. If X is a homogeneous variety the $\overline{M}_{0,n}(X,\beta)$ is a normal, projective variety of pure dimension. Furthermore if $X=\mathbb{P}^N$ then $\overline{M}_{0,n}(\mathbb{P}^N,d)$ is irreducible. On the other hand when $g\geqslant 1$, and even when $g=0$ for most schemes $X\neq\mathbb{P}^N$ the space $\overline{M}_{g,n}(X,\beta)$ may have many components of dimension greater that the expected dimension. To overcome this gap and give a rigorous definition of Gromov-Witten invariants *J. Li, G. Tian* in [LT1], [LT2], and *K. Behrend, B. Fantechi* in [BF] introduce the notions of *virtual fundamental class* and *virtual dimension*.

Recently *F. Poma* in [Po], using intersection theory on Artin stacks developed by *A. Kresch* in [Ke], constructed a perfect obstruction theory leading to a virtual class and then to a rigorous definition of Gromov-Witten invariants in positive and mixed characteristic, satisfying the axioms of Gromov-Witten invariants given by *M. Kontsevich* and *Y. Manin* in [KM], and the $WDVV$ equations.

The Gromov-Witten potential, which as a function encoding the information carried by Gromov-Witten invariants, satisfies $WDVV$ equations. This is equivalent to the associativity of the quantum product. As a consequence turns out that the quantum cohomology ring QH^*X to is a supercommutative algebra, and the complex cohomology $H^*(X,\mathbb{C})$ has a structure of Frobenius manifold.

1. Moduli of Stable Maps

Let X be a projective variety, $\beta\in H_2(X,\mathbb{Z})$ be a homology class, and $Z_1,...,Z_n\subset X$ cycles in general position. We want to study the following set of curves

$$\{C\subset X\ of\ genus\ g,\ homology\ \beta,\ and\ C\cap Z_i\neq\emptyset\ for\ any\ i\}. \tag{1.1}$$

In [Ko] *M. Kontsevich* observed that the curve $C\subset X$ should be replaced by a pointed curve $(C,\{x_1,...,x_n\})$ and a holomorphic map $f:C\rightarrow X$ such that $f(x_i)\in Z_i$ for any $i=1,...,n$. The key idea, in order to give an algebraic definition of *Gromov-Witten classes* and *invariants*, is to introduce a suitable compactification done by *stable maps* of the space of curves 1.1.

Definition 1.1. An n-pointed, genus g, *quasi-stable* curve $[C,\{x_1,...,x_n\}]$ is a projective, connected, reduced, at most nodal curve of arithmetic genus g, with n distinct, and smooth marked points.

A family of n-pointed genus g quasi-stable curve parametrized by a scheme S over $\mathbb{C}$ is a flat, projective morphism $\pi:\mathcal{C}\rightarrow S$, with n-sections $x_1,...,x_n:S\rightarrow\mathcal{C}$, such that the fiber $[C_s,\{x_1(s),...,x_n(s)\}]$ is a n-pointed, genus g, quasi-stable curve, for any geometric point $s\in S$.

Let X be a scheme over $\mathbb{C}$. A family of maps over S to X is a collection

$$(\pi:\mathcal{C}\rightarrow S,\{x_1,...,x_n\},\alpha:\mathcal{C}\rightarrow X)$$

such that

- $(\pi:\mathcal{C}\rightarrow S,\{x_1,...,x_n\})$, is a family of n-pointed genus g quasi-stable curve parametrized by S.
- $\alpha:\mathcal{C}\rightarrow X$ is a morphism.

The families $(\pi:\mathcal{C}\rightarrow S,\{x_1,...,x_n\},\alpha)$ and $(\pi^{'}:\mathcal{C}^{'}\rightarrow S,\{x_1^{'},...,x_n^{'}\},\alpha^{'})$ are isomorphic if there is a isomorphism of schemes $\varphi:\mathcal{C}\rightarrow\mathcal{C}^{'}$ such that $\pi=\pi^{'}\circ\varphi$, $x_i^{'}=\varphi\circ x_i$ for any $i=1,...,n$, and $\alpha=\alpha^{'}\circ\varphi$.

Let $(C,\{x_1,...,x_n\},\alpha)$ be a map from an n-pointed genus g curve to X, the *special points* of an irreducible component $E \subseteq C$ are the marked points of C on E and the points in $E \cap \overline{C \setminus E}$.

Definition 1.2. A map $(C,\{x_1,...,x_n\},\alpha)$ from an n-pointed genus g quasi-stable curve to X is *stable* if:

- any component $E \cong \mathbb{P}^1$ of C contracted by α contains at least three special points,
- any component $E \subseteq C$ of arithmetic genus 1 contracted by α contains at least one special point.

A family $(\pi : \mathcal{C} \to S, \{x_1,...,x_n\},\alpha)$ is stable if each geometric fiber is stable.

Remark 1.3. In the case $X = \mathbb{P}^N$ the map $(\pi : \mathcal{C} \to S, \{x_1,...,x_n\},\alpha)$ is stable if and only if $\omega_{\mathcal{C}/S}(x_1 + ... + x_n) \otimes \alpha^*(\mathcal{O}_{\mathbb{P}^N}(3))$ is π-ample.

Let X be a scheme over $\mathbb{C}$, and let $\beta \in A_1X$. To any scheme S over $\mathbb{C}$ we associate the set of isomorphism classes of stable families $(\pi : \mathcal{C} \to S, \{x_1,...,x_n\},\alpha)$ parametrized by S of n-pointed genus g curves to X such that $\alpha_*(C_s) = [\beta]$, where $[\beta]$ denotes the fundamental class of β. In this way we get a controvariant functor

$$\overline{\mathcal{M}}_{g,n}(X,\beta) : \mathfrak{Schemes} \to \mathfrak{Sets}.$$

If X is a projective scheme over $\mathbb{C}$ then there exists a projective scheme $\overline{M}_{g,n}(X,\beta)$ coarsely representing the functor $\overline{\mathcal{M}}_{g,n}(X,\beta)$, [FP, Theorem 1]. The spaces $\overline{M}_{g,n}(X,\beta)$ are called *moduli spaces of stable maps*, or *Kontsevich moduli spaces*.
Recall that a smooth variety X is said to be *convex* if $H^1(\mathbb{P}^1, \alpha^*T_X) = 0$ for any morphism $\alpha : \mathbb{P}^1 \to X$.

Remark 1.4. The tangent bundle of an homogeneous variety is generated by global section, so it is convex. On the other hand to be convex for an uniruled variety is a strong condition, as instance the blow-up of a convex variety is not convex.

Let X be a projective, nonsingular, convex variety, then $\overline{M}_{0,n}(X,\beta)$ is a normal, projective variety of pure dimension

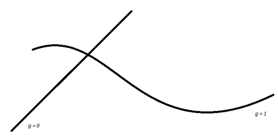
$$\dim(X) + \int_\beta c_1(T_X) + n - 3.$$

Furthermore $\overline{M}_{0,n}(X,\beta)$ is locally a quotient of a nonsingular variety by a finite group, that is $\overline{M}_{0,n}(X,\beta)$ has at most finite quotient singularities, [FP, Theorem 2].
In the special case $X = \mathbb{P}^N$ we have $\beta \sim d[\text{line}]$ for some integer d and the scheme $\overline{M}_{0,n}(\mathbb{P}^N, d)$ is irreducible.

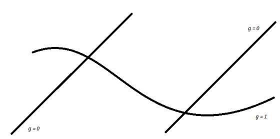
Examples. In the following we give a list of examples in which moduli of stable maps have a clear geometric description.

- The moduli space of stable maps to a point is isomorphic to the moduli space of curves
$$\overline{M}_{g,n}(\mathbb{P}^0, 0) \cong \overline{M}_{g,n}.$$
For the space of degree zero stable maps we have
$$\overline{M}_{g,n}(X, 0) \cong \overline{M}_{g,n} \times X.$$
- The moduli space of degree one maps to $\mathbb{P}^N$ is the Grassmannian
$$\overline{M}_{0,0}(\mathbb{P}^N, 1) \cong \mathbb{G}(1, N),$$
and similarly the moduli space of degree one maps to a smooth quadric hypersurface $Q \subset \mathbb{P}^N$, with $N \geqslant 3$, is the orthogonal Grassmannian
$$\overline{M}_{0,0}(Q, 1) \cong \mathbb{OG}(1, N).$$

- The Kontsevich moduli space $\overline{M}_{0,0}(\mathbb{P}^2, 2)$ is isomorphic to the space of complete conics that is to the blow up of the $\mathbb{P}^5$ parametrizing conics in $\mathbb{P}^2$ along the Veronese surface V of double lines
$$\overline{M}_{0,0}(\mathbb{P}^2, 2) \cong Bl_V\mathbb{P}^5.$$
- Consider now $\overline{M}_{1,0}(\mathbb{P}^2, 3)$. Smooth plane cubic are parametrized by an open subset of $\mathbb{P}^9 = \mathbb{P}(k[x_0, x_1, x_2]_3)$. On the other hand we have maps from a reducible curve with a component of genus zero and a component of genus one, contracting the genus one component and of degree three on the genus zero component.



For any curve of genus one we have a 1-dimensional choice for the genus zero component, namely the connecting node. So we get a component of dimension 10 of $\overline{M}_{1,0}(\mathbb{P}^2, 3)$. Finally we have curve with three component: an elliptic curve and two rational tails. The map contracts the elliptic curve and maps the rational tails to a line and a conic.



Here we have a 2-dimensional choice for the two nodes on the elliptic curve, a 2-dimensional choice for the line, and a 5-dimensional choice for the conic. We conclude that $\overline{M}_{1,0}(\mathbb{P}^2, 3)$ has three irreducible component: two of dimension 9 and one of dimension 10.
- Let $X \subset \mathbb{P}^7$ be a smooth degree seven hypersurface containing a $\mathbb{P}^3$. Writing down an explicit equation for X one can see that $\overline{M}_{0,0}(X, 2)$ has two irreducible component: one component is 5-dimensional and cover X, the second component parametrizes conics in the $\mathbb{P}^3$ and so has dimension $5 + 3 = 8$.
Generalizing this construction one can show that $\overline{M}_{0,0}(X, 2)$ can have a component of dimension arbitrary larger that the dimension of the main component even if X is a Fano hypersurface in $\mathbb{P}^N$.

Natural maps. Kontsevich moduli spaces, as moduli spaces of curves, admits natural morphisms.

- Forgetful morphisms
$$\pi_I : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n-||I|}(X, \beta),$$
forgetting the the points marked by $I = i_1, ..., i_j$ for $h \leqslant n$.
- Evaluation morphisms
$$ev_i : \overline{M}_{g,n}(X, \beta) \to X,$$
mapping $(C, \{x_1, ..., x_n, \alpha\})$ to $\alpha(x_i)$.
- If $2g + n - 3 \geqslant 0$ we have morphisms forgetting the map α,
$$\rho : \overline{M}_{g,n}(X, \beta) \to \overline{M}_{g,n}.$$

1.1. **The stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$.** In this section we follow the clear and detailed discussion worked out by *F. Poma* in [Po]. The construction of the moduli of stable maps can be transposed into the realm of algebraic stacks. Let k be a field. Consider the functor

$$\mathcal{F} : \mathfrak{Schemes}_{/k} \rightarrow \mathfrak{Groupoids},$$

associating to a scheme S the groupoids $\mathcal{F}(S)$ of flat projective families $\pi : C \rightarrow S$ of nodal curves of genus g,

$$\begin{tikzcd} C \arrow[r, "\alpha"] \arrow[d, "\pi"] & X \\ S \arrow[u, bend left, "s_i"] & \end{tikzcd}$$

where s_i are disjoint smooth sections of π, $\alpha_*[C_s] = \beta$ for any fiber $C_s = \pi^{-1}(s)$, and $\mathrm{Aut}(C,\alpha,\pi,s_i)$ is finite over S.

Theorem 1.5. *(Abramovich-Oort '01) There exists a proper algebraic stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of finite type over k which represents $\mathcal{F}$.*

Theorem 1.6. *(Kontsevich '95, Behrend-Fantechi '97) If $\mathrm{char}\, k = 0$, then $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is of Deligne-Mumford type.*

Recall that a *Dedekind domain* D is an integral domain which is not a field, satisfying one of the following equivalent conditions:

- D is noetherian, and the localization at each maximal ideal is a Discrete Valuation Ring.
- D is an integrally closed, noetherian domain with Krull dimension one.
- Every nonzero proper ideal of D factors into primes ideals.
- Every fractional ideal of D is invertible.

Example 1.7. Let C be an affine smooth curve over a field k. The coordinate ring $A(C)$ of C is a finitely generated k-algebra, and so noetherian, it has dimension one since C is a curve, one and being C smooth and so normal means that $A(C)$ is integrally closed. So $A(C)$ is a Dedekind domain.

Consider now the functor

$$\mathcal{F}_D : \mathfrak{Schemes}_{/D} \rightarrow \mathfrak{Groupoids},$$

exactly defined as $\mathcal{F}$ but from the category of schemes over a Dedekind domain D.

Theorem 1.8. *(Abramovich-Oort '01) There exists a proper algebraic stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ of finite type over D which represents $\mathcal{F}_D$.*

In the case $\mathrm{char}\, k = p$, in general $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a proper *Artin stack*. As instance consider the element $(\mathbb{P}^1,\alpha) \in \overline{\mathcal{M}}_{0,0}(\mathbb{P}^1,p)$ given by

$$\alpha : \mathbb{P}^1 \rightarrow \mathbb{P}^1,\ [x_0,x_1] \mapsto [x_0^p, x_1^p].$$

Then $\mathrm{Aut}(\mathbb{P}^1,\alpha) = \mu_p = \mathrm{Spec}\, k[\xi]/(\xi^p - 1) = \mathrm{Spec}\, k[\xi]/(\xi - 1)^p$, which is not reduced over $\mathrm{Spec}\, k$. However even in the characteristic p case the stack $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is a global quotient stack and the functor

$$\theta : \overline{\mathcal{M}}_{g,n}(X,\beta) \rightarrow \mathfrak{M}_{g,n}$$

is representable. This led *A. Kresch* to define an intersection theory for Artin stacks over a field [Kr].

Recall that a *ring of mixed characteristic* is a commutative ring R having characteristic zero, having an ideal I such that R/I has positive characteristic. As instance the ring of integers $\mathbb{Z}$ have characteristic zero, and for any prime number p, $\mathbb{Z}/(p)$ is a finite field of characteristic p.

Recently *F. Poma* in [Po] extended the construction of the virtual fundamental class of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ in [BF] to schemes in positive and mixed characteristic. This lead to a rigorous definition of Gromov-Witten invariants for these classes of schemes.

1.2. **Virtual dimension of $\overline{M}_{g,n}(X,\beta)$.** If X is a homogeneous variety then it is smooth and its tangent bundle is generated by global sections, in particular X is convex. In this case $\overline{M}_{0,n}(X,\beta)$ is a normal, projective variety of pure dimension. Furthermore if $X=\mathbb{P}^N$ then $\overline{M}_{0,n}(\mathbb{P}^N,d)$ is irreducible. On the other hand when $g\geqslant 1$, and even when $g=0$ for most schemes $X\neq\mathbb{P}^N$ the space $\overline{M}_{g,n}(X,\beta)$ may have many components of dimension greater that the expected dimension. To overcome this gap and to give a rigorous definition of Gromov-Witten invariants we have to introduce the notions of *virtual fundamental class* and *virtual dimension*.

1.2.1. *The normal cone.* In this section we follow [BF]. Let E be a rank r vector bundle on a smooth variety Y, $s\in H^0(E)$ a section, and $Z=Z(s)\subset Y$ the zero scheme of s. As s varies Z can become reducible or even of non pure dimension. Let $\mathcal{I}$ be the ideal sheaf of Z in Y, the *normal cone* of Z in Y is the affine cone over Z defined by

$$C_ZY=\mathrm{Spec}(\bigoplus_{k=0}^{\infty}\mathcal{I}^k/\mathcal{I}^{k+1}).$$

Note that the C_ZY has pure dimension $n=\dim Y$. Multiplication by s induces a surjective map

$$\bigoplus_k Sym^k(\mathcal{O}(E^*/\mathcal{I}\mathcal{O}(E^*)))\rightarrow\bigoplus_k\mathcal{I}^k/\mathcal{I}^{k+1},$$

and applying Spec we get an embedding

$$C_ZY\rightarrow E_{|Z}.$$

The normal cone gives a class $[C_ZY]\in A_n(E_{|Z})$, so we have $s^*[C_ZY]\in A_{n-r}(Z)$.
Let $\mathcal{M}$ be a Deligne-Mumford stack. Since $\mathcal{M}$ admits an étale open cover by schemes we can consider a scheme U and take an embedding $U\hookrightarrow W$, where W is a smooth scheme. Now, consider the ideal sheaf $\mathcal{I}$ of U in W, and form the normal cone C_UW. The differentiation map

$$\bigoplus_k\mathcal{I}^k\rightarrow\Omega^1_W,\ f\mapsto df$$

induces a map

$$\bigoplus_k\mathcal{I}^k/\mathcal{I}^{k+1}\rightarrow\bigoplus_k Sym^k(\Omega^1_W/\mathcal{I}\Omega^1_W),$$

finally applying Spec we get a map

$$T_{W|U}=\mathrm{Spec}(\bigoplus_k Sym^k(\Omega^1_W/\mathcal{I}\Omega^1_W))\rightarrow C_UW.$$

The intrinsic normal cone $\mathcal{C}_U$ is defined as the stack quotient $[C_UW/T_{W|U}]$. Now, given an étale open cover $\{U_i\}$ of $\mathcal{M}$ the intrinsic normal cones C_{U_i} glue to give the intrinsic normal cone $\mathcal{C}_{\mathcal{M}}$ of $\mathcal{M}$.
If $L^{\bullet}_{\mathcal{M}}$ is the *cotangent complex* of $\mathcal{M}$, an *obstruction theory* for $\mathcal{M}$ is a complex of sheaves $\mathcal{E}^{\bullet}$ on $\mathcal{M}$ with a morphism $\mathcal{E}^{\bullet}\rightarrow L^{\bullet}_{\mathcal{M}}$, which is an isomorphism on h^0 and a surjection on h^{-1}.
Given an arbitrary complex $\mathcal{E}^{\bullet}$ we define $h^1/h^0(\mathcal{E}^{\bullet})$ to be the quotient stack of the kernel of $\mathcal{E}^1\rightarrow\mathcal{E}^2$ by the cokernel of $\mathcal{E}^{-1}\rightarrow\mathcal{E}^0$.
By the definition of perfect obstruction theory the intrinsic normal cone $\mathcal{C}_{\mathcal{M}}$ embeds in $h^1/h^0((\mathcal{E}^{\bullet})^*)$. Let C be the fiber product of $(E^{-1})^*$ with $\mathcal{C}_{\mathcal{M}}$ over $h^1/h^0((\mathcal{E}^{\bullet})^*)$, where $\mathcal{O}(E^{-1})=\mathcal{E}^{-1}$. This is a cone contained in the vector bundle $(E^{-1})^*$. The *virtual fundamental class* is defined to be the intersection of C with the zero section of $(E^{-1})^*$.

In this part we mainly follow [De] and [Po]. Let X be a smooth connected projective scheme, $\mathfrak{M}_{g,n}$ the Artin stack parametrizing pre-stable n-pointed genus g connected nodal curves, and C its universal curve. We define an algebraic stack $Mor(C,X)$ as follows:

- for any scheme S objects in $Mor(C,X)(S)$ are pre-stable curves $(C_S \to S, s_i)$ over S with a morphism $f_S : C_S \to X$,
- for any scheme S a morphism from $(C_S \to S, s_i)$ to $(C_S^{'} \to S, s_i^{'})$ is an isomorphism α of pre-stable curves such that $f_S^{'} \circ \alpha = f_S$.

There is a natural functor $\theta : Mor(C,X) \to \mathfrak{M}_{g,n}$ forgetting the map to X, furthermore $\overline{\mathcal{M}}_{g,n}(X,\beta)$ is an open substack of $Mor(C,X)$. The fiber product $\overline{\mathcal{C}} \times_{\mathfrak{M}_{g,n}} Mor(C,X)$ is an universal family for $Mor(C,X)$ and we have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[rr, bend left, "\psi"] \arrow[d, "\pi"] & \overline{\mathcal{C}} \arrow[r, "\overline{\psi}"'] \arrow[d, "\overline{\pi}"] & X \\
\overline{\mathcal{M}}_{g,n}(X,\beta) \arrow[r] \arrow[u, bend left, "s_i"] & Mor(C,X) \arrow[u, bend left, "\overline{s}_i"] &
\end{tikzcd}$$

where $\mathcal{C} = \overline{\mathcal{C}} \times_{Mor(C,X)} \overline{\mathcal{M}}_{g,n}(X,\beta)$ is the universal stable map.
It turns out that considering the complex $F^{\bullet} = (R\overline{\pi}_* \overline{\psi}^* T_X)^*$ we get a vector bundle stack $h^1/h^0(F^{\bullet})$. Similarly $E^{\bullet} = (R\pi_* \psi^* T_X)^*$ gives a perfect obstruction theory for θ, and so a virtual fundamental class for $\overline{\mathcal{M}}_{g,n}(X,\beta)$.

In what follows we try to understand more concretely the tangent and the obstruction spaces to $Mor(Y,X)$, where X,Y are projective varieties over a field. The scheme $Mor(Y,X)$ of parametrizing morphisms $Y \to X$ is a locally noetherian scheme having countably many components. However fixing an ample divisor H on X we can consider the scheme $Mor(P)(Y,X)$ parametrizing morphism $Y \to X$ with fixed Hilbert polynomial $P(m) = \chi(Y, mf^*H)$. This is a quasi-projective scheme.
The tangent space $T_{[f]}Mor(Y,X)$ in a point $[f] \in Mor(Y,X)$ parametrizes morphisms $\mathrm{Spec}\, k[\epsilon]/(\epsilon^2) \to Mor(Y,X)$, and hence $k[\epsilon]/(\epsilon^2)$-morphisms

$$f_\epsilon : Y \times \mathrm{Spec}\, k[\epsilon]/(\epsilon^2) \to X \times \mathrm{Spec}\, k[\epsilon]/(\epsilon^2),$$

which should be interpreted as first order deformations of f.

Proposition 1.9. *Let X,Y be projective varieties. The tangent space to $Mor(Y,X)$ in a point $[f]$ is given by*

$$T_{[f]}Mor(Y,X) = H^0(Y, \mathcal{H}om(f^*\Omega_X, \mathcal{O}_Y)).$$

Proof. Assume $X = \mathrm{Spec}(A), Y = \mathrm{Spec}(B)$ to be affine, where A,B are finitely generated k-algebras. Let $f^\sharp : A \to B$ be the morphism induced by f. We are looking for $k[\epsilon]/(\epsilon^2)$-algebras homomorphisms $f_\epsilon^\sharp : A[\epsilon] \to B[\epsilon]$ of the type $f_\epsilon^\sharp(a) = f^\sharp(a) + \epsilon g(a)$. Notice that the since $f_\epsilon^\sharp(aa^{'}) = f_\epsilon^\sharp(a) f_\epsilon^\sharp(a^{'})$ we get $\epsilon g(aa^{'}) = (f^\sharp(a) + \epsilon g(a))(f^\sharp(a^{'}) + \epsilon g(a^{'})) - f^\sharp(a) f^\sharp(a^{'}) = \epsilon(f^\sharp(a) g(a^{'}) + f^\sharp(a^{'}) g(a))$. Then $f_\epsilon^\sharp(aa^{'}) = f_\epsilon^\sharp(a) f_\epsilon^\sharp(a^{'})$ is equivalent to

$$g(aa^{'}) = f^\sharp(a) g(a^{'}) + f^\sharp(a^{'}) g(a),$$

that is $g : A \to B$ is a k-derivation of the A-module B and then it has to factorize as $g : A \to \Omega_A \to B$. Such extensions are therefore parametrized by $Hom_A(\Omega_A, B) = Hom_B(\Omega_A \otimes_A B, B)$.
In general cover X by open affine $U_i = \mathrm{Spec}(A_i)$ and Y by open affine $V_i = \mathrm{Spec}(B_i)$ such that

$f(V_i) \subseteq U_i$. By the previous part of the proof first order deformations of $f_{|V_i}$ are parametrized by $h_i \in Hom_{B_i}(\Omega_{A_i} \otimes_{A_i} B_i, B_i) = H^0(V_i, \mathcal{H}om(f^*\Omega_X, \mathcal{O}_Y))$. To glue these together we need the compatibility condition $h_{i|V_{ij}} = h_{j|V_{ij}}$ which means that the collection $\{h_i\}$ defines a global section on Y. □

Notice that when X is smooth along the image of f we have

$$T_{[f]}Mor(Y,X) = H^0(Y, f^*T_X).$$

Furthermore when Y is smooth $H^0(Y,T_Y)$ is the tangent space to the automorphisms group of Y at the identity, its element are called infinitesimal automorphisms. The images of the morphism $H^0(Y,T_Y) \rightarrow H^0(Y, f^*T_X)$ parametrizes deformation of f by reparametrizations.

Let $0 \mapsto I \rightarrow R \rightarrow R/I \mapsto 0$ be a semi-small extension in the category of local artinian k-algebras. That is $I \subseteq \mathfrak{M}$ and $I\mathfrak{M} = 0$, where $\mathfrak{M}$ is the maximal ideal of R. Let $f: Y \rightarrow X$ be a morphism. Assume as before X,Y affine. Since X is smooth along the image of f and $I^2 = 0$ by the infinitesimal lifting property [Ha, Exercise 8.6 - Chap 2], there exists a lifting of $f_{R/I}^{\sharp} : A \otimes_k R/I \rightarrow B \otimes_k R/I$ to a morphism $f_R^{\sharp} : A \otimes_k R \rightarrow B \otimes_k R$, and two different liftings differ by an R-derivation $A \otimes_k R \rightarrow B \otimes_k I$, that is by an element of $H^0(Y, f^*T_X) \otimes_k I$.
In the general case we need to glue two extensions h_i, h_j on each $V_i \cap V_j$. These two extension differs by an element $\nu_{ij} \in H^0(V_i \cap V_j, f^*T_X) \otimes_k I$. We have $\nu_{ij} h_{i|V_{ij}} = h_{j|V_{ij}}$. On the triple intersection $V_i \cap V_j \cap V_k$ we have $\nu_{jk}\nu_{ij}h_{i|V_{ijk}} = \nu_{jk}h_{j|V_{ijk}} = h_{k|V_{ijk}} = \nu_{ik}h_{i|V_{ijk}}$. So $\nu_{ik} = \nu_{jk}\nu_{ij}$ and the collection $\{\nu_{ij}\} \in C^1(\{V_i\}, f^*T_X \otimes_k I)$ is a cocycle. We have a global lifting if and only if $\nu_{ij} = 0$, and the *obstruction space* is $H^1(Y, f^*T_X) \otimes I$.

Locally around a point $[f] \in Mor(Y,X)$ the space $Mor(Y,X)$ can be defined by a set of polynomial $\{P_i\}$ is some affine space $\mathbb{A}^N$. The rank r of the Jacobian $J(P_i)$ is the codimension of the Zariski tangent space $T_{[f]}Mor(Y,X) \subseteq k^N$. Let V be a variety defined by r equations among the P_i for which the corresponding rows in the Jacobian have rank r, then V is smooth at $[f]$ and has the same Zariski tangent space of $Mor(Y,X)$. By 1.9 the variety V has dimension $h^0(Y, f^*T_X)$ in $[f]$. We want to show that in the regular local ring $R = \mathcal{O}_{V,[f]}$ the ideal I of regular functions vanishing on $Mor(Y,X)$ can be generated by $h^1(Y, f^*T_X)$ elements.
Since the Zariski tangent spaces are the same the ideal I is contained in the square of the maximal ideal $\mathfrak{M}$ of R. Furthermore by Nakayama's lemma it is enough to show that the k-vector space $I/\mathfrak{M}I$ has dimension at most h^1.
The morphism $\mathrm{Spec}(R/I) \rightarrow Mor(Y,X)$ corresponds to an extension $f_{R/I} : Y \times \mathrm{Spec}(R/I) \rightarrow X \times \mathrm{Spec}(R/I)$ of f. We know that the obstruction to lift this extension to an extension $f_{R/\mathfrak{M}I}$: $Y \times \mathrm{Spec}(R/\mathfrak{M}I) \rightarrow X \times \mathrm{Spec}(R/\mathfrak{M}I)$ lies in

$$H^1(Y, f^*T_X) \otimes_k I/\mathfrak{M}I.$$

Let $\sum_{i=1}^{h_1} a_i \otimes \overline{b}_i$ be the obstruction, where $b_i \in I$. Since the obstruction vanishes modulo the ideal $(b_1, ..., b_{h^1})$ the morphism $\mathrm{Spec}(R/I) \rightarrow Mor(Y,X)$ lifts to a morphism $\mathrm{Spec}(R/\mathfrak{M}I + (b_1, ..., b_{h^1})) \rightarrow Mor(Y,X)$. In other words the identity $R/I \rightarrow R/I$ factors through the projection as $R/I \rightarrow R/\mathfrak{M}I + (b_1, ..., b_{h^1}) \rightarrow R/I$. Then $I = \mathfrak{M}I + (b_1, ..., b_{h^1})$, which means that $I/\mathfrak{M}I$ is generated by the classes of $b_1, ..., b_{h^1}$.

Remark 1.10. Locally around $[f]$ the space $Mor(Y,X)$ can be defined by at most $h^1(Y, f^*T_X)$ equations in a smooth variety of dimension $h^0(Y, f^*T_X)$. In particular any irreducible component of $Mor(Y,X)$ through $[f]$ has dimension at least

$$h^0(Y, f^*T_X) - h^1(Y, f^*T_X).$$

The equations defining $Mor(Y,X)$ in a locally around $[f]$ can intersect badly so that the actual dimension is not the expected one. My naive way of understanding the deformation to the normal cone and the virtual fundamental class is to imagine a deformation of these equations that make the intersection transverse. If there is such a deformation, which formally means that exists a perfect obstruction theory, then the object we obtain would be a virtual fundamental class.

Spectral sequence of Ext functors. Let $\mathcal{E} \in \mathfrak{Coh}(X)$ be a coherent sheaf on a scheme X. Consider the functor

$$\mathcal{H}om(\mathcal{E},-) : \mathfrak{Coh}(X) \rightarrow \mathfrak{Coh}(X),\ \mathcal{Q} \mapsto \mathcal{H}om(\mathcal{E},\mathcal{Q}),$$

and the global section functor

$$\Gamma_X : \mathfrak{Coh}(X) \rightarrow \mathfrak{Ab},\ \mathcal{Q} \mapsto \Gamma_X(\mathcal{Q}).$$

Note that $\Gamma_X \circ \mathcal{H}om(\mathcal{E},-) = Hom(\mathcal{E},-)$. By Grothendieck spectral sequence we have $(R^h\Gamma_X \circ R^k\mathcal{H}om(\mathcal{E},-))(\mathcal{Q}) \Longrightarrow R^{h+k}(Hom(\mathcal{E},-)(\mathcal{Q})$ for any $\mathcal{Q} \in \mathfrak{Coh}(X)$, that is

$$H^h(X, \mathcal{E}xt^k(\mathcal{E},\mathcal{Q})) \Longrightarrow Ext^{h+k}(\mathcal{E},\mathcal{Q}).$$

The corresponding sequence of low degrees is

$$0 \mapsto H^1(X,\mathcal{H}om(\mathcal{E},\mathcal{Q})) \rightarrow Ext^1(\mathcal{E},\mathcal{Q}) \rightarrow H^0(X,\mathcal{E}xt^1(\mathcal{E},\mathcal{Q})) \rightarrow H^2(X,\mathcal{H}om(\mathcal{E},\mathcal{Q})) \rightarrow Ext^2(\mathcal{E},\mathcal{Q}).$$

Theorem 1.11. *Let X be a smooth projective variety. The virtual dimension of the moduli space $\overline{M}_{g,n}(X,\beta)$ is given by*

$$\mathrm{virdim}(\overline{M}_{g,n}(X,\beta)) = (1-g)(\dim(X)-3) - \int_\beta \omega_X + n.$$

Proof. Consider the stable map $(C,\{x_1,...,x_n\},\alpha\}) \in \overline{M}_{g,n}(X,\beta)$. Let $Def(C,\{x_1,...,x_n\},\alpha\})$ be the space of first order deformations of $(C,\{x_1,...,x_n\},\alpha\})$, and let $Def_\alpha(C,\{x_1,...,x_n\},\alpha\})$ be the space of first order deformations with C held rigid. There is an exact sequence

$$0 \mapsto Def(C,\{x_1,...,x_n\}) \rightarrow Def(C,\{x_1,...,x_n\},\alpha\}) \rightarrow Def_\alpha(C,\{x_1,...,x_n\},\alpha\}) \mapsto 0.$$

Note that since $(C,\{x_1,...,x_n\},\alpha\})$ is stable it does not have infinitesimal automorphisms, and this gives the injectivity of the map on the left.

- First we compute the dimension of $Def(C,\{x_1,...,x_n\})$. The curve C is a stable nodal curve. By 1.2.1 we have a sequence
$$0 \mapsto H^1(C,\mathcal{H}om(\Omega_C,\mathcal{O}_C)) \rightarrow Ext^1(\Omega_C,\mathcal{O}_C) \rightarrow H^0(C,\mathcal{E}xt^1(\Omega_C,\mathcal{O}_C)) \mapsto 0,$$
there being no H^2 on a curve. We denote by δ the number of nodes in C. Since the sheaf Ω_C is locally free on the smooth locus of C, the sheaf $\mathcal{E}xt^1(\Omega_C,\mathcal{O}_C))$ is just k at each node, then $\dim(H^0(C,\mathcal{E}xt^1(\Omega_C,\mathcal{O}_C))) = \delta$. The curve C is l.c.i, then the dualizing sheaf ω_C is an invertible sheaf, and since $\omega_C \cong \Omega_C$ on the open set of regular points, we have an injective morphism $\check{\omega_C} \rightarrow \mathcal{H}om(\Omega_C,\mathcal{O}_C)$, and an exact sequence
$$0 \mapsto \check{\omega_C} \rightarrow \mathcal{H}om(\Omega_C,\mathcal{O}_C) \rightarrow \mathcal{O}_Z \mapsto 0,$$
where $Z = Sing(C)$. Since C is stable $h^0(\mathcal{H}om(\Omega_C,\mathcal{O}_C)) = 0$, by the cohomology exact sequence we get $h^0(\check{\omega_C}) = 0$, and
$$0 \mapsto H^0(C,\mathcal{O}_Z) \rightarrow H^1(C,\check{\omega_C}) \rightarrow H^1(\mathcal{H}om(\Omega_C,\mathcal{O}_C)) \mapsto 0.$$
By Riemann-Roch for singular curves we get $h^1(\check{\omega_C}) = 3g-3$, and since $h^0(\mathcal{O}_Z) = \delta$ we get $h^1(\mathcal{H}om(\Omega_C,\mathcal{O}_C)) = 3g-3-\delta$. Finally
$$\dim(Ext^1(\Omega_C,\mathcal{O}_C)) = h^1(T_C) + h^0(\mathcal{E}xt^1(\Omega_C,\mathcal{O}_C)) = 3g-3-\delta+\delta = 3g-3.$$
and
$$\dim Def(C,\{x_1,...,x_n\}) = 3g-3+n.$$

- By Remark 1.10 the expected dimension of $Def_{\alpha}(C,\{x_1,...,x_n\},\alpha\})$ is $h^0(\alpha^*T_X)-h^1(\alpha^*T_C)$. By Riemann-Roch theorem we get

$$\operatorname{expdim} Def_{\alpha}(C,\{x_1,...,x_n\},\alpha\}) = h^0(\alpha^*T_X)-h^1(\alpha^*T_C) = \chi(\alpha^*T_C) = -K_X\cdot\alpha_*C+(1-g)\dim(X).$$

We conclude that

$$\operatorname{expdim} Def(C,\{x_1,...,x_n\},\alpha\}) \geqslant -K_X\cdot\alpha_*C+(1-g)\dim(X)+3g-3+n,$$

and the virtual dimension of $\overline{M}_{g,n}(X,\beta)$ is given by

$$\operatorname{virdim}(\overline{M}_{g,n}(X,\beta)) = -K_X\cdot\alpha_*C+(1-g)\dim(X)+3g-3+n = (1-g)(\dim(X)-3)-\int_{\beta}\omega_X+n.$$

□

2. Gromov-Witten Invariants

Let X be a projective variety, $\beta\in H_2(X,\mathbb{Z})$ be a homology class, and $Z_1,...,Z_n\subset X$ cycles in general position. We want to study the following set of curves

$$\{C\subset X \text{ of genus } g,\ \text{homology } \beta,\ \text{and } C\cap Z_i\neq\emptyset \text{ for any } i\}.$$

In [Ko] *M. Kontsevich* observed that the curve $C\subset X$ should be replaced by a pointed curve $(C,\{x_1,...,x_n\})$ and a holomorphic map $f:C\rightarrow X$ such that $f(x_i)\in Z_i$ for any $i=1,...,n$. The idea is that Gromov-Witten classes should give a subset of $\overline{M}_{g,n}$ which in turn gives a cohomology class in $H^*(\overline{M}_{g,n},\mathbb{Q})$. Here we use rational cohomology because $\overline{M}_{g,n}$ exists as a smooth Deligne-Mumford stack that is as a smooth orbifold when $2g+n-3\geqslant 0$.
Let $\alpha_1,...,\alpha_n\in H^*(X,\mathbb{Q})$ be the cohomology classes dual to $Z_1,...,Z_n$. The *Gromov-Witten class*

$$I_{g,n,\beta}(\alpha_1,...,\alpha_n)\in H^*(\overline{M}_{g,n},\mathbb{Q})$$

is supposed to be the cohomology class represented by the set of pointed curves

$$\{(C,\{x_1,...,x_n\}) \text{ genus } g,\ \text{homology } \beta,\ \text{and } f(x_i)\in Z_i \text{ for any } i\}. \tag{2.1}$$

From this point of view Gromov-Witten classes are a system of maps

$$I_{g,n,\beta}: H^*(X,\mathbb{Q})^{\otimes n}\rightarrow H^*(\overline{M}_{g,n},\mathbb{Q}).$$

When $I_{g,n,\beta}(\alpha_1,...,\alpha_n)$ has a component of top degree in $H^*(\overline{M}_{g,n},\mathbb{Q})$ we define a *Gromov-Witten invariant* as

$$\langle I_{g,n,\beta}\rangle(\alpha_1,...,\alpha_n) = \int_{\overline{M}_{g,n}} I_{g,n,\beta}(\alpha_1,...,\alpha_n).$$

Intuitively in this case 2.1 should consist of finitely many curves and $\langle I_{g,n,\beta}\rangle(\alpha_1,...,\alpha_n)$ would be the number of such curves. Nevertheless Gromov-Witten invariants may be rational or even negative, so their enumerative meaning is not always straightforward.

2.1. **Rigorous definition of Gromov-Witten invariants.**

The are two construction of the virtual fundamental class: one given by *J. Li* and *G. Tian* in [LT1], and the other by *K. Behrend*, *B. Fantechi* in [BF]. Even in the algebraic setting there are two definitions of Gromov-Witten invariants. However it turns out that these two construction are equivalent. In the symplectic case these invariants were defined by *J. Li* and *G. Tian* in [LT2].
Now we have the virtual fundamental class $\xi=[\overline{M}_{g,n}(X,\beta)]^{vir}$, and we can give a rigorous definition of Gromov-Witten classes and invariants. Consider the following maps:

$$\begin{tikzcd}
\overline{M}_{g,n}(X,\beta) \arrow[r, "\pi:=ev_1\times\ldots\times ev_n\times\rho"] & X^n\times\overline{M}_{g,n} \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
X^n & & \overline{M}_{g,n}
\end{tikzcd}$$

Definition 2.1. Let $\beta \in H^2(X,\mathbb{Z})$ be a homology class and $\alpha_1, ..., \alpha_n \in H^*(X,\mathbb{Q})$ be cohomology classes.

- If $2g+n-3 \geqslant 0$, the Gromov-Witten class $I_{g,n,\beta}(\alpha_1, ..., \alpha_n) \in H^*(\overline{M}_{g,n},\mathbb{Q})$ is defined by
$$I_{g,n,\beta}(\alpha_1, ..., \alpha_n) = PD^{-1}p_{2*}(p_1^*(\alpha_1 \otimes ... \otimes \alpha_2) \cap \pi_*(\xi)),$$
where PD is the Poincaré duality.
- If $n, g \geqslant 0$, the Gromov-Witten invariant $\langle I_{g,n,\beta}\rangle(\alpha_1, ..., \alpha_n)$ is the rational number defined by
$$\langle I_{g,n,\beta}\rangle(\alpha_1, ..., \alpha_n) = \int_\xi ev_1^*(\alpha_1) \cup ... \cup ev_n^*(\alpha_n).$$

Remark 2.2. The number $\langle I_{g,n,\beta}\rangle(\alpha_1, ..., \alpha_n)$ would be zero when $ev_1^*(\alpha_1) \cup ... \cup ev_n^*(\alpha_n)$ does not have a component of top degree. If $2g+n-3 \geqslant 0$ one can show that
$$\langle I_{g,n,\beta}\rangle(\alpha_1, ..., \alpha_n) = \int_{\overline{M}_{g,n}} I_{g,n,\beta}(\alpha_1, ..., \alpha_n).$$
So Gromov-Witten invariants are determined by the corresponding Gromov-Witten classes when $2g+n-3 \geqslant 0$.

2.2. **Axioms of Gromov-Witten classes.** *M. Kontsevich* and *Y. Manin* proposed in [KM] a system of axioms for Gromov-Witten classes. It is known that Gromov-Witten classes satisfy these axioms both in the algebraic and in the symplectic setting. In the algebraic case this is proved in [BM], [LT1], [LT2], [BF], [Be]. A proof for the symplectic case can be found in [LT2].
If X is a smooth projective variety, $g, n \geqslant 0$ and $2g+n-3 \geqslant 0$ we defined Gromov-Witten classes as maps
$$I_{g,n,\beta} : H^*(X,\mathbb{Q})^{\otimes n} \to H^*(\overline{M}_{g,n},\mathbb{Q}).$$
For $g, n \geqslant 0$ Gromov-Witten invariants are maps
$$\langle I_{g,n,\beta}\rangle : H^*(X,\mathbb{Q})^{\otimes n} \to \mathbb{Q},$$
and when $2g+n-3 \geqslant 0$ these are related by
$$\langle I_{g,n,\beta}\rangle(\alpha_1, ..., \alpha_n) = \int_{\overline{M}_{g,n}} I_{g,n,\beta}(\alpha_1, ..., \alpha_n).$$
The Gromov-Witten classes axioms are the following.

- **Linearity axiom**. The class $I_{g,n,\beta}$ is linear in each variable. Intuitively this is because a sum of cycles is their union.
- **Effectivity axiom**. On a smooth projective variety X, $I_{g,n,\beta} = 0$ if β is not effective. This is because $f_*[C]$ is effective if $f : C \to X$ is a holomorphic map.
- **Degree axiom**. If $\alpha_1, ..., \alpha_n \in H^*(X,\mathbb{Q})$ are homogeneous classes, the cohomology class $I_{g,n,\beta}(\alpha_1, ..., \alpha_n) \in H^*(\overline{M}_{g,n},\mathbb{Q})$ has degree
$$2(g-1)\dim X + 2\int_\beta \omega_X + \sum_{i=1}^n \deg \alpha_i.$$
Since the virtual fundamental class has the expected dimension this fact is a consequence of the definition. Infact by Poincaré duality
$$PD : H^*(X,\mathbb{Q}) \to H_{2\dim X - *}(X,\mathbb{Q})$$
the cohomology class α_i corresponds to a cycle Z_i of codimension $\frac{\deg \alpha_i}{2}$. So we are cutting the push-forward of the virtual fundamental class on $X^n \times \overline{M}_{g,n}$ with $\frac{1}{2}\sum_{i=1}^n \deg \alpha_i$

equations, and pushing-forward to $\overline{M}_{g,n}$ we get a cycle of dimension $(1-g)(\dim X-3)-\int_\beta \omega_X + n - \frac{1}{2}\sum_{i=1}^n \deg\alpha_i$. Now, by Poincaré duality on $\overline{M}_{g,n}$

$$PD : H^*(\overline{M}_{g,n},\mathbb{Q}) \to H_{2\dim \overline{M}_{g,n}-*}(\overline{M}_{g,n},\mathbb{Q})$$

we get

$$\dim \overline{M}_{g,n} - \frac{\deg I_{g,n,\beta}}{2} = (1-g)\dim X - \int_\beta \omega_X - \frac{1}{2}\sum_{i=1}^n \deg\alpha_i + \dim \overline{M}_{g,n},$$

that is

$$\deg I_{g,n,\beta} = 2(g-1)\dim X + 2\int_\beta \omega_X + \sum_{i=1}^n \deg\alpha_i.$$

The degree axiom implies that $I_{g,n,\beta}(\alpha_1,...,\alpha_n)$ is a top degree class if and only if

$$\sum_{i=1}^n \deg\alpha_i = 2(1-g)\dim X - 2\int_\beta \omega_X + 2\dim \overline{M}_{g,n}. \tag{2.2}$$

- **Equivariance axiom**. The symmetric group S_n acts on both $H^*(X,\mathbb{Q})^{\otimes n}$ and $H^*(\overline{M}_{g,n},\mathbb{Q})$, where the latter action corresponds to permuting the marked points on the curves. This axiom asserts that the map
$$I_{g,n,\beta} : H^*(X,\mathbb{Q})^{\otimes n} \to H^*(\overline{M}_{g,n},\mathbb{Q})$$
is S_n equivariant.
- **Fundamental class axiom**. When $2g+n-4 \geqslant 0$ we have a forgetful map $\pi_n : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$. Let $[X] \in H^0(X,\mathbb{Q})$ be the fundamental class of X, the axiom asserts that
$$I_{g,n,\beta}(\alpha_1,...,\alpha_{n-1},[X]) = \pi_n^* I_{g,n-1,\beta}(\alpha_1,...,\alpha_{n-1}).$$
This make sense because $f(p_n) \in X$ does not give any condition on p_n.

Remark 2.3. This axiom implies that $\langle I_{g,n,\beta}\rangle(\alpha_1,...,\alpha_{n-1},[X]) = 0$. Infact $\langle I_{g,n,\beta}\rangle(\alpha_1,...,\alpha_{n-1},[X])$ is nonzero only if $I_{g,n,\beta}(\alpha_1,...,\alpha_{n-1},[X])$ is a class of top degree, but in this case $I_{g,n-1,\beta}(\alpha_1,...,\alpha_{n-1})$ is zero, since $\overline{M}_{g,n-1}$ has smaller dimension.

- **Divisor axiom**. We consider again the case $2g+n-4 \geqslant 0$ and the forgetful map $\pi_n : \overline{M}_{g,n} \to \overline{M}_{g,n-1}$. If $\alpha_n \in H^2(X,\mathbb{Q})$, then
$$\pi_{n*} I_{g,n,\beta}(\alpha_1,...,\alpha_{n-1},\alpha_n) = \left(\int_\beta \alpha_n\right) I_{g,n-1,\beta}(\alpha_1,...,\alpha_{n-1}).$$
To understand the meaning of this axiom consider a stable map $f : (C,\{x_1,...,x_n\}) \to X$ such that $f_*[C] = \beta$ and $f(x_i) \in Z_i$ for any $i = 1,...,n-1$. The last point $f(p_n)$ must lie in $f(C)\cap Z_n = \beta \cap Z_n$, and this means that there are $\int_\beta \alpha_n$ possible choices for $f(p_n)$. This formula reflects on Gromov-Witten invariants giving the following:
$$\pi_{n*}\langle I_{g,n,\beta}\rangle(\alpha_1,...,\alpha_{n-1},\alpha_n) = \left(\int_\beta \alpha_n\right)\langle I_{g,n-1,\beta}\rangle(\alpha_1,...,\alpha_{n-1}).$$
- **Point mapping axiom**. If $\beta = 0$ and $g = 0$, this axiom states that given $\alpha_1,...\alpha_n$ homogeneous cohomology classes, then
$$I_{0,n,0}(\alpha_1,...,\alpha_n) = \begin{cases} \left(\int_X \alpha_1 \cup ... \cup \alpha_n\right)[\overline{M}_{g,n}] & \text{if } \sum_{\mathrm{i=1}}^{\mathrm{n}} \deg\alpha_{\mathrm{i}} = 2\dim \mathrm{X}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that a map satisfying $f_*[C] = 0$ is constant. Then we have $f(C) \in Z_1 \cap ... \cap Z_n$. By the degree axiom when $\sum_{i=1}^n \deg\alpha_i = 2\dim X$ this class has degree zero and gives the point mapping formula. Comparing the condition $\sum_{i=1}^n \deg\alpha_i = 2\dim X$ to 2.2 we get $g\dim X = 2\int_\beta \omega_X + 3g - 3 + n$ and substituting $g = \beta = 0$ gives $n = 3$. So for the Gromov-Witten invariants we have the following special behavior

$$\langle I_{0,n,0}\rangle(\alpha_1, ..., \alpha_n) = \begin{cases} \int_X \alpha_1 \cup \alpha_2 \cup \alpha_3 & \text{if } n = 3, \\ 0 & \text{otherwise.} \end{cases}$$

This reasoning works only when $g = 0$. The point is that the expected dimension of $\overline{M}_{g,n}(X, 0) = \overline{M}_{g,n} \times X$ is $(1-g)(\dim X - 3) + n$, and it coincides with the actual dimension $3g - 3 + n + \dim X$ if and only if $g = 0$.

- **Splitting axiom**. Consider two curves $(C_1, \{x_1, ..., x_{n_1+1}\})$ and $(C_2, \{y_1, ..., y_{n_2+1}\})$ of genus g_1, g_2, such that $g_1 + g_2 = g$, $n_1 + n_2 = n$ and $2g_i + n_i \geqslant 2$ for any i. We obtain a curve $(C = C_1 \cup C_2, \{x_1, ..., x_{n_1}, y_1, ..., y_{n_2}\})$ by identifying x_{n_1+1} with y_{n_2+1}. This give a map
$$\psi : \overline{M}_{g_1,n_1+1} \times \overline{M}_{g_2,n_2+1} \to \overline{M}_{g,n}.$$
The splitting axiom asserts that $\psi^* I_{g,n,\beta}(\alpha_1, ..., \alpha_n)$ is given by
$$\sum_{\beta=\beta_1+\beta_2} \sum_{i,j} g^{ij} I_{g_1,n_1+1,\beta_1}(\alpha_1, ..., \alpha_{n_1}, T_i) \otimes I_{g_2,n_2+1,\beta_2}(T_j, \alpha_{n_1+1}, ..., \alpha_n),$$
where T_i is a homogeneous basis of the cohomology $H^*(X, \mathbb{Q})$ and (g^{ij}) is the inverse of the matrix (g_{ij}) defined by $g_{ij} = \int_X T_i \cup T_j$. Notice that thanks to the effectivity axiom the above sum is finite. The cohomology class of the diagonal in $H^*(X \times X, \mathbb{Q})$ is $\sum_{i,j} g^{ij} T_i \otimes T_j$.
The inverse image under ψ of a map $f : C \to X$ such that $f(p_i) \in Z_i$ and $f_*[C] = \beta$ consists of maps
$$(f_1, f_2) : (C_1 \cup C_2, \{x_1, ..., x_{n_1}, x, x_{n_1+1}, ..., x_n, y\}) \to X$$
such that $f_*[C_1] + f_*[C_2] = \beta$, $f(x_i) \in Z_i$ and $f(x) = f(y)$. The first condition correspond to $\beta = \beta_1 + \beta_2$ in the formula, while the last means that $(f(x), f(y))$ must be in the diagonal.
- **Reduction axiom**. Gluing together the last two marked points we get a map $\varphi : \overline{M}_{g-1,n+2} \to \overline{M}_{g,n}$. In the same notations of the previous axiom we have
$$\varphi^* I_{g,n,\beta}(\alpha_1, ..., \alpha_n) = \sum_{i,j} g^{ij} I_{g-1,n+2,\beta}(\alpha_1, ..., \alpha_n, T_i, T_j).$$
The inverse image under φ of a map $f : C \to X$ such that $f(p_i) \in Z_i$ and $f_*[C] = \beta$ consists of maps
$$g : (\tilde{C}, \{x_1, ..., x_{n+2}\}) \to X$$
of genus $g - 1$ such that $g_*[\tilde{C}] = \beta$, $g(x_i) \in Z_i$ and $g(x_{n+1}) = g(x_{n+2})$. The last means that $(g(x_{n+1}), g(x_{n+2}))$ must be in the diagonal.
- **Deformation axiom**. Let $\pi : \mathcal{X} \to S$ be a smooth proper morphism with connected fibers, and let $X_s = \pi^{-1}(s)$. For any $s \in S$ and $\beta_s \in H_2(X_s, \mathbb{Z})$ we have a map
$$I^{X_s}_{g,n,\beta_s} : H^*(X_s, \mathbb{Q})^{\otimes n} \to H^*(\overline{M}_{g,n}, \mathbb{Q}).$$
In this setting if β_s is a locally constant section of $H_2(X, \mathbb{Z})$ and $\alpha_1, ..., \alpha_n$ are locally constant sections of $H^*(X_s, \mathbb{Q})$, then $I^{X_s}_{g,n,\beta_s}$ is constant.

The Splitting and the Reduction axioms are very important, they are fundamental to prove the associativity of quantum product.

2.3. **Tree-Level Gromov-Witten classes.** Stable curves of genus $g = 0$ are trees of $\mathbb{P}^1$'s, because of this the classes $I_{0,n\beta}$ are called *tree-level Gromov-Witten classes*, and the invariants $\langle I_{0,n,\beta}\rangle$ are called *tree-level Gromov-Witten invariants*.

Theorem 2.4. *(Kontsevich-Manin) Let X be a smooth projective variety. Assume that $H^*(X,\mathbb{Q})$ is generated by $H^2(X,\mathbb{Q})$, and that the Gromov-Witten invariants $\langle I_{0,3,\beta}\rangle(\alpha_1,\alpha_2,\alpha_3)$ are known for all $\beta \in H_2(X,\mathbb{Z})$ satisfying $-\int_\beta \omega_X \leqslant \dim X + 1$ and $\deg \alpha_3 = 2$. Then we can determine all tree-level Gromov-Witten classes $I_{0,n,\beta}(\alpha_1,...,\alpha_n)$ for all $\beta \in H_2(X,\mathbb{Z})$.*

Proof. We can assume $n \geqslant 4$. The image of the map

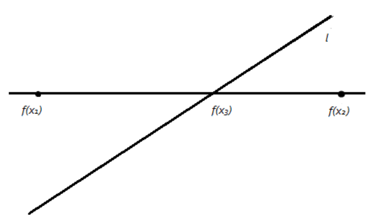
$$\psi : \overline{M}_{0,n_1+1} \times \overline{M}_{0,n_2+1} \to \overline{M}_{0,n},$$

is a divisor in $\overline{M}_{0,n}$. By permuting the markings $\{x_1,...,x_n\}$ we get other divisors, and by [Ke] the cohomology $H^*(\overline{M}_{0,n},\mathbb{Q})$ is generated by the classes of these divisors.
We first want to show that tree-level Gromov-Witten classes can be reconstructed from Gromov-Witten invariants. We proceed by induction on $n \geqslant 4$.
Let $I_{0,n,\beta}(\alpha_1,...,\alpha_n)$ be a Gromov-Witten class. If it lies in top degree then it is $\langle I_{0,n\beta}(\alpha_1,...,\alpha_n)\rangle[pt]$. Otherwise it is determined by its intersection with the divisor described above. Intersecting the given divisor with $\psi^* I_{0,n\beta}(\alpha_1,...,\alpha_n)$, by the Splitting axiom, we get Gromov-Witten classes with smaller n. Furthermore the Equivariance axiom implies that the same is true for the intersection with the other divisors. Then $I_{0,n\beta}(\alpha_1,...,\alpha_n)$ is determined by Gromov-Witten classes with smaller n, and by induction these are determined by Gromov-Witten invariants.
Quadratic relations among the Gromov-Witten invariants $\langle I_{0,n,\beta}\rangle$ for different n,β are given by some linear relations among the divisors mentioned above. Thanks to the relations we can express Gromov-Witten invariants in therms of those listed in the statement. To explain the inequality appearing in the statement notice that given α_i homogeneous classes, $\langle I_{0,3,\beta}\rangle = 0$ unless

$$\sum_{i=1}^{n} \deg \alpha_i = 2\dim[\overline{M}_{0,n}(X,\beta)]^{vir} = 2\left(-\int_\beta \omega_X + \dim X\right).$$

This equality combined with the trivial inequality $\sum_{i=1}^n \deg \alpha_i \leqslant 2\dim X + 2$, assuming $\deg \alpha_3 = 3$, gives the inequality in the statement. $\square$

2.3.1. *Tree-Level invariants of* $\mathbb{P}^2$. Consider $\beta = d[l]$ where $l \subset \mathbb{P}^2$ is a line and $d \geqslant 0$. Clearly $\langle I_{0,3,1}\rangle([pt],[pt],[l]) = 1$.



By Theorem 2.4 this is the only invariant we need to compute. By the Point mapping axiom we can assume $d \geqslant 1$. To compute $\langle I_{0,n,d}\rangle(\alpha_1,...,\alpha_n)$, we can assume α_i is $[\mathbb{P}^2],[l]$ or $[pt]$. Furthermore by the Degree axiom we have $\sum_{i=1}^n \deg \alpha_i = 6d + 2n - 2$. Notice that for $n = 0,1$ there are not solutions, and for $n = 2,3$ the only Gromov-Witten invariants are $\langle I_{0,3,1}\rangle([pt],[pt],[l]) = \langle I_{0,2,1}\rangle([pt],[pt]) = 1$.
When $n \geqslant 4$ we have $\langle I_{0,n,d}\rangle(\alpha_1,...,\alpha_n) = 0$ if $\alpha_i = [\mathbb{P}^2]$ for some i, by the Fundamental class axiom. When $\alpha_n = [l]$, by the Divisor axiom we know that $\langle I_{0,n,d}\rangle(\alpha_1,...,\alpha_{n-1},[l]) =$

$d\langle I_{0,n-1,d}\rangle(\alpha_1,...,\alpha_{n-1})$, and then we proceed inductively. The invariants that remains to compute are $\langle I_{0,n,d}\rangle([pt],...,[pt])$. Since $\deg[pt]=4$, $\sum_{i=1}^n \deg\alpha_i = 6d+2n-2$ yields $n=3d-1$. So we want to compute

$$N_d = \langle I_{0,3d-1,d}\rangle([pt],...,[pt])$$

for $d \geqslant 1$. The number N_d is the number of plane rational curves of degree d through $3d-1$ points in general position, or alternatively the degree of the *Severi variety* of degree d rational plane curves. Interpreting N_d as a Gromov-Witten invariant one gets the following.

Theorem 2.5. *The number N_d is given by the recursive formula*

$$N_d = \sum_{d=d_1+d_2,\ d_1,d_2>0} N_{d_1}N_{d_2}\left(d_1^2d_2^2\binom{3d-4}{3d_1-2} - d_1^3d_2\binom{3d-4}{3d_1-1}\right). \tag{2.3}$$

Since $N_1=\langle I_{0,2,1}\rangle([pt],[pt])=1$, this formula implies

$$N_2=1, N_3=12, N_4=620, N_5=87304,...$$

Proof. Consider the Gromov-Witten class

$$\theta = I_{0,3d,d}(\underbrace{[pt],...,[pt]}_{3d-2-times},[l],[l]]) \in H^*(\overline{M}_{0,3d},\mathbb{Q}).$$

The Degree axiom implies that this is a class of degree $2\dim\overline{M}_{0,3d}-2$, and its intersection with a divisor is a rational number. We index the marked points on stable curve by $I=\{1,...,3d-4,p,q,r,s\}$, where $\alpha_1=...=\alpha_{3d-4}=\alpha_p=\alpha_q=[pt]$ and $\alpha_r=\alpha_s=[l]$. A partition of I into disjoint subsets A,B gives a map

$$\varphi_{A,B}:\overline{M}_{0,|A|+1}\times\overline{M}_{0,|B|+1}\to\overline{M}_{0,3d}.$$

The images of the maps $\varphi_{A,B}$ are boundary divisors in $\overline{M}_{0,3d}$ and since both are fibers on boundary points on $\overline{M}_{0,4}$ of the forgetting morphism $\overline{M}_{0,3d}\to\overline{M}_{0,4}$ we have the linear equivalence

$$\sum_{r,s\in A,\ p,q\in B} Im(\varphi_{A,B}) \cong \sum_{p,r\in A,\ q,s\in B} Im(\varphi_{A,B}). \tag{2.4}$$

Notice that the diagonal in $\mathbb{P}^2\times\mathbb{P}^2$ is given by $[pt]\otimes[\mathbb{P}^2]+[l]\otimes[l]+[\mathbb{P}^2]\otimes[pt]$. By the Splitting and the Equivariance axiom one can compute that the intersection of θ with the left hand side of 2.4 is given by

$$N_d + \sum_{d=d_1+d_2,\ d_1,d_2>0} N_{d_1}N_{d_2}d_1^3d_2\binom{3d-4}{3d_1-1}.$$

Similarly the intersection with the right hand side of 2.4 is

$$\sum_{d=d_1+d_2,\ d_1,d_2>0} N_{d_1}N_{d_2}d_1^2d_2^2\binom{3d-4}{3d_1-2}.$$

Since the last two numbers are equal this complete the proof. $\square$

3. Quantum Cohomology

A quantum cohomology ring is an extension of the ordinary cohomology ring of a variety. While the cup product of ordinary cohomology describes how varieties of the variety intersect each other, the quantum cup product of quantum cohomology describes how subspaces intersect in a different way. More precisely, they intersect if they are connected via one or more pseudoholomorphic curves. Gromov-Witten invariants, which count these curves, appear as coefficients in expansions of the quantum cup product. Quantum cohomology has important implications for enumerative geometry, mathematical physics and mirror symmetry.

Let $\alpha_0 = 1 \in A^0X$, $\alpha_1, ..., \alpha_p$ be a basis of A^1X, and let $\alpha_{p+1}, ..., \alpha_m$ be a basis for the other cohomology groups. Consider the Gromov-Witten invariant

$$N(n_{p+1}, ..., n_m; \beta) = \langle I_{0,n,\beta} \rangle (\alpha_{p+1}^{n_{p+1}}, ..., \alpha_m^{n_m}),$$

for $n_i \geqslant 0$. This invariant is nonzero only when $\sum n_i(\operatorname{codim}(\alpha_1) - 1) = \dim X + \int_\beta c_1(T_X) - 3$. In this case the invariant gives the number of pointed rational maps meeting n_i general representatives of α_i for each $p+1 \leqslant i \leqslant m$. In this section, for simplicity, we use the following notation:

$$N(n_{p+1}, ..., n_m; \beta) = \langle I_{0,n,\beta} \rangle (\alpha_{p+1}^{n_{p+1}}, ..., \alpha_m^{n_m}) = I_\beta(\alpha_{p+1}^{n_{p+1}}, ..., \alpha_m^{n_m}).$$

Define

$$g_{ij} = \int_X \alpha_i \cup \alpha_j,$$

and let (g^{ij}) be the inverse matrix of (g_{ij}). That is the class of the diagonal in $X \times X$ is given by $[\Delta] = \sum_{e,f} g^{ef} \alpha_e \otimes \alpha_f$.

Remark 3.1. The class of Schubert varieties gives a natural basis for homogeneous varieties. In this case for any Schubert class α_i these is a unique j such that $g_{ij} \neq 0$ and for such a j we have $g_{ij} = 1$.

In $A^*(X \times X) = A^*X \otimes A^*X$ the following equality hold

$$\alpha_i \cup \alpha_j = \sum_{e,f} \left(\int_X \alpha_i \cup \alpha_j \cup \alpha_e \right) g^{ef} \alpha_f = \sum_{e,f} \langle I_0 \rangle (\alpha_i, \alpha_j, \alpha_e) g^{ef} \alpha_f. \tag{3.1}$$

We want to define a *quantum deformation* of the cup product 3.1 by allowing nonzero classes β. The idea is to consider a potential function, called *Gromov-Witten potential*, encoding all the enumerative informations. For a class $\gamma \in A^*X$ we define

$$\Phi(\gamma) = \sum_{n \geqslant 3} \sum_\beta \frac{1}{n!} I_\beta(\gamma^n). \tag{3.2}$$

If X is a homogeneous variety then any effective class in A_1X is a nonnegative linear combination of finitely many nonzero effective classes $\beta_1, ...\beta_p$. Using this fact one can prove that for any integer n there are only finitely many effective classes $\beta \in A_1X$, such that $I_\beta(\gamma^n)$ is nonzero [FP, Lemma 15]. Write $\gamma = \sum y_i \alpha_i$, by [FP, Lemma 15] the expression $\Phi(\gamma) = \Phi(y_0, ..., y_m)$ is a formal power series in $\mathbb{Q}[[y]] = \mathbb{Q}[[y_0, ..., y_m]]$

$$\Phi(y_0, ..., y_m) = \sum_{n_0 + ... + n_m \geqslant 3} \sum_\beta I_\beta(\alpha_0^{n_0}, ..., \alpha_m^{n_m}) \frac{y_0^{n_0}}{n_0!} ... \frac{y_m^{n_m}}{n_m!}. \tag{3.3}$$

Now we consider the partial derivatives

$$\Phi_{ijk} = \frac{\partial^3 \Phi}{\partial y_i \partial y_j \partial y_k}, \ 0 \leqslant i, j, k \leqslant m,$$

and define the *quantum product* as:

$$\alpha_i * \alpha_j = \sum_{e,f} \Phi_{ije} g^{ef} \alpha_f. \tag{3.4}$$

Extending $\mathbb{Q}[[y]]$-linearly the quantum product 3.4 to the $\mathbb{Q}[[y]]$-module $A^*X \otimes_{\mathbb{Z}} \mathbb{Q}[[y]]$, makes this into a $\mathbb{Q}[[y]]$-algebra. Notice that since the partial derivatives are symmetric in i, j, k the quantum product is *commutative*. Furthermore $\alpha_0 = 1$ is a unit for the $*$-product.

Theorem 3.2. [FP, Theorem 4] *The $\mathbb{Q}[[y]]$-algebra $A^*X \otimes_{\mathbb{Z}} \mathbb{Q}[[y]]$ endowed with the quantum product $*$ is a commutative, associative $\mathbb{Q}[[y]]$-algebra, with unit α_0.*

Remark 3.3. Since

$$(\alpha_i * \alpha_j) * \alpha_k = \sum_{e,f} \Phi_{ije} g^{ef} \alpha_f * \alpha_k = \sum_{e,f} \sum_{c,d} \Phi_{ije} g^{ef} \Phi_{fke} g^{cd} \alpha_d,$$

$$\alpha_i * (\alpha_j * \alpha_k) = \sum_{e,f} \Phi_{jke} g^{ef} \alpha_i * \alpha_f = \sum_{e,f} \sum_{c,d} \Phi_{jke} g^{ef} \Phi_{ifc} g^{cd} \alpha_d,$$

and the matrix (g^{cd}) is nonsingular, the equality $(\alpha_i * \alpha_j) * \alpha_k = \alpha_i * (\alpha_j) * \alpha_k$ is equivalent to the equations

$$\sum_{e,f} \Phi_{ije} g^{ef} \Phi_{fkl} = \sum_{e,f} \Phi_{jke} g^{ef} \Phi_{ifl}. \tag{3.5}$$

So the associativity of the quantum product $*$ is equivalent to the fact that the *Gromov-Witten potential* 3.2 satisfies the third-order differential equations 3.5, known as the *WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) equations.*

The definition of *quantum cohomology ring* depends upon a choice of a basis $\alpha_0, ..., \alpha_m$ of A^*X the rings do not, in the sense that rings obtained form different basis are canonically isomorphic. Let V be the underlying free abelian group of A^*X, and let $\mathbb{Q}[[V^*]]$ be the completion of the graded polynomial ring $\bigoplus_{i=0}^{\infty} Sym^i(V^*) \otimes \mathbb{Q}$ at its unique maximal graded ideal. In other words reintroducing coordinates we are taking the completion of the ring $\bigoplus_{i=0}^{\infty} \mathbb{Z}[y_0, ..., y_m]_i \otimes \mathbb{Q}$ at the maximal ideal $\mathfrak{M} = (y_0, ..., y_m)$, that is the ring $\mathbb{Q}[[y_0, ..., y_m]]$ of formal power series.
The quantum product defines a canonical ring structure on the free $\mathbb{Q}[[V^*]]$-module $V \otimes_{\mathbb{Z}} \mathbb{Q}[[V^*]]$. We denote by

$$QH^*X = (V \otimes_{\mathbb{Z}} \mathbb{Q}[[V^*]], *)$$

the *quantum cohomology ring* of X.

Remark 3.4. The canonical injection of abelian groups

$$i : A^*X \hookrightarrow QH^*X, \; \alpha \mapsto \alpha \otimes 1$$

is not compatible with the cup product $\cup$ and the quantum product $*$. The quantum cohomology ring QH^*X is not in general a formal deformation of A^*X over the local ring $\mathbb{Q}[[V^*]]$.

Example 3.5. For the details of this example see [FP, Section 9]. The potential function can be write as a sum

$$\Phi(y_0, ..., y_m) = \Phi_c(y) + \Phi_q(y),$$

where Φ_c is the classical part involving the terms with $\beta = 0$

$$\Phi_c(y) = \sum_{n_0 + ... + n_m = 3} \int_X (\alpha_0^{n_0} \cup ... \cup \alpha_m^{n_m}) \frac{y_0^{n_0}}{n_0!} ... \frac{y_m^{n_m}}{n_m!},$$

and the quantum part Φ_q can be replaced by the function

$$\Gamma(y) = \sum_{n_{p_1+1} + ... + n_m \geqslant 0} \sum_{\beta \neq 0} N(n_{p_1+1}, ..., n_m; \beta) \prod_{i=1}^{p} e^{(\int_\beta \alpha_i) y_i} \prod_{i=1}^{m} \frac{y_i^{n_i}}{n_i!},$$

where $N(n_{p_1+1}, ..., n_m; \beta) = I_\beta(\alpha_{p+1}^{n_p+1}, ..., \alpha_m^{n_m})$. Let us consider the case $X = \mathbb{P}^2$. Take $\alpha_0 = 1$, α_1 the class of a line, and α_2 the class of a point. Note that

$$g_{ij} = \begin{cases} 1 & i + j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

So $\alpha_i * \alpha_j = \Phi_{ij0}\alpha_2 + \Phi_{ij1}\alpha_1 + \Phi_{ij2}\alpha_0$. As instance $\alpha_1 * \alpha_1 = \alpha_2 + \Gamma_{111}\alpha_1 + \Gamma_{112}\alpha_0$.
In general the quantum product can be written as a deformation of the cup product as

$$\alpha_i * \alpha_j = \alpha_i \cup \alpha_j + \sum_{i=1}^{m} \Gamma_{ijl} g^{lk} \alpha_k.$$

Remark 3.6. Thus the quantum product contains the ordinary cup product. In general, the Poincaré dual of $\alpha_i * \alpha_j$ corresponds to the space of curves of class β passing through the Poincaré duals of α_i and α_j. So while the ordinary cohomology considers α_i and α_j to intersect only when they meet at one or more points, the quantum cohomology records a nonzero intersection for α_i and α_j whenever they are connected by one or more curves.

The quantum cohomology ring of $\mathbb{P}^2$ is given by

$$QH^*\mathbb{P}^2 \cong \mathbb{Q}[[y_0, y_1, y_2]][\alpha_1]/(\alpha_1^3 - \Gamma_{111}\alpha_1^2 - 2\Gamma_{112}\alpha_1 - \Gamma_{122}).$$

Determining Γ one can show that

$$QH^*\mathbb{P}^2 \otimes_{\mathbb{Q}[[V^*]]} \mathbb{Q}[[V^*]]/\mathfrak{M} = \mathbb{Q}[\alpha_1]/(\alpha_1^3 - 1),$$

which does not specialize to the usual cohomology ring $A^*_{\mathbb{Q}}\mathbb{P}^2 = \mathbb{Q}[\alpha_1]/\alpha_1^3$.

3.1. **Small quantum cohomology.** The small quantum cohomology ring involves only 3-pointed Gromov-Witten invariants. This ring QH^*_sX is obtained by restricting the $*$-product to divisors classes. The modified quantum potential is defined as

$$\overline{\Gamma}_{ijk} = \sum_{n\geqslant 0} \frac{1}{n!} \sum_{\beta\neq 0} I_\beta(\gamma^n, \alpha_i, \alpha_j, \alpha_k),$$

where $\gamma = y_1\alpha_1 + ... + y_p\alpha_p$ is a divisor class. By the Divisor axiom of Gromov-Witten classes we have

$$\overline{\Gamma}_{ijk} = \sum_{\beta\neq 0} I_\beta(\alpha_i \cdot \alpha_j \cdot \alpha_k) q_1^{\int_\beta \alpha_1} ... q_p^{\int_\beta \alpha_p}, \tag{3.6}$$

where $q_i = e^{y_i}$. Notice that only 3-pointed invariants occur. The product

$$\alpha_i * \alpha_j = \sum_{e,f} \overline{\Phi}_{ije} g^{ef} \alpha_f = \alpha_i \cup \alpha_j + \sum_{e,f} \overline{\Gamma}_{ije} g^{ef} \alpha_f$$

where $\overline{\Phi}_{ije} = \Phi_{ije}(y_0, ..., y_p, 0, ..., 0)$, makes the $\mathbb{Z}[q]$-module $QH^*_sX := A^*X \otimes_{\mathbb{Z}} \mathbb{Z}[q]$ into a commutative, associative $\mathbb{Z}[q]$-algebra with unit α_0. From 3.6 we see that setting the variables $q_i = 0$ we recover the usual cohomology ring A^*X.

Small quantum cohomology of $\mathbb{P}^N$. Take $X = \mathbb{P}^N$. Let α_i be the class of a linear subspace of codimension i, and let β be d times the class of a line. By the Degree axiom the number $I_\beta(\alpha_i, \alpha_j, \alpha_k)$ can be nonzero only if $i + j + k = N + (N+1)d$. This happens only for $d = 0, 1$, and in each case $I_\beta(\alpha_i, \alpha_j, \alpha_k) = 1$. Then the $*$-product is given by

- if $i + j \leqslant N$, then $\alpha_i * \alpha_j = \alpha_{i+j}$,
- if $N + 1 \leqslant i + j \leqslant 2N$, then $\alpha_i * \alpha_j = q_1\alpha_{i+j-N-1}$.

From this we get the relation $\alpha_1^{N+1} = q_1$ in $QH^*_s\mathbb{P}^N$, and the small quantum cohomology ring of $\mathbb{P}^N$ is given by:

$$QH^*_s\mathbb{P}^N = \mathbb{Z}[\alpha_1, q_1]/(\alpha_1^{N+1} - q_1).$$

Example 3.7. To fix ideas take $N = 3$. We have $i + j + k = 3 + 4d$, so 4 divides $i + j + k - 3$, and $i + j + k \leqslant 9$ forces $i + j + k = 7$ and $d = 1$. There are only two possible cases:

i	j	k	$I_\beta(\alpha_i, \alpha_j, \alpha_k)$
1	3	3	1
2	2	3	1

Because give a general plane and two general points there is a unique line passing through the points and intersecting the plane, and given two general lines and a general point there is a unique line through the point an intersecting the two given lines. Furthermore

$$\alpha_1^4 = \alpha_1^2\alpha_2 = \alpha_1\alpha_3 = q_1\alpha_0 = q_1,$$

and the small quantum cohomology ring of $\mathbb{P}^3$ is

$$QH_s^*\mathbb{P}^3 = \mathbb{Z}[\alpha_1, q_1]/(\alpha_1^4 - q_1).$$

4. Dubrovin connection

We begin giving the definitions of superalgebra, supermanifold and Frobenius manifold.

4.0.1. *Superalgebas.* Let R be a commutative ring. A *superalgebra* over R is a R-module A with a direct sum decomposition

$$A = A_0 \oplus A_1$$

together with a bilinear multiplication $A \times A \to A$ such that

$$A_iA_j \subseteq A_{i+j},$$

where the subscripts are modulo 2. A superring, or $\mathbb{Z}_2$-graded ring, is a superalgebra over the ring of integers $\mathbb{Z}$. The elements of A_i are said to be homogeneous. The parity of a homogeneous element a is 0 or 1 according to whether it is in A_0 or A_1, the parity of an element is denoted by $|a|$. An associative superalgebra is one whose multiplication is associative and a unital superalgebra is one with a multiplicative identity element. The identity element in a unital superalgebra is necessarily even.
A *commutative superalgebra* is one which satisfies a graded version of commutativity, that is A is commutative if

$$a_1a_2 = (-1)^{|a_1||a_2|}a_1a_2,$$

for any $a_1, a_2 \in A$.

Example 4.1. Any exterior algebra over R is an example of supercommutative algebra.

4.0.2. *Supermanifolds.* We first introduce the local model for supermanifolds. A *superdomain* $U^{p,q}$ is the ringed space $(U^p, \mathcal{C}^{\infty p|q})$, where U^p is an open subset of $\mathbb{R}^p$ and $\mathcal{C}^{\infty p|q}$ is the sheaf of suppercommutative rings defined by

$$\mathcal{C}^{\infty p|q}(V) = \mathcal{C}^\infty(V)[\xi_1, ..., \xi_p]$$

for any open subset $V \subseteq U$, where the ξ_i are anticommuting indeterminates such that

$$\xi_i^2 = 0,\ \xi_i\xi_j = -\xi_j\xi_i\ (i \neq j) \Longleftrightarrow \xi_i\xi_j = \xi_j\xi_i\ (1 \leqslant i, j \leqslant q).$$

The dimension of the superdomain is defined to be $p|q$. A *supermanifold* of dimension $p|q$ is a superringed space which is locally isomorphic to $\mathbb{R}^{p|q}$. The coordinates x_i on $\mathbb{R}^p$ are called *even coordinates*, while the ξ_i are called *odd coordinates*.

Remark 4.2. We defined supermanifolds in the smooth category. The same definition can be rephrased in the complex analytic category. Actually one can define more general object like superanalytic spaces and superschemes.

4.1. **Frobenius manifolds.** A complex *Frobenius manifold* $\mathcal{F}$ consists of four structures:

- a complex m-dimensional manifold M,
- a holomorphic, symmetric, non-degenerate quadratic form g on the complex tangent bundle TM,
- a holomorphic symmetric 3-tensor
$$A : TM \otimes TM \otimes TM \rightarrow \mathcal{O}_M,$$
- a holomorphic vector field $\mathbf{1}$ on M.

A and g define a commutative product $*$ on TM by:
$$g(X * Y, Z) = A(X, Y, Z),$$
where X, Y, Z are holomorphic vector fields.

Definition 4.3. A *complex Frobenius manifold* $\mathcal{F}$ is a quadruple $(M, g, A, \mathbf{1})$ satisfying the following conditions:

- *Flatness*: g is a flat holomorphic metric.
- *Potential*: M is covered by open subsets U each equipped with a commuting basis of g-flat holomorphic vector fields:
$$X_1, ..., X_m \in \Gamma(U, TM),$$
and a holomorphic potential function $\Phi \in \Gamma(U, \mathcal{O}_U)$ such that
$$A(X_i, X_j, X_k) = X_i X_j X_k(\Phi).$$
- *Associativity*: $*$ is an associative product.
- *Unit*: $\mathbf{1}$ is a g-flat unit vector field.

The associativity condition is equivalent to the $WDVV$ equations:
$$g((X_i * X_j) * X_k, X_h) = g(X_i * (X_j * X_k), X_h)$$
fro any i, j, k, h. Let ∇ be the holomorphic Levi-Civita connection induces by the metric g. For any $\lambda \in \mathbb{C}^*$ define the *Dubrovin connection* ∇_λ by
$$\nabla_{\lambda, X} Y = \nabla_X Y - \frac{1}{\lambda} X * Y.$$
The $WDVV$ equations are equivalent to the flatness of ∇_λ for any $\lambda \in \mathbb{C}*$.

Remark 4.4. A $\mathcal{C}^\infty$ Frobenius manifold is defined by requiring all the structures to be defined in the $\mathcal{C}^\infty$ category.

In the rest of the section we mainly follows the treatment of *B. Dubrovin* [Du1], [Du2]. Let $T_0 = 1, ..., T_m$ be a basis of $H^*(X, \mathbb{C})$, and let $t_0, ..., t_m$ be the corresponding supercommutative variables:
$$t_i t_j = (-1)^{\deg t_i \deg t_j} t_j t_i,$$
by the supercommutativity of the quantum product
$$T_i T_j = (-1)^{\deg T_i \deg T_j} T_j T_i.$$
We want to consider an arbitrary potential function $F : H^*(X, \mathbb{C}) \rightarrow \mathbb{C}$ instead of the Gromov-Witten potential Φ. Then F is an even formal power series in the t_i, we define the tensor A_{ijk} by
$$A_{ijk} = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k}.$$
Then we set
$$A_{ij}^k = \sum_l A_{ijl} g^{lk}.$$

We define two operations

- $T_i * T_j = \sum_k A_{ijl} g^{lk}$,
- The *Dubrovin connection*
$$\nabla^{\lambda}_{\frac{\partial}{\partial t_i}} = \lambda \sum_k A^k_{ij} \frac{\partial}{\partial t_k},$$
where $\lambda \in \mathbb{C}^*$.

The properties of the operation $*$ and of the connection ∇^{λ} are closely related.

Torsion and Commutativity. We have the equality $A^k_{ij} = (-1)^{\deg t_i \deg t_j} A^k_{ji}$. This forces $*$ to be supercommutative and ∇^{λ} to have zero torsion.

Curvature and Associativity. The connection ∇^{λ} is flat if and only if F satisfies the $WDVV$ equation, which is equivalent to the associativity of $*$.

Identity. The class T_0 is the identity for $*$ if and only if $A_{0ij} = g_{ij}$ for any i, j, and this is equivalent to $\nabla^{\lambda}_{\frac{\partial}{\partial t_0}}(\frac{\partial}{\partial t_i}) = \lambda \frac{\partial}{\partial t_i}$ for all i.

Proposition 4.5. *If F satisfies $WDVV$ equation and $A_{0ij} = g_{ij}$ for all i, j then $H^*(X, \mathbb{C})$ is a Frobenius algebra under $*$ with identity T_0.*

Proof. Under this hyphotesis $H^*(X, \mathbb{C})$ is a supercommutative algebra with identity T_0. Since F is an even function in the t_i we have $g(T_i * T_j, T_k) = g(T_i, T_j * T_k)$. So $H^*(X, \mathbb{C})$ is a Frobenius algebra under $*$. $\square$

The supermanifold $H^*(X, \mathbb{C})$ with the metric g and the even potential function F satisfying $WDVV$ equation is a an example of *Frobenius manifold*.

5. GW - Invariants, DT - Invariants and counting boxes

From the inaugural lecture given by Rahul Pandharipande at the ETH of Zürich on Tuesday March 20, 2012.

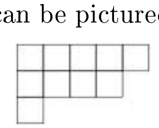
Let $n \in \mathbb{N}$ be a natural number, and let $p(n)$ be the number of partitions of n, that is the number of ways that we have to write n as a sum of positive natural numbers. As instance:

- For $n = 3$ we have 3, $2 + 1$ and $1 + 1 + 1$. Then $p(3) = 3$.
- For $n = 4$ we have 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$ and $1 + 1 + 1 + 1$. Then $p(4) = 5$.

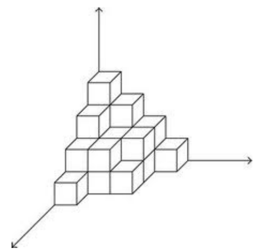
There is no direct formula for $p(n)$, however there is a formula for the generating series:
$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1 - q^k} \right).$$

Expanding the right hand side as geometric series $\frac{1}{1-q^k} = \sum_{h=0}^{\infty} q^{kh}$, we get $\prod_{k=1}^{\infty} \left(\frac{1}{1-q^k} \right) = \prod_{k=1}^{\infty} \left(\sum_{h=0}^{\infty} q^{kh} \right) = (1 + q + q^2 + q^3 + ...)(1 + q^2 + q^4 + q^6 + ...)(1 + q^3 + q^6 + q^9 + ...)... = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + ...$ and
$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{k=1}^{\infty} \left(\frac{1}{1 - q^k} \right) = 1 + q + 2q^2 + 3q^3 + 5q^4 + 7q^5 + ...$$

This formula was found by *Leonhard Euler* $(1707 - 1783)$. Partitions can be expressed as diagrams. As instance the partition $10 = 5 + 4 + 1$ can be pictured as



Such a diagram may be viewed as stacking squares in a 2-dimensional corner. Now We would like to stack boxes in a 3-dimensional corner.



Let $P(n)$ be the number of 3-dimensional partitions of n, that is the number of ways of stacking n boxes in a 3-dimensional corner. As instance $P(1)=1$, $P(2)=3$ and $P(3)=6$. Again, there is no direct formula for $P(n)$, but there is a formula for the generating series:

$$\sum_{n=0}^{\infty} P(n)q^n = \prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right)^k.$$

The formula is due to *Percy MacMahon* $(1854-1929)$. Before his mathematical career, he was a Lieutenant in the British army. He was said to be at least partially inspired by stacking cannon balls. MacMahon proposed

$$\prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right)^{\binom{k+1}{2}}$$

for the generating series of 4-dimensional partitions. But he was wrong, formulas for dimensions 4 and higher are still unknown.

In quantum field theories (and string theory), path integrals arise: integrals over the spaces of functions. Sometimes, in the presence of supersymmetry and further constraints, such path integrals are related to integration over finite-dimensional moduli spaces in algebraic geometry. As instance in gauge theory and topological string theory. In 1990's, there was an effort made in algebraic geometry to define the integration on algebraic moduli spaces predicted by path integral techniques [Ko], [LT1], [LT2], [BF]. The idea is to use deformation theory in algebraic geometry. The outcome is a virtual fundamental class and a well-defined theory of integration on many algebraic moduli spaces including the Hilbert scheme of $\mathbb{C}^3$. Let $\mathrm{Hilb}(\mathbb{C}^3)$ be the Hilbert scheme of $\mathbb{C}^3$. We consider the components of $\mathrm{Hilb}(\mathbb{C}^3)$ where

$$\dim_{\mathbb{C}}\left(\frac{\mathbb{C}[x,y,z]}{I}\right) < \infty.$$

Basically this means that we consider 0-dimensional subschemes of $\mathbb{C}^3$.
In [MP1] and [MP2] *D. Maulik, N. Nekrasov, A. Okounkov, R. Pandharipande* proved the following formula

$$\int_{\mathrm{Hilb}(\mathbb{C}^3)} (-q)^{\dim_{\mathbb{C}}\left(\frac{\mathbb{C}[x,y,z]}{I}\right)} = \prod_{k=1}^{\infty}\left(\frac{1}{1-q^k}\right)^k \tag{5.1}$$

which is MacMahon's series for counting 3-dimensional partitions. The study of such integration over $\mathrm{Hilb}(\mathbb{C}^3)$ is called Donaldson-Thomas theory, viewed as a counting theory of sheaves. Donaldson-Thomas theory can be studied for any nonsingular 3-dimensional space, not just $\mathbb{C}^3$. For example the Calabi-Yau quintic $X = Z(x^5+y^5+z^5+w^5-1) \subset \mathbb{C}^4$ the outcome is a completely non-linear generalization of the box counting of MacMahon.

Donaldson-Thomas Theory. Let X be a nonsingular, projective, Calabi-Yau 3-fold.An ideal sheaf is a torsion-free sheaf of rank 1 with trivial determinant. Since each ideal sheaf $\mathcal{I}$ injects into its double dual, and $\mathcal{I}^{**}$ is reflexive of rank 1 with trivial determinant, we have $\mathcal{I}^{**} \cong \mathcal{O}_X$ and a short exact sequence

$$0 \mapsto \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \mapsto 0,$$

where $Y \subset X$ is a subscheme. The maximal dimensional components of Y determines an element $[Y] \in H_*(X, \mathbb{Z})$. Let $I_n(X, \beta)$ be the moduli space parametrizing ideal sheaves $\mathcal{I}$ such that $\chi(\mathcal{O}_Y) = n$ and $[Y] = \beta \in H_2(X, \mathbb{Z})$. Where χ denotes the holomorphic Euler characteristic. The moduli space $I_n(X, \beta)$ is isomorphic to the Hilbert scheme of curves in X.
The Donaldson-Thomas invariant is defined via integration against the zero-dimensional virtual class

$$\tilde{N}_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1.$$

We denote by

$$Z_{DT}(q, v) = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_{n \in \mathbb{Z}} \tilde{N}_{n,\beta} q^n v_\beta$$

the partition function of the Donaldson-Thomas theory of X. One can show that for fixed β, the invariant $\tilde{N}_{n,\beta}$ vanishes for sufficiently negative n since the corresponding moduli spaces of ideal sheaves are empty. The degree zero moduli space $I_n(X, 0)$ is isomorphic to the Hilbert scheme of n points on X. The degree zero partition function,

$$Z_{DT}(q, 0) = \sum_{n \geqslant 0} \tilde{N}_{n,\beta} q^n$$

plays a special role in the theory because by 5.1 it is related to MacMahon's series for counting 3-dimensional partitions.

Another counting question began in the 19^{th} century: the counting of algebraic curves. There was a long classical development of curve enumeration. But the subject has now been recast as Gromov-Witten theory, which is the study of integration over the moduli spaces of stable maps. Let

$$N_g = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1,$$

be the Gromov-Witten invariant virtually counting genus g curves in a projective variety X. We weight N_g with u^{2g-2} where u is a formal parameter, and form the series:

$$Z_{GW}(u) = \sum_g N_g u^{2g-2}.$$

Let X be any nonsingular 3-fold. Let $Z_{DT}(q)$ be the generating series for the Hilbert scheme integrals of Donaldson-Thomas theory. Let $Z_{GW}(u)$ be the generating series for the moduli space of map integrals of Gromov-Witten theory. The main conjectured correspondence is the following:

Conjecture 5.1. *(Maulik, Nekrasov, Okounkov, Pandharipande) After the change of variables* $-q = e^{iu}$ *we have*

$$Z_{DT}(q) = Z_{GW}(u).$$

This conjecture is proven for many geometries and is still open for many others [MP1], [MP2]. This correspondence togheter with equality 5.1 tells us that boxes and curves counting questions in 3-dimensions are equivalent.

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Alex Massarenti, SISSA, via Bonomea 265, 34136 Trieste, Italy
E-mail address: alex.massarenti@sissa.it